411 BIG QUIZ

February 22, 2016 **Name:** KEY.

By printing my name I pledge to uphold the honor code.

I. True/False, circle T or F as appropriate. Then explain your answer by citing specific theorems/definitions/computations/etc. as time permits.

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1. a) **T** **F** In a metric space, if the limit of a sequence exists, then it is unique.

ANY TWO POINTS HAVE A DISTANCE ε APART AND $B(x,\varepsilon/2)\cap B(y,\varepsilon/2)=\phi$ SO THEY CAN'T BOTH CONTAIN ∞LY MANY SEQ. ELEMENTS!

b) **T** **F** $\mathbb{Z}$ is a complete metric space with the LUB property.

YES! ALL CAUCHY SEQ. ARE EVENTUALLY CONSTANT SO CONVERGE & EVERY BOUNDED SET ACTUALLY CONTAINS ITS LUB (STRONGER THAN)

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c) **T** **F** If $|\{a_n\}|=\infty$, then a is a subsequential limit of the sequence $\{a_n\}$ if and only if a is a limit point of the set $\{a_n\}$.

~~NOT ALL SUBSEQ. LIMITS ARE LIMIT~~

$\{1,1/2,1,1/4,1,1/8,1,1/16,\cdots\}$ HAS 1 AS A SUBSEQ. LIMIT BUT NOT A LIMIT POINT OF THE SET!!

d) **T** **F** If A is sequencially compact in a metric space X, then every open set is finite.

EVERY OPEN COVER HAS A FINITE SUBCOVER. IN GENERAL, FINITE SETS ARE CLOSED, NOT OPEN. → IN A METRIC SPACE SEQ CPT ⇔ CPT.

e) **T** **F** Under a continuous function, the inverse image of a connected set is connected.

$f(x)=x^2$

$f^{-1}((1,2))=(-\sqrt{2},-1)\cup(1,\sqrt{2})$

$f:X\to Y$ $A\subset Y$ CONNECTED

$f^{-1}(A)=\alpha\cup\beta=B$ ~~BUT~~ $f(B)$ B CONN ⇒ $f(B)$ CONN

w/ $\bar\alpha\cap\beta=\alpha\cap\bar\beta=\phi$ $f(\alpha)\cup f(\beta)=f(B)$ AND $f(B)$ SEP ⇒ B SEP. ~~BUT NOT NECESS~~ $f(\bar\alpha\cap\beta)=f(\alpha\cap\bar\beta)=\phi$ $f(\phi)=\phi$.

BUT $f(\bar\alpha\cap\beta)\neq\overline{f(\alpha)}\cap f(\beta)$

f) **T** **F** If the image of a convergent sequence is always convergent, then the function was continuous.

UNLESS BETWEEN COMPLETE M.S.

WHAT IF SEQ. CONVERGES TO NOT $f(a)$? WOULDN'T BE CONTINUOUS.

MIGHT HOLD BETWEEN COMPLETE M.S. CONV ⇔ CAUCHY UNIF. CONT ⇒ CAUCHY ↦ CAUCHY.

g) **T** **F** Given a uniformly continuous function on a bounded set, we can always extend that function to a function on the closure of the set.

MAIN THEOREM ABOUT UNIF. CONT. BUT REQUIRES COMPLETE. (OTHERWISE WE CAN'T NECESS. DEFINE f(LIMIT POINT)

h) **T** **F** The proof for the ordinary Mean Value Theorem depends on $[a,b]$ being connected.

ACTUALLY IT DEPENDS ON CPT

i) **T** **F** The only change we need to make in order to define limits at infinity and infinite limits, is to the concept of an open ball to a 'neighborhood of infinity'.

'NEIGHBORHOOD OF INFINITY' IS (a,∞) REPLACES $B(y,\varepsilon)$ (OR $B(x,\delta)$)

j) **T** **F** Taylor's theorem is a generalization of the ordinary MVT.

WRITE OUT TAYLOR'S THM. FOR $n=1$

k) **T** **F** If $f(x)$ is a function such that its nth Taylor polynomial is zero at $x=0$, then $\lim\limits_{x\to 0}\dfrac{f(x)}{x^n}=0$.

$f(x)=P_{n-1}(x)+\dfrac{f^{(n)}(y)}{(n)!}\cdot x^{n}$

BUT $f^{(n)}=0$ AS WELL (SINCE $P_n=0$) SO YES!

II. Definitions/Fill in the blank Please define/state the following.

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1. What are our two different definitions of a continuous function? Give examples of a theorem that is easier to prove with one and one that is easier to prove with the other.

$f:X\to Y$ IS CONTINUOUS IF $\forall a\in X$, $\forall\varepsilon>0,\exists\delta>0$ ST. $d_X(x,a)<\delta\Rightarrow d_Y(f(x),f(a))<\varepsilon$.

IF $\forall A\subset Y$ OPEN $f^{-1}(A)\subset X$ OPEN) PROVING EVT OR IVT MUCH EASIER!

PROVING ACTUAL FUNCTION IS CONTINUOUS EASIER.

OPEN

2. A function is <u>differentiable</u> if ...

$\lim_{t \to x} \frac{f(t)-f(x)}{t-x}$ EXISTSS (AND IS FINITE).

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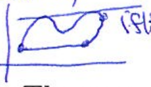
3. Please state the General Mean Value Theorem. What is the central idea behind its proof? f, g CONTINUOUS ON $[a,b]$ & DIFF ON (a,b), THEN $\exists\, c \in (a,b)$ w/ $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$

PICTURE IS  $(f(t),g(t))$ $\exists\, c \in (a,b)$ w/ TANGENT & SECANT PARALLEL $(f(t),g(t))$ PARAMETRIZED CURVE.

4. Please state Taylor's Theorem. What is the central idea in its proof?

f IS n TIMES DIFF ON $[a,b]$, THEN $\forall\, \alpha, \beta \in (a,b)$ $\exists\, \gamma$ BETWEEN α & β s.t.

$$f(\beta) = \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n$$

USE MVT n TIMES!

IDEA IS TO USE EVT ON A RELATED FUNCTION THEN ~~f(c)~~ → c LOCAL MAX/MIN $\Rightarrow f'(c)=0$

III. Short answer Please answer the following using a sentence or two.

1. Prove that if $A \subset \mathbb{R}^1$, A is connected implies A is convex.

INSTEAD PROVE NOT CONVEX $\Rightarrow$ NOT CONN.
NOT CONVEX MEANS $\exists\, x,y \in A$ ~~w/ x<~~ AND $z \notin A$ w/ $x<z<y$.
THEN $(-\infty,z)\cap A$ AND $(z,\infty)\cap A$ FORM A SEPARATION OF A (THE ONLY LIMIT POINT THEY COULD SHARE IS z & $z \notin A$.

2. Outline why every element of the Cantor set is a limit point.

FIX $\varepsilon>0$ PICK $c \in C$, CANTOR SET. W/ C_n nTH STAGE OF CANTOR CONSTRUCTION. PICK N s.t. $\frac{1}{3}^N<\varepsilon$, SO $x \in C_N$ AND INTERVAL IN C_N HAS LENGTH $<\varepsilon$ $\Rightarrow$ WHOLE INTERVAL IS IN $B(x,\varepsilon)$ $\Rightarrow$ BOTH ITS ENDPOINTS ARE IN INTERVAL & AT LEAST 1 IS $\neq x$. $\Rightarrow \forall\, x \in C\ \forall\, \varepsilon>0\ \exists\, y \in C$ w/ $y \in B(x,\varepsilon)$.

3. Describe how one would extend a uniformly continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ to a function on the closure. What do you need to prove about your definition? Will the result be continuous, uniformly continuous, etc.? (i.e. x IS A LIMIT PT)

LET x LIMIT POINT OF $A \Rightarrow \exists$ SEQ $\{a_n\} \subset A$ w/ $a_n \to a$. $\mathbb{R}^n$.
$\{f(a_n)\}$ IS CAUCHY IN A COMPLETE MS ($\mathbb{R}^m$ IS COMPLETE), f IS UNIF. CONT SO CONV. CALL THE LIMIT $\bar{f}(a)$ W/ $\bar{f}(a)=f(a)$ IF $a \in A$. NEED TO SHOW THIS IS WELL DEFINED.

4. What can you say about the derivative of a continuous function? Give an example that illustrates how bad the behavior of a derivative can get.

ITS <u>NOT</u> NECESSARILY CONTINUOUS, BUT IT DOES SATISFY IVT. EX. $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{IF } x \neq 0 \\ 0 & \text{IF } x=0 \end{cases}$

HAS DERIV. AT $x=0$ BUT $\lim_{x\to 0} f'(x)$ DNE.

$f'(x)$ IS $2x\sin(\frac{1}{x}) + \cancel{x^2}\cos\frac{1}{x}\cdot\frac{-1}{\cancel{x^2}}$

5. Give two different ways to intuitively explain L'Hoptial's rule for limits of type $\frac{0}{0}$.

RATES & TAYLOR POLYS.

6. Prove that $f'(x)$ bounded implies that f is uniformly continuous but that the converse is not necessarily true. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is uniformly continuous on all of $\mathbb{R}$ that is not a line.

$f'(x)$ BOUNDED $\Rightarrow \exists\, M$ w/ $|f'(x)|<M$ SO MVT SAYS $f'(c) = \frac{f(b)-f(a)}{b-a} < M$.
SO $|b-a|\cdot M < |f(b)-f(a)|$ SO FOR ANY $\varepsilon>0$, CAN USE $\delta = \frac{\varepsilon}{M}$ IN DEF OF UNIF CONT.

COUNTEREX TO CONVERSE

EX. SIN x SATISFIES THIS PROPERTY!