Math 411 Homework 2 Due Friday, March 4

1. Show that if $f(x)$ is a function whose derivative $f'(x)$ is monotonic, then $f'(x)$ is continuous. Hint: use the fact that derivatives satisfy IVT.

2. **L'Hôpital's rule** Suppose that f is defined in a neighborhood of x and suppose that $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show by example that the limit may exist even if $f''(x)$ does not (for the example, $f'(x)$ may not exist either, but $f(x)$ is always defined).

3. **Taylor's theorem** Suppose $a \in \mathbb{R}$, f is a twice differentiable function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ respectively on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$. Hint: If $h > 0$ Taylor's theorem shows that $f'(x) = \frac{1}{2h}(f(x+2h) - f(x)) - hf''(\zeta)$ for some $\zeta \in (x, x+2h)$. Hence $|f'(x)| \leq hM_2 + \frac{M_0}{h}$.

4. **Darboux sums**

a) Let $f(x) = x^2 - x$ and let $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. Compute $U(P, f)$ and $L(P, f)$.

b) Let $\alpha(x) = x^2$. Compute $U(P, f, \alpha)$ and $L(P, f, \alpha)$.

5. **Integrability of lines**

a) Use our integrability condition from class ($f \in \mathcal{R}(\alpha)[a, b]$ if for all $\epsilon > 0$, there exists P partition of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.) to show that $f(x) = 3x + 1 \in \mathcal{R}[a, b]$ for all $[a, b]$. (So for this part, use $\alpha(x) = x$.)

b) Use the same condition to show that $f \in \mathcal{R}(\alpha)$ for all α increasing on $[a, b]$.

6. Let $\alpha(x) = \begin{cases} 0 & a \leq x \leq c \\ 1 & c < x \leq b \end{cases}$ Show that $f \in \mathcal{R}(\alpha)$ if and only if f is continuous from the right at $x = c$.

7. **Riemann vs Riemann-Stilches Integration**

a) Let α increasing on $[a, b]$ with $x_0 \in [a, b]$ and α continuous at x_0. Let $f(x_0) = 1$ and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f(x)\, d\alpha = 0$.

b) Suppose that $f(x) \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

8. Let $f : (0, 1] \to \mathbb{R}$ and $f \in \mathcal{R}[c, 1]\ \forall\, c > 0$. Define $\displaystyle\int_0^1 f(x)\, dx = \lim_{c \to 0^+} \int_c^1 f(x) dx$ if the limit exists and is finite. If $f \in \mathcal{R}[0, 1]$, show that this definition agrees with the usual one.