# Math 411 Homework 2 Answers

**1.** Use induction and the definition of derivative to prove the power rule ($\frac{d}{dx}x^n = nx^{n-1}$).

Case $n = 1$:
$\dfrac{d}{dx}x^1 = \lim_{t\to x}\dfrac{t-x}{t-x} = \lim_{t\to x} 1 = 1 = 1x^0$ which starts the induction.

Assume that the theorem holds for $k = n-1$ and $k = n$, prove it holds for $k = n+1$ (this is 'semi-strong' induction).
Assume that $\dfrac{d}{dx}x^k = kx^{k-1}$, that is, $\lim_{t\to x}\dfrac{t^k - x^k}{t-x} = kx^{k-1}$ for $k = 1, 2, ..., n$. Therefore, using the formula for $\dfrac{d}{dx}x^n$, we get $\lim_{t\to x}\dfrac{(t^n - x^n)(t+x)}{t-x} = \lim_{t\to x}\dfrac{t^n - x^n}{t-x}\cdot \lim_{t\to x}(t+x) = nx^{n-1}(2x) = 2nx^n$. However, the same thing is also equal to $\lim_{x\to t}\dfrac{t^{n+1} - tx^n + xt^n - x^{n+1}}{t-x} = \lim_{t\to x}\dfrac{t^{n+1}-x^{n+1}}{t-x} + \lim_{t\to x}\dfrac{xt^n - tx^n}{t-x} = \dfrac{d}{dx}x^{n+1} + \lim_{t\to x}\dfrac{xt(t^{n-1}-x^{n-1})}{t-x} = \dfrac{d}{dx}x^{n+1} + \lim_{t\to x} xt\cdot\dfrac{d}{dx}x^{n-1} = \dfrac{d}{dx}x^{n+1} + x^2(n-1)x^{n-2} = \dfrac{d}{dx}x^{n+1} + (n-1)x^n$. Solving gives us $\dfrac{d}{dx}x^{n+1} = 2nx^n - (n-1)x^n = (2n-n+1)x^n = (n+1)x^n$.
Honestly, it is probably easier to use binomial coefficients with the other definition of derivative or to factor $t^{n+1} - x^{n+1}$ into $(t-x)(t^n + xt^{n-1} + x^2t^{n-2} + \cdots + x^n)$ and compute the limit directly, but the problem did say induction, so you got a proof by (strong) induction instead.

**2.** Let $a_1, ..., a_n \in \mathbb{R}$ constants. Find the value of $x$ that minimizes

$$f(x) = \sum_{i=1}^{n}(x-a_i)^2 = \sum_{i=1}^{n} x^2 - 2a_ix + a_i^2.$$

We're going to use the theorem that says that if $f$ is differentiable, then $x = c$ is a local max/min implies that $f'(c) = 0$. Since $f$ is a degree 2 polynomial, it is differentiable and moreover, its graph is a right side up parabola, so it's only critical point is also its local minimum.
Compute $f'(x) = \sum_{i=1}^{n} 2x - 2a_i = 2\left(\sum_{i=1}^{n} x - a_i\right) = 2nx - 2\sum_{i=1}^{n} a_i$. This is zero when $nx = \sum_{i=1}^{n} a_i$ or for $x = \frac{1}{n}\sum_{i=1}^{n} a_i$. By the above argument, this is the value of $x$ for which $f(x)$ is the global minimum.

**3.** Prove that if $f$ is differentiable on $(a,b)$ with $c \in (a,b)$ and $f'(c) < 0$, then there exists an interval around $c$ in which $f(x)$ is decreasing.

Since we only know that $f'(c) < 0$ and not that $f'(c) < 0$ near $x = c$, the most we can show is there exists some $\epsilon > 0$ such that $c - \epsilon < x_1 < c < x_2 < c + \epsilon$ implies that $f(x_1) > f(x_2)$.

Since $f'(c)$ exists, $\lim_{t\to c}\dfrac{f(t)-f(c)}{t-c}$ exists and is $< 0$, in particular, we have $\lim_{t\to c^-}\dfrac{f(t)-t(c)}{t-c} < 0$. Since $t \to c^-$, $t < c$ so the denominator is negative. Since the limit is negative, there exists some $\epsilon_1$ such that $t \in (c - \epsilon_1, c)$ implies that the numerator is positive, so for $t \in (c - \epsilon_1, c)$, $f(t) - f(c) > 0$ or $f(t) > f(c)$.

Similarly, since $\lim\limits_{t \to c^+} \dfrac{f(t) - f(c)}{t - c} < 0$ and the denominator is positive, there exists some $\epsilon_2 > 0$ such that if $t \in (c, c + \epsilon_2)$, then $f(t) - f(c) < 0$, or $f(c) > f(t)$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then if $x_1 \in (c - \epsilon, c), f(x_1) > g(c)$ and if $x_2 \in (c, c + \epsilon), f(c) > f(x_2)$, which implies that $f(x_1) > f(x_2)$ which is what we wanted to show.

**4.** The other definition of derivative:

**a)** Suppose that $|f(x + h) - f(x)| \leq K|h|^\alpha$ for some constants $K$ and $\alpha$ with $\alpha > 0$, prove that $f$ is continuous.

We want to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$. Let $x + h = c$, so $h = c - x$. Then our assumption is $|f(c) - f(x)| \leq K|c - x|^\alpha = K|x - c|^\alpha$. Since $\alpha > 0$, we can take $\alpha^{\text{th}}$ roots, so given $\epsilon > 0$, let $\delta = \sqrt[\alpha]{\frac{\epsilon}{K}}$, then $|x - c| < \sqrt[\alpha]{\frac{\epsilon}{K}}$ implies that $|x - c|^\alpha < \frac{\epsilon}{K}$ or $K|x - c|^\alpha < \epsilon$. Therefore, our assumption that $|f(x + h) - f(x)| \leq K|h|^\alpha$ implies that $|f(c) - f(x)| = |f(x) - f(c)| < K|x - c|^\alpha < \epsilon$ which is what we needed to show.
Notice that the issue about possibly not being able to take the $\alpha^{\text{th}}$ root of a negative number is not a problem because if $K$ is negative, then our assumption is that $|f(x + h) - f(x)| = 0$ and constant functions are continuous.

**b)** Suppose that $|f(x + h) - f(x)| \leq K|h|^\alpha$ for some constants $K$ and $\alpha$ with $\alpha > 1$, prove that $f$ is differentiable and $f'(x) = 0$.

For this problem, I'm going to use the other definition of derivative, but it is just as easy to translate to the usual definition like I did in part a. Namely, I'll use $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.
Our assumption above implies that $|\frac{f(x+h)-f(x)}{h}| \leq K|h|^{\alpha - 1}$ where $\alpha - 1 > 0$. We must show that the limit above exists and is equal to zero. To show that the limit exists, show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |h - 0| < \delta$ implies that $|\frac{f(x+h)-f(x)}{h} - 0| < \epsilon$.
Again, we don't have to worry about $K$ being negative because if it is, then we are assuming that $f(x)$ is constant and constant functions have zero derivative.
Fix $\epsilon > 0$ and let $\delta = (\frac{\epsilon}{K})^{\frac{1}{\alpha - 1}}$. Then $0 < |h| < \delta$ implies that

$$\left|\frac{f(x + h) - f(x)}{h}\right| \leq K|h|^{\alpha - 1} < K\delta^{\alpha - 1} = K\frac{\epsilon}{K} = \epsilon$$

which is what we wanted to show.

**c)** Assume that $|f(x + h) - f(x)| \leq |h|$ for $x = 0$. Must $f$ be differentiable at $x = 0$?
Hint: consider $f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$

Notice that this problem had a fairly major typo since if this equation needed to hold for all $x$ the given function certainly wouldn't satisfy this condition as by part a, the function would have to be continuous everywhere and the given function is only continuous at $x = 0$. I fixed the typo above and am going to answer the question as it was supposed to be posed. As it was posed, the answer is yes by part b.

We just need to check that the inequality is satisfied for the given function and that the function fails to be differentiable at $x = 0$.

We have two options here, $h \in \mathbb{Q}$ and $h \notin \mathbb{Q}$.

If $h \in \mathbb{Q}$, $|f(h) - f(0)| = |\frac{1}{2}h - 0| = \frac{1}{2}|h| < |h|$, so the assumption holds.

If $h \notin \mathbb{Q}$, then $|f(h) - f(0)| = |h - 0| = |h| \leq |h|$, so the assumption holds.

Now show that $f'(0)$ does not exist. $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$. Since $f$ is a piecewise function, we need to compute the right hand side for our two different cases, and notice that if $h \in \mathbb{Q}$, $\frac{f(h)-f(0)}{h} = \frac{1}{2}$ and if $h \notin \mathbb{Q}$, $\frac{f(h)-f(0)}{h} = 1$. Therefore, the limit does not exist and $f$ is not differentiable at $x = 0$. This implies that in order for our part a and part b conclusions to hold, the assumption needs to be true at more than just one point!

**5.** Derivatives and uniform continuity

**a)** Prove that if $f$ is differentiable and $|f'(x)| \leq M$ for some $M \in \mathbb{R}$, the $f$ is uniformly continuous.

This problem just screams MVT, doesn't it! We want to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. Fix $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. Let $|x - y| < \frac{\epsilon}{M}$. Then since $f$ is differentiable and hence continuous, the MVT applies on $[x, y]$, so there exists $c \in (x, y)$ with $\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(c)|$. Therefore, $|f(y) - f(x)| = |f(x) - f(y)| = |f'(c)| \cdot |x - y| < M \cdot \frac{\epsilon}{M} = \epsilon$ which is what we needed to show!

**b)** Give an example of a function that is differentiable and uniformly continuous on $(0, 1)$ but whose derivative is unbounded on $(0, 1)$.

There are many such examples, but a good one is $f(x) = \sqrt{x}$. This function is continuous on $[0, 1]$ which is a compact set, so is uniformly continuous on $[0, 1]$ and hence on $(0, 1)$. However, $f'(x) = \frac{1}{2\sqrt{x}}$ which takes values in $(1, \infty)$ for $x \in (0, 1)$.

**6.** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b) - \{c\}$ where $c \in (a, b)$.

**a)** Prove that if $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, that $f$ has a local maximum at $x = c$.

I suppose technically the theorem proved in class that if $f'(x) > 0$ on $[a, b]$, then $f$ is increasing on $[a, b]$ doesn't quite apply because we have a case where $f'(x) > 0$ on $[a, c)$, not on a closed interval, so we'd get if $a \leq x < y < c$, then $f(x) \leq f(y)$, but not that $f(x) \leq f(c)$. Therefore, we're going to reprove that theorem for a half open interval. *sigh*

We need to show that there exists $\epsilon > 0$ such that if $x \in (c - \epsilon, c + \epsilon)$ implies that $f(x) \leq f(c)$. Our assumption means that, in particular, that for all $c - a \geq \epsilon_1 > 0$ and for all $c - \epsilon_1 < x < c$, $f$ is continuous on $[x, c]$ and differentiable on $(x, c)$, so MVT applies! This means that there exists $d \in (x, c)$ such that $\frac{f(c)-f(x)}{c-x} = f'(d) > 0$. Since $x \in (c - \epsilon_1, c)$, $c - x > 0$ and therefore $f(c) - f(x) > 0$ which implies that $f(x) \leq f(c)$. Similarly, for all $b - c \geq \epsilon_2 > 0$ and for all $x \in (c, c + \epsilon_2)$, $f$ is continuous on $[c, x]$ and differentiable on $(c, x)$ so MVT applies and ..... we get $f(x) - f(c) < 0$ which implies that $f(c) \geq f(x)$. Simply take $\epsilon - \min\{\epsilon_1, \epsilon_2\}$ and we are done!

**b)** Prove that if $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, that $f$ has a local minimum at $x = c$.

This is pretty much identical to part a, so I refuse to write everything again with inequalities reversed. I support any student's decision to do the same!

**c)** Find the local extrema of $f(x) = x^{\frac{2}{3}}(8 - x)^2$ on $[-10, 10]$ and classify them as max or mins.

First we need to differentiate to find $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(8-x)^2 - 2x^{\frac{2}{3}}(8-x)$ which exists everywhere except $x = 0$. Therefore, the potential local extrema are where $f'(x) = 0$ and $x = 0$. $f'(8) = 0$ is clear, and, in addition to this, if $x \neq 8$, we can cancel to get $\frac{1}{3}x^{-\frac{1}{3}}(8-x) = x^{\frac{2}{3}}$ which gives $8 - x = 3x$ or $x = 2$.

Since $f'(x)$ is continuous away from $x = 0$, by the intermediate value theorem, we just need to check a single point between each of these 'critical points' to figure out if $f'(x)$ is positive or negative.

$f'(-1) = -\frac{2}{3}(81) - 2(9) = -54 - 18 < 0$, $f(1) = \frac{2}{3}(49) - 2(7) = 2(16\frac{1}{3} - 7) > 0$,

$f'(3) = 10(\frac{1}{3}\frac{5}{\sqrt[3]{3}} - \sqrt[3]{9}) = 10(\sqrt[3]{\frac{125}{81}} - \sqrt[3]{9}) < 0$

$f'(9) = 2(\frac{1}{3}\frac{1}{\sqrt[3]{9}} - \sqrt[3]{81}(-1)) > 0$

Therefore, $x = 0$ is a local minimum, $x = 2$ is a local maximum and $x = 8$ is a local minimum.

**7.** Assume that $f$ is a function whose derivative exists for every $x$ and that $f$ has $n$ distinct roots.

**a)** Prove that $f'$ has at least $n - 1$ distinct roots.

Let $x_1, x_2, ..., x_n$ be the $n$ distinct roots of $f$. By MVT applied to $[x_i, x_{i+1}]$ for $i = 1, 2, ..., n-1$, since $f(x_i) = f(x_{i+1}) = 0$, there exists $\widetilde{x_i} \in (x_i, x_{i+1})$ such that $f'(\widetilde{x_i}) = 0$. Since such points exist for all $i = 1, ..., n-1$ and each is a root, $f'(x)$ has at least $n - 1$ roots.

**b)** Is it possible for $f'$ to have *more* roots than $f$?

Yes, it is! It isn't even hard to find an example. Let $f(x) = (x-1)x(x+1) + 1000 = x^3 - x + 1000$. This has one root (at something a bit less than $-10$), but $f'(x) = 3x^2 - 1$ has two roots (at $\pm\sqrt{\frac{1}{3}}$).

**8.** Derivatives need not be continuous.

**a)** Assume that $f'$ exists on $(a, b)$ and $c \in (a, b)$. Show that there exists a sequence $\{x_n\}$ converging to $c$ such that $\{f'(x_n)\}$ converges to $f'(c)$.

Apparently I was *completely* overthinking this! We need to start with a sequence $c_n \to c$ any sequence (I'm doing the case $c_n \to c$ from the right, but left or alternating will also work). Then using the ordinary MVT on $[c, c_n]$ gives the existence of $x_n \in (c, c_n)$ such that $f'(x_n) = \dfrac{f(c_n) - f(c)}{c_n - c}$. Since $c_n \to c$, for any $\epsilon > 0$, there exists $N$ such that $c_n - c < \epsilon$ for all $n \geq N$. Since $c < x_n < c_n$, this implies that $x_n - c < \epsilon$ as well for all $n \geq N$ and therefore $x_n \to c$. By taking the limit as $n \to \infty$ of both sides of the inequality above, we get

$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} \dfrac{f(c_n) - f(c)}{c_n - c}$. Since $f'(c)$ exists, $\lim_{t\to c} \dfrac{f(t) - f(c)}{t - c}$ exists, let $F(t) = \frac{f(t)-f(c)}{t-c}$, so $\lim_{t\to c} F(t) = f'(c)$. In other words, for all $\widetilde{\epsilon} > 0$, there exists $\widetilde{\delta} > 0$ such that $|t - c| < \widetilde{\delta}$ implies $|F(t) - f'(c)| < \widetilde{\epsilon}$. Fix $\widetilde{\epsilon} > 0$ and let $\epsilon$ from the definition of $c_n \to c$ above be equal to the corresponding $\widetilde{\delta}$ that we get from the definition of the limit of $F(t)$. This means, there exists $N$ such that $|c_n - c| < \widetilde{\delta}$ if $n \geq N$. This implies that $|F(x_n) - f'(c)| < \widetilde{\epsilon}$ and therefore, $\{F(c_n)\}$ converges to $f'(c)$. However, the equality from MVT above says that $x_n$ is specifically chosen so that $F(c_n) = f'(x_n)$ and therefore, we have found a sequence $\{x_n\} \to c$ such that $\{f'(x_n)\} \to f'(c)$ which is what we wanted.

**b)** Find such a sequence for our example from class for $(a, b) = (-\infty, \infty)$ and $c = 0$ for $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Let $c_n = \frac{1}{\pi n}$. Then $f(c_n) = \frac{1}{\pi^2 n^2} \sin(\pi n) = 0$. Clearly $\{\frac{1}{\pi n}\}_{n=1}^{\infty}$ converges to 0, and $\{f(c_n)\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$, so the image sequence converges to $f'(0) = 0$.