Math 411 Homework 1 Answers

1. Continuity and Compactness Let $f : X \to Y$ metric spaces.

a) Show that if f is continuous and $A \subset X$ is compact, then $f(A)$ is compact.

We talked about this in class, and if you use the topological definition of continuity, it is quite easy!
To show that $f(A) \subset Y$ is compact, let $\{U_\alpha\}$ be an arbitrary open cover of $f(A)$ (so $U_\alpha \subset Y$ is open and $f(A) \subset \bigcup_\alpha U_\alpha$). Since f is continuous and U_α is open, we know that $f^{-1}(U_\alpha) \subset X$ is open for all α. Therefore, to show that $\{f^{-1}(U_\alpha)\}$ form an open cover, we only need show that $A \subset \bigcup_\alpha f^{-1}(U_\alpha)$. Let $a \in A$, then, by definition of the image of a set $f(a) \in f(A)$ and since $\{U_\alpha\}$ covers $f(A)$, there exists α such that $f(a) \in U_\alpha$ and therefore, $a \in f^{-1}(U_\alpha)$ by definition of the inverse image (see the solution to problem 5a for the definition). Therefore, $A \subset \bigcup_\alpha f^{-1}(U_\alpha)$ and $\{f^{-1}(U_\alpha)\}$ form an open cover of A. Since A is compact, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ that still covers A. Claim: $\{U_{\alpha_1}, U_{\alpha_2}, ..., U_{\alpha_n}\}$ form an open cover of $f(A)$ (which is, of course, a finite subcover of our original open cover). To prove our claim, let $y \in f(A)$. Then there exists $a \in A$ with $y = f(a)$ (by definition of the image of a set), and since $\{f^{-1}(U_{\alpha_i}\}_{i=1}^n$ form an open cover of A, there exists $i \in \{1, 2, ..., n\}$ with $a \in f^{-1}(U_{\alpha_i})$. Therefore, by the definition of a preimage, $y = f(a) \in U_{\alpha_i}$, so we can conclude that $f(A) \subset \bigcup_{i=1}^n U_{\alpha_i}$. Since our original cover was arbitrary, we have proven that every open cover has a finite subcover, and therefore $f(A)$ is compact.

b) Use the result of part (a) to prove the extreme value theorem.

The theorem we want to prove is as follows: Let $f : [a, b] \to \mathbb{R}$ continuous. Then there exists $m, M \in [a, b]$ such that for all $x \in [a, b]$, $f(m) \leq f(x) \leq f(M)$. ($f(m)$ is the minimum value of $f(x)$ on $[a, b]$ and $f(M)$ is the maximum value of $f(x)$ on $[a, b]$.)
Since $[a, b] \subset \mathbb{R}$ is closed an bounded in the Euclidean metric on $\mathbb{R}^n$ for some $n \in \mathbb{N}$, $[a, b]$ is compact, and since f is continuous, by **8a)** we get that $f([a, b]) \subset \mathbb{R}^1$ is compact as well. Therefore, by problem **1** of this homework, $f([a, b])$ contains both its supremium and its infemium. This means that $\sup f([a, b]) \in f([a, b])$ and therefore (by the definition of the image of a set) there exists at least one $M \in [a, b]$ with $f(M) = \sup f([a, b])$. By the definition of the supremium and of the image of a set, we have $f(x) \leq f(M)$ for all $f(x) \in f([a, b])$. Similarly, there exists at least one $m \in [a, b]$ with $f(m) = \inf f([a, b])$ and by the definitions of infemium and image of a set $f(m) \leq f(x)$ for all $f(x) \in f([a, b])$ which, together with the previous sentence, is what we wanted to prove.

2. Cantor Set This problem is all about the Cantor middle third set which is achieved by starting with the interval $[0, 1]$ and taking away first its middle third, then the middle thirds of the two remaining intervals, then the middle thirds of the four remaining intervals, etc..

a) Prove that the Cantor set is compact.

Let $E_0 = [0, 1]$ which is a closed and bounded subset of $\mathbb{R}^1$ so is compact. Assume that we have $E_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cup \cdots \cup [\frac{3^n-1}{3^n}, 1]$ (which is a closed an bounded subset of $\mathbb{R}$ so compact) and define E_{n+1} by removing the middle third of each subinterval. The result, is another compact subset of $\mathbb{R}$ and by induction, we have defined E_n for $n \in \mathbb{N}$ all of which are compact. Since, by a proof in class, an arbitrary intersection of compact sets is non-empty, closed and bounded and hence compact, the Cantor set $C = \bigcap_{n=1}^\infty E_n$ is non-empty and compact.

b) Prove that the Cantor middle third set has zero length.

Using the same notation as above, notice that the length of E_0 (written $|E_0|$) is $1-0=1$. When we go from E_0 to E_1, we remove the middle third of the interval, and get $|E_1 = 1-\frac{1}{3}$. Similarly, E_2 is constructed from E_1 by removing the middle thirds of the two remaining subintervals, each of which had length $\frac{1}{3}$, which means a total of $\frac{2}{9}$ is removed in this step. Namely $|E_2| = 1-\frac{1}{3}-\frac{2}{9}$. By induction $|E_n| = 1-\frac{1}{3}-\frac{2}{9}-\cdots-\frac{2^{n-1}}{3^n} = 1-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{2}{3}\right)^i$. Therefore, $|C| = 1-\frac{1}{2}\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^i = 1-\frac{1}{2}\left(\frac{1}{1-\frac{2}{3}}-1\right) = 1-1 = 0$. (We used the fact that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r|<1$.)

c) Prove that the Cantor set is uncountable.

There are several ways to go about this, one of which is in Abbott, but this one is my favorite!

We define the *trinary decimal expansion* of a real number $x \in [0,1)$ as $x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$ where $a_i \in \{0,1,2\}$. This works exactly like the ordinary decimal expansion of a number but with powers of 3 instead of powers of 10. With the ordinary base 10 decimal expansion of $x \in [0,1)$, if $x = .b_1b_2b_3...$ as its ordinary decimal expansion, if $b_1 = 0$, then $x \in [0,.1)$, if $b_1 = 1$, then $x \in [.1,.2)$, etc. Similarly, if $x = .a_1a_2a_2...$ as its trinary decimal expansion, then $a_1 = 0$ implies $x \in [0,\frac{1}{3})$, $a_1 = 1$ implies $x \in [\frac{1}{3},\frac{2}{3})$ and $a_1 = 2$ implies $x \in [\frac{2}{3},1)$. Therefore, if $x \in C$, either $x = 1$ or $x = \frac{1}{3}$ or $a_1 = 0$ or $a_1 = 2$. Similarly, the second decimal place tells us which $\frac{1}{3}$ of the intervals $[0,\frac{1}{3})$, $[\frac{1}{3},\frac{2}{3})$ or $[\frac{2}{3},1)$ is in. In order to be in the cantor set, either $x = 1, x = \frac{1}{3}, x = \frac{1}{9}$ or $x = \frac{8}{9}$ or $a_2 = 0$ or $a_2 = 2$. In fact, if a number $x = .a_1a_2a_3...$ as its trinary decimal expansion with $a_i \in \{0,2\}$, then x is not in any middle thirds of intervals, so is in the Cantor set. (Notice, as we saw above, the only things we are going to miss using this criterion are endpoints and since all endpoints of subintervals are rational there are only countably many of them.) Since every single trinary decimal with only 0s and 2s is in the Cantor set and there are uncountably many trinary decimals of this form.

(The Cantor diagionalization argument works to show this: assume the contrary and let

$c_1 = .a_{1,1}a_{1,2}a_{1,3}...$

$c_2 = .a_{2,1}a_{2,2}a_{2,3}...$

$c_3 = .a_{3,1}a_{3,2}...$

$\vdots$

be a list of all of the trinary decimals with $a_{i,j} \in \{0,2\}$. We form a new decimal $c = .c_1c_2c_3...$ where $c_i = \begin{cases} 0 & \text{if } a_{i,i} = 2 \\ 2 & \text{if } a_{i,i} = 0 \end{cases}$. Then $c \neq c_1$ since their first trinary digit don't agree and the only way to have two decimal expansions be equal is to either have all their digits agree or to have $.22222\cdots = 1.00000\cdots$ and all of our decimals are in $[0,1)$. Similarly, $c \neq c_2$ since their second trinary digits don't agree and the only way for two decimals to be the same with different second trinary digit is to have carrying of the form .1000... = .02222.... or .2000... = .12222..... or 1.000... = .2222.... and since we are not allowing any 1 digits, this can't happen. (As you can see, we do need to be slightly careful with this argument!) Similarly, the $c \neq c_n$ for any n because we are not allowing any trinary digits equal to 1. Therefore, c is not on our list, so our list can not be complete and therefore the set of all trinary decimals with only 0s and 2s is uncountable.)

Since this set is contained in (but not equal to) the Cantor set, the Cantor set must be uncountable as well.

One can also do the same type of argument with strings of Ls and Rs where each element of the Cantor set is specified by stating wether it is the left hand (L) or right hand (R) side of each iteration. For example, the number $\frac{1}{4}$ would start out with L since it's in $[0, \frac{1}{3}]$ and then an R since it's in $[\frac{2}{9}, \frac{1}{3}]$, etc. Then each element of the Cantor set is specified by a unique string of Ls and Rs and since there are uncountably infinitely many such strings, the Cantor set is uncountable.

3. Connected/Separated

a) Give an example of a set A such that A is not connected but $A \cup \{1\}$ is connected.

An example is $A = (0,1) \cup (1,2)$. $A \cup \{1\} = (0,2)$ which is connected.

b) Show that in part (a) 1 must be a boundary point of A. (Recall the technical definition of a boundary point as $\partial A = \overline{A} - \text{Int}(A)$.)

We are assuming that there exists $\alpha, \beta \subset \mathbb{R}^1$ such that $A = \alpha \cup \beta, \overline{\alpha} \cap \beta = \varnothing$ and $\alpha \cap \overline{\beta} = \varnothing$, but that there does not exist a separation of $A \cup \{1\}$. In particular, this means that neither $\alpha \cup \{1\}$ and β nor α and $\beta \cup \{1\}$ form separations of $A \cup \{1\}$. Since $\alpha \cup \{1\}$ and β don't form a separation, either $\overline{\alpha \cup \{1\}} \cap \beta \neq \varnothing$ or $(\alpha \cup \{1\}) \cap \overline{\beta} \neq \varnothing$. Now, $1 \notin A$ (since A was separated and $A \cup \{1\}$ was not), so $1 \notin \alpha$ and $1 \notin \beta$. Since $1 \notin \beta$ and $\overline{\alpha \cup \{1\}} = \overline{\alpha} \cup \overline{\{1\}} = \overline{\alpha} \cup \{1\}$ by homework 7 problem 5b and $\overline{\alpha} \cap \beta = \varnothing$, we can conclude that $\overline{\alpha \cup \{1\}} \cap \beta = \varnothing$, so it must be the case that $(\alpha \cup \{1\}) \cap \overline{\beta} \neq \varnothing$. Hence, $1 \in \overline{\beta}$, but $1 \notin \beta$, so $1 \in \partial\beta$. Similarly, the fact that α and $\beta \cup \{1\}$ don't form a separation of A gives $1 \in \partial\alpha$. Therefore, $\partial A = \overline{A} - \text{Int}(A) = \overline{\alpha \cup \beta} - \text{Int}(A) = \overline{\alpha} \cup \overline{\beta} - \text{Int}(A)$. Since $1 \notin A$, $1 \notin \text{Int}(A)$ and therefore, $1 \in \partial A$.

4. Continuity and Connectedness Let $f : X \to Y$ metric spaces.

a) Show that if f is continuous and A is connected, then $f(A)$ is connected. (Recall: A is connected if there doesn't exist a separation. A separation is a pair of sets α and β with $A = \alpha \cup \beta$ such that $\overline{\alpha} \cap \beta = \varnothing$ and $\alpha \cap \overline{\beta} = \varnothing$. Also recall that proofs using the contrapositive are often profitable for proving connectedness. In addition, you may want to use the version of topological continuity given above (inverse image of closed sets are closed) and the fact that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.)

Proof by contrapositive. Assume that $f(A)$ is separated, prove that either f is not continuous or that A is not connected. Since it is part of our hypothesis that f is continuous, this would imply that A is separated, and therefore, we could conclude that f continuous and A connected implies $f(A)$ is connected.

That is, assume that $f(A) \subset Y$ is separated. That is, assume there exist α and β in Y such that $f(A) = \alpha \cup \beta$ with $\overline{\alpha} \cap \beta = \varnothing$ and $\alpha \cap \overline{\beta} = \varnothing$. Assume that f is continuous.

Claim: $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ form a separation of A.

Proof of claim:
Continuity of f implies t-continuity. Namely, that the inverse image of open sets are open. Since $f^{-1}(A^c) = X - f^{-1}(A) = f^{-1}(A)^c$, this is the same as saying that the inverse image of all closed sets are closed (strictly speaking, we should be speaking about relatively open and relatively closed as in the previous problem). Therefore, $f^{-1}(\overline{\alpha})$ is closed and so $f^{-1}(\overline{\alpha}) \supset \overline{f^{-1}(\alpha)}$. Similarly, $f^{-1}(\overline{\beta}) \supset \overline{f^{-1}(\beta)}$.

In addition, $f^{-1}(C \cap D) = \{x \in X \mid f(x) \in C \cap D\} = \{x \in X \mid f(x) \in C \text{ and } f(x) \in D\} = \{x \in X \mid x \in f^{-1}(C) \text{ and } x \inf^{-1}(D)\} = f^{-1}(C) \cap f^{-1}(D)$. In order to check that $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ form a separation, we need to compute $\overline{f^{-1}(\alpha)} \cap f^{-1}(\beta)$ and $f^{-1}(\alpha) \cap \overline{f^{-1}(\beta)}$. However, $\overline{f^{-1}(\alpha)} \cap f^{-1}(\beta) \subset f^{-1}(\overline{\alpha}) \cap f^{-1}(\beta) = f^{-1}(\overline{\alpha} \cap \beta) = f^{-1}(\varnothing) =$

$\varnothing$. Similarly, $f^{-1}(\alpha) \cap \overline{f^{-1}(\beta)} = \varnothing$ and they do, indeed, form a separation.

b) Use the result of part (a) to prove the intermediate value theorem.

Let $f : [a, b] \to \mathbb{R}$ continuous and $K \in \mathbb{R}$ with K between $f(a)$ and $f(b)$, show that there exists $c \in [a, b]$ with $f(c) = K$. Since $[a, b]$ connected (we proved outside that connected if and only if convex in $\mathbb{R}^1$ and $[a, b]$ is convex practically by definition), therefore, by part a), $f([a, b])$ is connected in $\mathbb{R}^1$. This implies that $f([a, b])$ is convex, and since $f(a) \in f([a, b])$ and $f(b) \in f([a, b])$, the entire line segment connecting $f(a)$ and $f(b)$ is in $f([a, b])$. This means that for all K between $f(a)$ and $f(b)$, $K \in f([a, b])$. This means, by definition of the image of a set, that there exists $c \in [a, b]$ such that $f(c) = K$ which is the intermediate value theorem.

5. Show that any function whose domain is the integers is uniformly continuous.

Let $f : \mathbb{Z} \to Y$ where Y is any metric space. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that $d_{\mathbb{Z}}(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$.
This will be true for any $\epsilon > 0$ as long as we pick $\delta \leq 1$! Since the only pairs of elements $x, y \in \mathbb{Z}$ satisfying $d_{\mathbb{Z}}(x, y) < 1$ are $x = y$ and if $x = y$, then $d_Y(f(x), f(y) = 0 < \epsilon$, the function will be uniformly continuous.

6. Prove that if $f(x)$ is uniformly continuous on (a, b) and continuous at $x = a$ and at $x = b$, then $f(x)$ is uniformly continuous on $[a, b]$.

If let $f : [a, b] \to Y$ where Y is any metric space. Then $f : (a, b) \to Y$ uniformly continuous implies that f is continuous on (a, b) as well (so for every $x \in (a, b)$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $f((x-\delta, x+\delta)) \subset B_Y(f(x), \epsilon)$. Similarily, since f is continuous at a, for every $\widetilde{\epsilon} > 0$, there exists a $\widetilde{\delta} > 0$ such that $f((a - \widetilde{\delta}, a + \widetilde{\delta})) \subset B_Y(f(a), \widetilde{\epsilon})$. In addition, since f is continuous at b, for every $\hat{\epsilon} > 0$ there exists $\hat{\delta} > 0$ such that....
Basically, what we are saying is that f is continuous on all of $[a, b]$ and since $[a, b]$ is a closed bounded interval in $\mathbb{R}^1$, then $[a, b]$ is compact, so f is continuous on a compact set so is uniformly continuous!

7. Uniform Continuity and Boundedness

a) Show that if f is uniformly continuous on a bounded interval (*not* necessarily closed!), then f is bounded.

Assume that $f : (a, b) \to Y$ is uniformly continuous on (a, b) with $-\infty < a < b < \infty$. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$. Prove that there exists $p \in Y$ and M such that $\{f(x) \mid x \in (a, b)\} \subset B_Y(p, M)$.
Let $p = f(\frac{a+b}{2})$ and let $\epsilon = 1$. Then there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $d_Y(f(x), f(y)) < 1$. Notice that $(a, b) = B(\frac{a+b}{2}, \frac{b-a}{2})$ and let $\frac{b-a}{2} = N \cdot \delta$. Then if $x \in (\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta)$, then $f(x) \in B(p, 1)$. Similarly, if $x \in (\frac{a+b}{2} - 2\delta, \frac{a+b}{2})$, then $f(x) \in B(f(\frac{a+b}{2} - \frac{\delta}{2}, 1)$ and since $f(\frac{a+b}{2} - \frac{\delta}{2}) \in B(p, 1)$, $f(x) \in B(p, 2)$ by the triangle inequality. In this way, we can see that if $x \in (a, b)$, then $f(x) \in B(p, N)$, so the set of $\{f(x) \mid x \in (a, b)\}$ is bounded in Y.

b) Give an example of a continuous function on $(0, 1)$ that is not bounded.

$f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ since it is the ratio of two non-zero continuous functions.

c) Give an example of a bounded continuous function on $(0,1)$ that is not uniformly continuous.

$f(x) = \frac{1}{x}$ works for this as well since if $f(x)$ were uniformly continuous, then there would exists $C \in \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ C & \text{if } x = 0 \end{cases}$ is continuous on $[0,1]$ (a uniformly continuous function on A can be extended to a continuous function on $\overline{A}$). Since the only possible answer is $C = \infty$ and $\infty \notin \mathbb{R}$, f can't be uniformly continuous.

8. Continuity and Cauchy

a) Show that if f is uniformly continuous on (a,b) and $\{x_n\} \subset (a,b)$ is a Cauchy sequence, then $\{f(x_n)\}$ is a Cauchy sequence.

We did this in class, just in a slight bit more generality!
Let $f : (a,b) \to Y$. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, for all $\widetilde{\epsilon} > 0$, there exists $\widetilde{N}$ such that $|x_n - x_m| < \widetilde{\epsilon}$ whenever $n, m \geq \widetilde{N}$. We also know that f is uniformly continuous, so for every $\hat{\epsilon} > 0$, there exits a $\hat{\delta} > 0$ such that $|x - y| < \hat{\delta}$ implies that $d_Y(f(x), f(y)) < \hat{\epsilon}$. We want to show that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Namely, show that for all $\epsilon > 0$, there exits N such that $d_Y(f(x_n), f(x_m)) < \epsilon$ whenever $n, m \geq N$.

Let $\hat{\epsilon} = \epsilon$ and let the $\hat{\delta}$ we get from the definition of uniformly continuous be equal to $\widetilde{\epsilon}$ from the definition of a Cauchy sequence in $\mathbb{R}$. Then let the resulting $\widetilde{N}$ be our N. Then, if $n, m \geq N = \widetilde{N}$, then $|x_n - x_m| < \widetilde{\epsilon} = \hat{\delta}$. Therefore, since f is uniformly continuous, $d_Y(f(x_n), f(x_m)) < \widetilde{\epsilon} = \epsilon$ which is what we wanted to show!

b) Give an example of a continuous function f on $(0,1)$ and a Cauchy sequence $\{x_n\} \subset (0,1)$ such that $\{f(x_n)\}$ is not Cauchy.

Using our example from the previous problem, let $f(x) = \frac{1}{x}$ and let our Cauchy sequence be $\{\frac{1}{n}\}_{n=1}^{\infty}$. The image sequence is $\{f(\frac{1}{n}) = n\}_{n=1}^{\infty}$ which diverges in $\mathbb{R}$ so is not Cauchy.

c) Show that if f is continuous but not uniformly continuous on (a,b), then there exists a Cauchy sequence $\{x_n\} \subset (a,b)$ such that $\{f(x_n)\}$ is not Cauchy.

This is the hardest problem on the homework!

One important thing to notice is that if f was continuous at a and continuous at b, then f would be continuous on a compact set, so uniformly continuous, so this feat would be impossible by part a of this problem. Therefore, f must fail to be continuous at either $x = a$ or at $x = b$.

Case 1. f is not continuous at $x = a$.

This means that it is not true that for all $\widetilde{\epsilon} > 0$, there exits $\widetilde{\delta} > 0$ such that $f((a - \widetilde{\delta}, a + \widetilde{\delta})) \subset B_Y(f(a), \epsilon)$. Since one of the things that could cause this to not hold is $f(a)$ not existing (which isn't very helpful), we're going to instead negate the definition of uniform continuity but specify that it fails at a. Namely, there exists $\widetilde{\epsilon} > 0$ such that for all $\widetilde{\delta} > 0$ there exists x, y such that $|x - y| < \widetilde{\delta}$ but $d_Y(f(x), f(y)) \geq \widetilde{\epsilon}$. Let $\delta_n = \frac{1}{n}$. Then this statement holds for all of these δ_n, but only for x, y near a. Namely, there exists $x_n, y_n \in (a,b)$ with $x_n, y_n \in (a, a + \delta_n)$ such that $d_Y(f(x_n), f(y_n)) \geq \widetilde{\epsilon}$. Consider the sequence $\{z_n\}_{n=1}^{\infty} = \{x_1, y_1, x_2, y_2, x_3, y_3, ...\}$. This sequence is Cauchy in (a,b) since $x_i, y_i \in (a,b)$ and for any $\epsilon > 0$, there exists N such that $\frac{2}{N} < \epsilon$ and therefore, for all $n \geq N$, $z_n = x_{\frac{n+1}{2}}$ or $z_n = y_{n/2}$ is inside $(a, a + \frac{2}{n}) \subset (a, a + \epsilon)$, so $|a - z_n| < \epsilon$ (sequence converges to a and is therefore Cauchy). Furthermore, since $d_Y(f(x_n), f(y_n)) \geq \widetilde{\epsilon}$, there doesn't exists M such that $d(z_n, z_m) < \widetilde{\epsilon}$ for every $n, m \geq M$, so the image sequence fails to be Cauchy.

d) If we changed (a, b) to (a, ∞), does the previous result still hold?

No it doesn't! Use $f(x) = x^2$ as a function $f : (0, \infty) \to \mathbb{R}$ which we saw in class is not uniformly continuous. Let $\{x_n\}_{n=1}^{\infty}$ be any Cauchy sequence in $(0, \infty)$, prove that $\{x_n^2\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Since the original sequence is Cauchy, for every $\widetilde{\epsilon} > 0$, there exists a $\widetilde{N}$ such that $|x_n - x_m| < \widetilde{\epsilon}$ whenever $n, m \geq \widetilde{N}$. Show that for every $\epsilon > 0$, there exists N such that $|x_n^2 - x_m^2| < \epsilon$ whenever $n, m \geq N$. Since the original sequence is Cauchy, it is also bounded, so there exists M such that $x_n \leq M$ for all n. Let $\widetilde{\epsilon} = \frac{\epsilon}{2M}$ and $N = \widetilde{N}$. Then, if $n, m \geq N = \widetilde{N}$, $|x_n - x_m| < \frac{\epsilon}{2M}$ and therefore $|x_n^2 - x_m^2| = |x_n - x_m| \cdot |x_n + x_m| < \frac{\epsilon}{2M}|x_n + x_m| < \frac{\epsilon}{2M}|M + M| = \epsilon$ which is what we wanted to show.