Additional Suggested HW for Section 3.5

1. Let β be the ordered basis $\{x^2 + x + 1, x^2 - x + 1, x^2 + 2\}$ for $\mathbb{P}_2$.

   (a) Find $[p]_\beta$ if $p = 4x^2 + 3x + 2$.

   (b) Find q if $[q]_\beta = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$

2. Let γ be the ordered basis $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \right\}$ of $M(2,2)$.

   (a) Find A is $[A]_\gamma = \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix}$.

   (b) Find $[C]_\gamma$ if $C = \begin{bmatrix} 6 & 6 \\ 4 & 11 \end{bmatrix}$.

3. Let n be a positive integer. Suppose $\mathcal{V}$ is any n-dimensional vector space and $\beta = \{X_1, X_2, \ldots, X_n\}$ is any ordered basis for $\mathcal{V}$. Define the **point or basis transformation** $T_\beta^P : \mathbb{R}^n \to \mathcal{V}$ by

$$T_\beta^P\left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) = c_1X_1 + c_2X_2 + \ldots + c_nX_n \qquad \text{(i.e. } T_\beta^P([X]_\beta) = X\text{)}.$$

   Show that T_β^P is a linear transformation.

4. True or False: If a statement is true, explain why. If a statement is false, what would you have to change about it to turn it into a true statement.

   (a) Every vector space $\mathcal{V}$ is isomorphic to some $\mathbb{R}^n$.

   (b) $\mathbb{P}_{10}$ is isomorphic to $\mathbb{R}^{10}$.

   (c) $M(2,3)$ is isomorphic to $\mathbb{R}^6$.

   (d) If $\beta = \{X_1, X_2, \ldots, X_n\}$ is an ordered basis for a finite-dimensional vector space $\mathcal{V}$, then T_β^P is an isomorphism from $\mathbb{R}^n$ onto $\mathcal{V}$.

5. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 17x_1 - 8x_2 - 12x_3 \\ 16x_1 - 7x_2 - 12x_3 \\ 16x_1 - 8x_2 - 11x_3 \end{bmatrix}.$$

Let α be the standard basis of $\mathbb{R}^3$ and let β be the ordered basis of $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$ of $\mathbb{R}^3$.

(a) Find $[T]_\alpha^\alpha$.
(b) Find $[T]_\alpha^\beta$.
(c) Find $[T]_\beta^\beta$.
(d) If $[X]_\alpha = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, find $[T(X)]_\beta$.
(e) Suppose γ is the ordered basis of $\mathbb{R}^3$ formed by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find $[T]_\gamma^\beta$.

6. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Let $\mathcal{E}_2$ be the standard basis for $\mathbb{R}^2$, $\mathcal{E}_3$ be the standard basis for $\mathbb{R}^3$, and β be the ordered basis for $\mathbb{R}^3$ formed by $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$.

(a) Find $[T]_\beta^{\mathcal{E}_2}$
(b) Find $[T]_{\mathcal{E}_3}^{\mathcal{E}_2}$.
(c) If $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $T(X)$.

7. Suppose that X_1, X_2, X_3 form a basis α for a vector space $\mathcal{V}$ and that T is a linear transformation such that $T(X_1) = X_1 - X_2$, $T(X_2) = X_2 - X_3$, and $T(X_3) = X_3 - X_1$.

(a) Find $[T]_\alpha^\alpha$.
(b) Find $[T(X)]_\alpha$ if $X = X_1 - 2X_2 + 3X_3$.
(c) Use the result of part b to find $T(X)$ in terms of X_1, X_2, X_3.