Math 411 Homework 2 Due Friday, February 19

1. Use induction and the definition of derivative to prove the power rule ($\frac{d}{dx}x^n = nx^{n-1}$).

2. Let $a_1, ..., a_n \in \mathbb{R}$ constants. Find the value of x that minimizes

$$f(x) = \sum_{i=1}^{n}(x - a_i)^2.$$

3. Prove that if f is differentiable on (a, b) with $c \in (a, b)$ and $f'(c) < 0$, then there exists an interval around c in which $f(x)$ is decreasing.

4. The other definition of derivative:

a) Suppose that $|f(x + h) - f(x)| \leq K|h|^\alpha$ for some constants K and α with $\alpha > 0$, prove that f is continuous.

b) Suppose that $|f(x + h) - f(x)| \leq K|h|^\alpha$ for some constants K and α with $\alpha > 1$, prove that f is differentiable and $f'(x) = 0$.

c) Assume that $|f(x + h) - f(x)| \leq |h|$. Must f be differentiable at $x = 0$? Hint: consider $f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$

5. Derivatives and uniform continuity

a) Prove that if f is differentiable and $|f'(x)| \leq M$ for some $M \in \mathbb{R}$, the f is uniformly continuous.

b) Give an example of a function that is differentiable and uniformly continuous on $(0, 1)$ but whose derivative is unbounded on $(0, 1)$.

6. Assume that f is continuous on $[a, b]$ and differentiable on $(a, b) - \{c\}$ where $c \in (a, b)$.

a) Prove that if $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, that f has a local maximium at $x = c$.

b) Prove that if $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, that f has a local minimium at $x = c$.

c) Find the local extrama of $f(x) = x^{\frac{2}{3}}(8 - x)^2$ on $[-10, 10]$ and classify them as max or mins.

7. Assume that f is a function whose derivative exists for every x and that f has n distinct roots.

a) Prove that f' has at least $n - 1$ distinct roots.

b) Is it possible for f' to have *more* roots than f?

8. Derivatives need not be continuous.

a) Assume that f' exists on (a, b) and $c \in (a, b)$. Show that there exists a sequence $\{x_n\}$ converging to c such that $\{f'(x_n)\}$ converges to $f'(c)$.

b) Find such a sequence for our example from class for $(a, b) = (-\infty, \infty)$ and $c = 0$ for $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.