411 BIG QUIZ 2

April 25, 2016 Name: KEY

By printing my name I pledge to uphold the honor code.

I. True/False, circle T or F as appropriate. Then explain your answer by citing specific theorems/definitions/computations/etc. as time permits.

1. a) T **F** There are no substantially new metric spaces in Math 411.

$C_X(Y)$ IS A NEW TYPE OF METRIC SPACE! (THOUGH WE HAVE SEEN METRICS THAT COME FROM NORMS BEFORE.

b) T **F** All sets that are Lebesgue measurable but not Borel measurable have Lebesgue outer measure zero.

THE DIFFERENCE BETWEEN AN ~~ELE~~ ELEMENT OF $\mathcal{L}-\mathcal{B}$ AND NEAREST ELEMENT OF $\mathcal{B}$ IS A SET OF MEASURE ZERO

c) **T** F If $A \in \mathcal{L}$ where $\mathcal{L}$ is the Lebesgue σ-algebras, then the Lebesgue measure of A is equal to the Lebesgue outer measure of A.

IF $A \in \mathcal{L}$ THEN LEBESGUE OUTER MEASURE IS ACTUALLY ADDITIVE INSTEAD OF JUST SUB-ADDITIVE.

d) **T** F If the uniform limit exists, it is always also a pointwise limit.

~~A~~ IT IS POINTWISE LIMIT WHERE N DOESN'T DEPEND ON x!

e) **T** F If $f_n \to f$ where $f_n, f \in \mathcal{C}_Y(K)$ with K compact, then the convergence was uniform.

WE ~~PROVED~~ THIS IN CLASS. NOT TRUE IF K ISN'T MENTIONED CPT.

f) ~~T~~ **F** All finite sets of continuous functions are equicontinuous.

~~SS~~ UNIFORMLY!

g) **T** F The Cantor function is constant almost everywhere.

IT'S NOT CONSTANT ON THE CANTOR SET AND $\lambda(C) = 0$

h) **T** F There exists an example of continuous functions converging (pointwise) to the Dirchlet function (1 for rationals, 0 for irrationals).

$f_m = \lim_{n\to\infty} (\underbrace{\cos m!\pi x}_{\text{CONTINUOUS}})$ $\lim_{m\to\infty} f_m(x) = $ DIRCHLET FCTN.

i) **T** F The integral of the uniform limit of functions is the pointwise limit of the integrals. IT'S ALSO THE UNIFORM LIMIT OF INTEGRALS, BUT UNIFORM LIMIT IS ALWAYS ALSO A POINTWISE LIMIT.

j) T **F** The Weierstrass theorem implies that the set of polynomials with coefficients in $\mathbb{Q}$ is dense in $\mathcal{C}_{\mathbb{R}}(\mathbb{R})$

TOO MUCH TO ASK. IT IS TRUE FOR $\mathcal{C}_{\mathbb{R}}([a,b])$ $a<b$, $a,b \in \mathbb{R}$

II. Definitions/Fill in the blank Please define/state the following.

1. Please define the Lebesgue outer measure. Is it a measure? Why/why not?

$\lambda^*(E) = \inf\{\sum b_i - a_i \mid E \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\}$

NOT A MEASURE UNLESS WE RESTRICT OURSELVES TO $\mathcal{L}$ (LEBESGUE σ-ALG) ~~...~~ AS IT IS ONLY COUNTABLY SUB ADDITIVE $\lambda^*(\bigcup_{i=1}^{\infty} B_i) \le \sum_{i=1}^{\infty} \lambda^*(B_i)$

2. What was our concluding theorem for Riemann-Stiltches integration? Why did it 'solve the problem' of R-S integration? α MONOTONIC INC, $\alpha' \in \mathcal{R}([a,b])$

$\Rightarrow \int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$

AND $f \in \mathcal{R}(\alpha)[a,b] \Leftrightarrow f\alpha' \in \mathcal{R}[a,b]$

BECAUSE NOW THAT IT'S AN ORDINARY RIEMANN INTEGRAL, WE CAN USE THE FTC.

3. We proved that a function with only finitely many discontinuities on $[a, b]$ is in $\mathcal{R}[a, b]$. What is the general, measure theoretic version of this result? What is the central idea in its proof? MEASURE THEORETIC VERSION IS RIEMANN-LEBESGUE TH. $f \in \mathcal{R}([a,b]) \Leftrightarrow$ DISCONTINUITIES OF f HAS MEASURE 0.

IDEA OF PROOF: SEPARATE OUT REALLY BAD DISCONTINUITIES, MAKE THE PARTITION ELEMENTS CONTAINING THESE V. SMALL. THE FACT THAT THE REST IS 'CLOSE TO CONTINUOUS' ALLOWS US TO FIND A P FOR THAT PART TOO.

4. What is the difference between a pointwise bounded set of functions and uniformly bounded one? Please provide an example. Why did we define these in the first place?

$\{f_\alpha(x)\}_{\alpha\in I}$ POINTWISE BOUNDED IF $\exists\ \varphi: X \to \mathbb{R}$ ST. $|f_\alpha(x)| < \varphi(x)\ \forall x\in X, \alpha \in I$.

$\{f_\alpha(x)\}_{\alpha\in I}$ UNIF. BOUNDED IF $\exists M$ W/ $|f_\alpha(x)| < M\ \ \forall x \in X, \alpha\in I$.

$\{y=x^2\}$ IS POINTWISE BOUNDED $\{y=\sin x\}$ IS UNIF. BDD. DEFINED THEM IN OUR 1ST ATTEMPT AT COMPACTNESS ON $\mathcal{C}_{\mathbb{R}}(X)$.

5. Please state the real version of the Stone-Weierstrass theorem (be sure to define all of your terms). What is one consequence of the Stone-Weierstrass theorem involving $\mathbb{Q}[x]$? $\mathcal{A} \subset \mathcal{C}_{\mathbb{R}}(K)$ SUB-ALGEBRA (CLOSED UNDER $+, \cdot$, SCALAR MULT) IS DENSE IN $\mathcal{C}_{\mathbb{R}}(K) \Leftrightarrow \mathcal{A}$ CONTAINS A NON-ZERO CONSTANT FCTN & SEPARATES POINTS (IF $x \neq y \in K$, $\exists f \in \mathcal{A}$ W/ $f(x) \neq f(y)$)

$\mathbb{Q}[x]$ CONTAINS NON-ZERO CONSTANT & IF $a \neq b \Rightarrow f(x) = 2x$ SEPARATES a & b ($a \neq b \Rightarrow 2a \neq 2b$)

III. Short answer Please answer the following using a sentence or two.

1. What does it mean for a set to have Lebesgue outer measure zero? *Prove* that any countable set has Lebesgue measure zero.

$\lambda^*(E) = 0 \Leftrightarrow \forall \varepsilon > 0\ \exists\ \{[a_i, b_i]\}_{i=1}^{\infty}$ W/ $E \subset \bigcup (a_i, b_i)$ AND $\sum b_i - a_i < \varepsilon$.

(THIS USES DEF OF INFIMUM TO BE EQUIVALENT TO II.1)

LET $E \subset \mathbb{R}$ COUNTABLE (WE ONLY DEFINED λ^* FOR SUBSETS OF $\mathbb{R}$)

$\Rightarrow E = \{a_1, a_2, \cdots a_n\}$ THEN FOR ANY $\varepsilon > 0$, CHOOSE $0 < \varepsilon_1 < \varepsilon$ AND

$E \subset (a_1 - \frac{\varepsilon_1}{2n}, a_1 + \frac{\varepsilon_1}{2n}) \cup (a_2 - \frac{\varepsilon_1}{2n}, a_2 + \frac{\varepsilon_1}{2n}) \cup \cdots \cup (a_n - \frac{\varepsilon_1}{2n}, a_n + \frac{\varepsilon_1}{2n})$

AND $\sum a_i + \frac{\varepsilon_1}{2n} - a_i + \frac{\varepsilon_1}{2n} = \varepsilon_1 < \varepsilon$.

2. Outline the proof of FTC part 1.

FTC 1 SAYS IF $f \in \mathcal{R}[a,b] \Rightarrow$ $F(x) = \int_a^x f(t)\,dt$ IS CONTINUOUS ON $[a,b]$ & IF f IS CONTINUOUS AT x_0, F IS DIFF. AT x_0 W/ $F'(x_0) = f(x_0)$. $s < x_0 < t$ $\left|\frac{F(t) - F(s)}{t-s} - f(x_0)\right| = \left|\frac{\int_s^t f(t)\,dt - f(x_0)}{t-s}\right|$

$= \left|\frac{\int_s^t f(t) - \int_s^t f(x_0)\,dt}{t-s}\right| = \left|\frac{\int_s^t f(t) - f(x_0)\,dt}{t-s}\right|$ CHOOSE s, t SO THIS IS V. SMALL

3. Prove the uniform limit of bounded functions is bounded.

LET $f_n: X \to Y$ W/ $|f_n(x)| < M_n$ AND $f_n \to f$ UNIF.

$\Rightarrow \forall \varepsilon > 0\ \exists N$ ST. $|f_n(x) - f(x)| < \varepsilon\ \ \forall n \geq N, \forall x \in X$. LET $\varepsilon = 1$

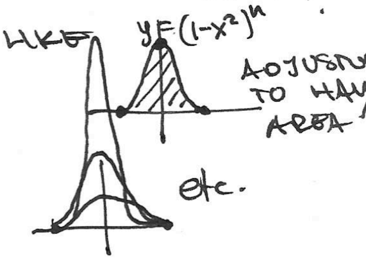
$\Rightarrow |f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N\ \ \forall x \in X$ SO f IS BOUNDED AS WELL.

4. Why was finding a subsequence converging on a countable subset so helpful? What was the missing ingredient to finding a full convergent subsequence and why?

EVERY COMPACT SET HAS A COUNTABLE DENSE SUBSET, SO IF $\{f_n\} \subset \mathcal{C}_{\mathbb{R}^n}(K)$ & $E \subset K$ COUNTABLE $\Rightarrow \exists$ SUBSEQ. CONV. ON E, CHOOSE E TO BE DENSE & OUR SUBSEQ CONVERGES WITHIN ε OF ANY POINT OF K, $\forall \varepsilon > 0$. MISSING INGREDIENT IS EQUICONTINUITY WHICH ALLOWS US TO USE CONTINUITY OF OUR SUBSEQ. TO GET CONTINUITY OF POINTWISE LIMIT.

5. Describe the 'basis' of functions used in the proof of the Weierstrass theorem. How was this set of functions helpful in the proof?

$Q_n(x) = c_n(1-x^2)^n$ W/ $c_n = \frac{1}{\int_{-1}^{1}(1-x^2)^n\,dx}$ SO Q_n LOOK LIKE



WE USED THEM TO DESIGN OUR POLYNOMIALS $p_n(x) = \int_{-1}^{1} f(x+t) Q_n(t)\,dt$ CONVERGING TO f

IT'S A POLY SINCE $Q_n(t)$ IS (MULTIPLY IT OUT) & MOST OF Q IS CLOSE TO 0.