Math 411 Homework 3 Answers

1. Show that if $f(x)$ is a function whose derivative $f'(x)$ is monotonic, then $f'(x)$ is continuous. Hint: use the fact that derivatives satisfy IVT.

Case 1: Assume that $f'(x)$ exists and that $x < y$ implies that $f'(x) \le f'(y)$ (f' is increasing). Fix c and $\epsilon > 0$ and show that there exists a $\delta > 0$ such that $c-\delta < x < c+\delta$ implies that $f'(c) - \epsilon < f'(x) < f'(c) + \epsilon$. First, let $\epsilon_1 < \epsilon$ (this is necessary because f' is increasing, but not necessarily strictly increasing). Next, pick $y < c$ arbitrary. Then $f'(y) \le f'(c)$ (f' is increasing) and either $f'(y) \le f'(c) - \epsilon_1 \le f'(c)$ or $f'(c) - \epsilon_1 \le f'(y) \le f'(c)$. If the first holds, since f' satisfies the IVT, there exists $t \in (y, c)$ with $f'(t) = f'(c) - \epsilon_1$, let $c - t = \delta_1$. If the second holds, let $\delta_1 = c - y$. Similarly, pick z with $c < z$ arbitrarily. Then since f' is increasing, either $f'(c) \le f'(c) + \epsilon_1 \le f'(z)$ or $f'(c) \le f'(z) \le f'(c) + \epsilon_1$. Define δ_2 as either $s - c$ with $f'(s) = f'(c) + \epsilon_1$ if the first holds or as $z - c$ if the second holds. Then let $\delta = \min\{\delta_1, \delta_2\}$.
Then if $c - \delta \le c - \delta_1 < x < c + \delta_2 \le c + \delta$, then, by all of the above inequalities,

$$f'(c) - \epsilon < f'(c) - \epsilon_1 \le f'(c - \delta_1) \le f'(x) \le f'(c + \delta_2) \le f'(c) + \epsilon_1 < f'(c) + \epsilon$$

so we are done!

Case 2 is very similar (f' is now decreasing, so any inequalities involving f' become $\ge$ instead of $\le$. For example, our final inequality becomes

$$f'(c) - \epsilon < f'(c) - \epsilon_1 \le f'(c + \delta_1) \le f'(x) \le f'(c - \delta_2) \le f'(c) + \epsilon_1 < f'(c) + \epsilon$$

2. **L'Hôpital's rule** Suppose that f is defined in a neighborhood of x and suppose that $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show by example that the limit may exist even if $f''(x)$ does not (for the example, $f'(x)$ may not exist either, but $f(x)$ is always defined).

Lets first try to do this without theorems (naively). We know that $f''(x)$ exists, so $f'(x)$ must exist as well and therefore,

$$f''(x) = \lim_{t\to 0} \frac{\lim_{h\to 0} \frac{f((x+t)+h) - f(x+t)}{h} - \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}{t}$$

$$f''(x) = \lim_{t\to 0} \frac{\lim_{h\to 0} \left(\frac{f((x+t+h) - f(x+t)}{h} - \frac{f(x+h)-f(x)}{h} \right)}{t}$$

$$f''(x) = \lim_{t\to 0} \frac{\lim_{h\to 0} \left(\frac{f((x+t+h) - f(x+t) - f(x+h) + f(x)}{h} \right)}{t}$$

$$f''(x) = \lim_{t\to 0}\lim_{h\to 0}\frac{f((x+t+h) - f(x+t) - f(x+h) + f(x)}{h\cdot t}$$

then, somehow, let $t = -h$, and

$$f''(x) = \lim_{h\to 0}\frac{f((x) - f(x-h) - f(x+h) + f(x)}{-h^2} = \lim_{h\to 0}\frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

which is what we wanted to show! So, how to do the 'somehow'? Well, I didn't use the implied hint, which is the title of the problem, namely, use L'Hôpital's rule. Maybe that's how I can do this!

Notice that $\lim_{h\to 0}\dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2}$ is a limit where (since f is twice differentiable and therefore differentiable and therefore continuous at $x = 0$), the limit of the numerator is $\lim_{h\to 0} f(x+h) + f(x-h) - 2f(x) = f(x) + f(x) - 2f(x) = 0$ and the limit of the denominator is $\lim_{h\to 0} h^2 = 0$ so the only remaining condition we need to check in order to use L'Hôpital's rule on this limit is that the limit of the ratios of the derivatives exists. Namely, $\lim_{h\to 0}\dfrac{f'(x+h) + f'(x-h)}{2h}$ exists.

Notice that this looks a lot like the definition of $f''(x)$, namely,

$$\lim_{h\to 0}\frac{f'(x+h) + f'(x-h)(-1)}{2h} = \frac{1}{2}\lim_{h\to 0}\left(\frac{f'(x+h) - f'(x) + f'(x) - f'(x-h)}{h}\right)$$
$$= \frac{1}{2}\left(\lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h} + \lim_{h\to 0}\frac{f'(x) - f'(x-h)}{h}\right)$$

Now the right half of the sum is just the definition of derivative where we've just the points $(x, f'(x))$ and $((x-h), f'(x-h))$ to define the secant line and difference in y over difference in x is as above.

Therefore, this is $\frac{1}{2}(f''(x) + f''(x)) = f''(x)$ as we wanted to show.

Notice that a combination of the two methods shows that, at least in this case, we don't need to take the limits $h \to 0$ and $t \to 0$ separately. In fact, we can even let $t = -h$ and approach $(0,0)$ on the $h - t$ plane from a single combined direction and get the same answer (provable by another method) as we would by reducing to the line $h = 0$ first and then approaching $(0,0)$ along the t axis (namely $\lim_{t\to 0}\lim_{h\to 0}$). For an example why this is so special, look at something like $\lim_{x\to 0^+}\lim_{t\to 0} x^t = 1$. This limit is only 1 from the particular direction specified and things like switching the limits of integration give different answers. In particular, approaching from $x = -t$ can give us $\lim_{t\to 0}(-t)^t$ which isn't even well defined!

For example, pick a function which is odd but not twice differentiable at $x = 0$. Then $f(0) = 0$ and $f(0+h) = -f(0-h)$, so the limit becomes $\lim_{h\to 0}\frac{0}{h^2} = 0$, but since f' is not twice differentiable at 0, $f''(0)$ can't exist. One easy way to do this is to pick a function that is not continuous at $x = 0$, so a piecewise function like $f(x) = x + 1$ if $x > 0$, $f(0) = 0$ and $f(x) = x - 1$ if $x < 0$. You can also make an odd function out of any function on $[0,1)$ which has concavity non-zero at $x = 0$ (for example $y = \sqrt{1-(x-1)^2}$). For such a function, $f'(0)$ may exist, but $f''(0)$ will not because its left hand and right hand concavity won't agree.

3. **Taylor's theorem** Suppose $a \in \mathbb{R}$, f is a twice differentiable function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ respectively on (a,∞). Prove that $M_1^2 \le 4M_0M_2$. Hint: If $h > 0$ Taylor's theorem shows that $f'(x) = \frac{1}{2h}(f(x+2h) - f(x)) - hf''(\zeta)$ for some $\zeta \in (x, x+2h)$. Hence $|f'(x)| \le hM_2 + \frac{M_0}{h}$.

Assume M_0 and M_2 are finite.

Taylor's theorem says that if f is twice differentiable in $[a,b]$, then for any $\alpha \neq \beta \in [a,b]$, $f(\beta) = f(\alpha) + f'(\alpha)(\beta-\alpha)^1 + \frac{f''(\gamma)}{2}(\beta-\alpha)^2$ for some γ between α and β.

Further Taylor polynomials should be unnecessary as we are only comparing bounds on f, f', and f''.

The hint suggests we should be considering $(x, x+2h)$ as (α,β), so let $[A,B] \subset (a,\infty)$ and $(x, x+2h) \in [A,B]$. Then Taylor's theorem states that

$f(x+2h) = f(x) + f'(x)(2h) + \frac{1}{2}f''(\zeta)(2h)^2$ for some $\zeta \in (x, x+2h)$ or

$f'(x) = \frac{1}{2h}(f(x+2h) - f(x) - 2f''(\zeta)h^2) = \frac{f(x+2h)-f(x)}{2h} - hf''(\zeta)$. Taking absolute values gives us

$|f'(x)| = |\frac{f(x+2h)-f(x)}{2h} - hf''(\zeta)| \leq |\frac{f(x+2h)-f(x)}{2h}| + |hf''(\zeta)| \leq \frac{M_0+M_0}{2h} + hM_2 = \frac{M_0}{h} + hM_2$ which implies that

$M_1 \leq \frac{M_0}{h} + hM_2$ which holds for all $h > 0$.

Apparently, we can choose a particular $h = \sqrt{\frac{M_0}{M_2}}$ and then get

$M_1 \leq \frac{M_0}{\sqrt{\frac{M_0}{M_2}}} + \sqrt{\frac{M_0}{M_2}}M_2 = 2\sqrt{M_0M_2}$ and hence $M_1^2 \leq 4M_0M_2$ (because all of the terms are non-negative, squaring preserves inequalities).

4. Darboux sums

a) Let $f(x) = x^2 - x$ and let $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$. Compute $U(P,f)$ and $L(P,f)$.

Please excuse any arithmetic mistakes with these problems. I hope there are none, but it's hard to be sure. First notice that $\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \frac{1}{2}$. Also notice that $f(x) = x(x-1)$ so has its global min at $x = \frac{1}{2}$ and is decreasing then increasing. Therefore, $M_1 = 0, m_1 = -\frac{1}{4}, M_2 = 0, m_2 = -\frac{1}{4}, M_3 = \frac{3}{4}, m_3 = 0, M_4 = 2, m_4 = \frac{3}{4}$, so $U(P, x^2-x) = \sum_{i=1}^4 M_i \Delta x_i = \frac{11}{8}$ and $L(P, x^2-x) = \sum_{i=1}^4 m_i \Delta x_i = \frac{1}{8}$.

b) Let $\alpha(x) = x^2$. Compute $U(P,f,\alpha)$ and $L(P,f,\alpha)$.

$\Delta\alpha_1 = \frac{1}{4}, \Delta\alpha_2 = \frac{3}{4}, \Delta\alpha_3 = \frac{5}{4}, \Delta\alpha_4 = \frac{7}{4}$. Using the M_i, m_i from part a) above, we get
$U(P, x^2-x, x^2) = \sum_{i=1}^4 M_i\Delta\alpha_i = 0 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{5}{4} + 2 \cdot \frac{7}{4} = \frac{71}{16}$ and
$L(P, x^2-x, x^2) = \sum_{i=1}^4 m_i\Delta\alpha_i = -\frac{1}{4} \cdot \frac{1}{4} + -\frac{1}{4} \cdot \frac{3}{4} + 0 \cdot \frac{5}{4} + \frac{3}{4} \cdot \frac{7}{4} = \frac{17}{16}$

5. Integrability of lines

a) Use our integrability condition from class ($f \in \mathcal{R}(\alpha)[a,b]$ if for all $\epsilon > 0$, there exists P partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.) to show that $f(x) = 3x+1 \in \mathcal{R}[a,b]$ for all $[a,b]$. (So for this part, use $\alpha(x) = x$.)

We basically did this in class, but here are some details. Let $[a,b]$ be any (non-empty) interval and fix $\epsilon > 0$. We need to show that there exists a partition, and the easiest kind of partition to describe is one with equally spaced subintervals (called a *regular partition*) and then choose the size of the subinterval small based on ϵ (and $[a,b]$). Also not that our function is increasing, so if our partition is $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b\}$, then the ith subinterval is $[x_{i-1}, x_i]$ and $M_i = 3x_i + 1$ and $m_i = 3x_{i-1} + 1$. Let $\Delta x_i = \frac{b-a}{n}$.

Then $U(P, 3x+1) = \frac{b-a}{n}(3x_1+1+3x_2+1+\cdots+3x_n+1) = \frac{3(b-a)}{n}(x_1+x_2+\cdots+x_n)+n$ and $L(P, 3x+1) = \frac{b-a}{n}(3x_0+1+3x_1+1+\cdots+3x_{n-1}+1) = \frac{3(b-a)}{n}(x_0+x_1+\cdots+x_{n-1})+n$

and $U(P,3x+1)-L(P,3x+1)=\frac{3(b-a)}{n}(x_n-x_0)=\frac{3(b-a)}{n}(b-a)=\frac{3(b-a)^2}{n}$. To insure that this will be less than ϵ, simply let $n>\frac{3(b-a)^2}{\epsilon}$. (Remember, that a and b are fixed constants and we fixed $\epsilon>0$ as well, so this is a well defined number).

b) Use the same condition to show that $f\in\mathcal{R}(\alpha)$ for all α increasing on $[a,b]$.

For this one, we will still want to use an equal sized partition (for simplicity), but we can not cancel nearly so easily!

Again, fix $[a,b]$ and $\epsilon>0$ and let $P=\{a=x_0,x_1,x_2,...,x_{n-1},x_n=b\}$ be our partition of $[a,b]$ and notice that we are doing this for a fixed $\alpha(x)$ increasing. Then we can recycle the M_i and m_i as above, but $\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1})$ this time, and all we can conclude is that $\Delta\alpha_i>0$.

Then

$$U(P,3x+1,\alpha)-L(P,3x+1,\alpha)=\sum_{i=1}^{n}(3x_i+1)\Delta\alpha_i-\sum_{i=1}^{n}(3x_{i-1}+1)\Delta\alpha_i=\sum_{i=1}^{n}3(x_i-x_{i-1})(\alpha(x_i)-\alpha(x_{i-1}))$$

Now, writing out this sum (and keeping in mind that P is a regular partition, so $x_i-x_{i-1}=\frac{b-a}{n}$ fpr all i) we get

$$U(P,f,\alpha)-L(P,f,\alpha)=3\frac{b-a}{n}(\alpha(x_1)-\alpha(x_0)+\alpha(x_2)-\alpha(x_1)+\cdots+\alpha(x_n)-\alpha(x_{n-1}))$$

which is a telescoping sum, so equal to $3\frac{b-a}{n}(\alpha(b)-\alpha(a))$, so to make this less than ϵ, we just need to choose $n>\frac{3}{\epsilon}(b-a)(\alpha(b)-\alpha(a))$ and since a,b,ϵ,α are all fixed, this is certainly possible, so $f\in\mathcal{R}(\alpha)$.

6. Let $\alpha(x)=\begin{cases}0 & a\le x\le c\\ 1 & c<x\le b\end{cases}$ Show that $f\in\mathcal{R}(\alpha)$ if and only if f is continuous from the right at $x=c$.

Let P be a partition of $[a,b]$ such that $x_{i-1}=c$. Then $\Delta\alpha_1=0-0=0,...,\Delta\alpha_{i-1}=0-0=0,\Delta\alpha_i=1-0=1,\Delta\alpha_{i+1}=1-1=0,...,\Delta\alpha_n=1-1=0$. Therefore, $L(P,f,\alpha)=\sum_{k=1}^{n}m_k\Delta\alpha_k=m_i$ and $U(P,f,\alpha)=\sum_{k=1}^{n}M_k\Delta\alpha_k=M_i$ where $m_i=\inf\{f(x)\mid c\le x\le x_i\}$ and $M_i=\sup\{f(x)\mid c\le x\le x_i\}$. Therefore, $\underline{\int_a^b} f(x)d\alpha=\sup\{m_i\mid m_i=\inf\{f(x)\mid c\le x\le x_i\},c<x_i\}$ and $\overline{\int_a^b} f(x)d\alpha=\inf\{M_i\mid M_i=\sup\{f(x)\mid c\le x\le x_i\},c<x_i\}$

Notice that if $c<x_i<x_i'$, then $[c,x_i]\subsetneq[c,x_i']$ and therefore $m_i\ge m_i'$ since we are taking an infemium over a strictly smaller set and similarly $M_i\le M_i'$.

$\Rightarrow$ Assume that for all $\epsilon_1>0$, there exists P^* partition of $[a,b]$ such that $U(P^*,f,\alpha)-L(P^*,f,\alpha)<\tilde{\epsilon}$, show that for all $\epsilon>0$, there exists a $\delta>0$ such that $c\le x<c+\delta$ implies that $f(c)-\epsilon<f(x)<f(c)+\epsilon$ (namely, f is right continuous at $x=c$.

Fix $\epsilon>0$ and pick $\epsilon>\epsilon_1>0$. Let P^* be such that $U(P,f,\alpha)-L(P,f,\alpha)<\epsilon_1$. Let P be the refinement of P^* given by $P=P^*\cup\{c\}$. Then $U(P,f,\alpha)-L(P,f,\alpha)=M_i-m_i<\epsilon_1$ where M_i and m_i are defined as above. Then since $c\in[c,x_i]$, $m_i\le f(c)\le M_i$, and since $M_i-m_i<\epsilon_1$, $f(c)-\epsilon<f(c)-\epsilon_1\le m_i\le f(x)\le M_i\le f(c)+\epsilon_1<f(c)+\epsilon$. Let $\delta=x_i-c$ and we are done.

$\Leftarrow$ Assume that for all $\epsilon_1>0$, there exists $\delta_1>0$ such that $c\le x<x+\delta_1$ implies that $f(c)-\epsilon_1<f(x)<f(c)+\epsilon_1$. Prove that for all $\epsilon>0$, there exists a partition P such that $U(P,f,\alpha)-L(P,f,\alpha)<\epsilon$.

Fix $\epsilon > 0$ and let $\epsilon_1 = \epsilon/2$. Then pick $0 < \delta < \delta_1$ where δ_1 is as above. Let $P = \{a, c, c+\delta, b\}$. Then $U(P, f, \alpha) = \sup\{f(x) \mid c \le x \le c+\delta\}$ and $L(P, f, \alpha)$ is the infemium of the same set. Since for all $x \in [c, c+\delta] \subset [c, c+\delta_1)$, $f(x) \in (f(c)-\epsilon/2, f(c)+\epsilon/2)$, we must have that $f(c) - \epsilon/2 < m_i$ and $M_i < f(c) + \epsilon/2$. Therefore, $U(P, f, \alpha) - L(P, f, \alpha) = M_i - m_i < f(c) + \epsilon/2 - (f(c) - \epsilon/2) = \epsilon$ which is what we wanted to show.

7. Riemann vs Riemann-Stieltjes Integration

a) Let α increasing on $[a, b]$ with $x_0 \in [a, b]$ and α continuous at x_0. Let $f(x_0) = 1$ and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and $\int_a^b f(x)\, d\alpha = 0$.

Since f is continuous away from x_0 and α increasing is continuous at x_0, $f \in \mathcal{R}(\alpha)$. Also, $L(P, f, \alpha) = 0$ for all partitions P of $[a, b]$. (To see this, let x_0 in the ith subinterval $[x_{i-1}, x_i]$. Since $x_{i-1} < x_i$, there exists $x \in [x_{i-1}, x_i]$ with $x \ne x_0$, so $m_i = 0$ and $m_j = 0$ for $i \ne j$, so $L(P, f, \alpha) = 0$ no matter what α is.) Therefore, $\underline{\int_a^b} f(x)\, d\alpha = 0$ and since $f \in \mathcal{R}(\alpha)$, $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha = 0$.

b) Suppose that $f(x) \ge 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof by contradiction. Assume that there exists $x\_0 \in [a, b]$ with $f(x_0) > 0$. Since f is continuous on $[a, b]$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$. Let P be a partition of $[a, b]$ such that $x_{i-1} = x_0 - \delta$ and $x_i = x_0 + \delta$. Then $L(P, f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_i 2\delta + \cdots + m_n\Delta x_n \le \frac{f(x_0)}{2}\delta$ (since $m_k \ge 0$, $\Delta x_k > 0$ and $\frac{f(x_0)}{2} \le f(x) \le \frac{3f(x_0)}{2}$ for all $x_0 - \delta \le x \le x_0 + \delta$). Therefore, $\underline{\int_a^b} f(x)dx = \sup\{L(P, f) \mid P\} \ge \frac{1}{2} f(x_0)\delta > 0$ which is a contradiction to our assumption that $\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = 0$

8.

Let $f : (0, 1] \to \mathbb{R}$ and $f \in \mathcal{R}[c, 1]\ \forall\, c > 0$. Define $\displaystyle\int_0^1 f(x)\, dx = \lim_{c\to 0^+} \int_c^1 f(x)dx$ if the limit exists and is finite. If $f \in \mathcal{R}[0, 1]$, show that this definition agrees with the usual one.

Since $f \in \mathcal{R}[0, 1]$, for all $\epsilon_0 > 0$, there exists a P_0 partition of $[0, 1]$ such that $U(P_0, f) - L(P_0, f) < \epsilon_0$. If necessary, replace P_0 with a partition that contains c and let $P_c = P_0 \cap [c, 1]$. Then we get

$$U(P_0, f) - \epsilon_0 < L(P_0, f) \le L(P_c, f) \le U(P_c, f) \le U(P_0, f) < L(P_0, f) + \epsilon_0.$$

Also, since $f \in \mathcal{R}[0, 1]$, f must be bounded on $[0, 1]$ and let M be an upper bound for $|f(x)|$ on $[0, 1]$.

Fix $\epsilon > 0$ and let $\epsilon_0 = \epsilon/4$ and $\delta = \frac{\epsilon}{4M}$ and P_0 and P_c as above with $x_i = c$.

Let $P_{0c} = P_0 \cap [0, c]$ and notice that we have specifically chosen c so that

$$-\epsilon/4 = -M\frac{\epsilon}{4M} < -M(c-0) \le L(P_{0c}, f) \le \int_0^c f(x)dx \le U(P_{0c}, f) \le M(c-0) < M\frac{\epsilon}{4M} = \epsilon/4.$$

Then

$U(P_0, f) - \epsilon/4 < L(P_0, f) \le \int_0^1 f(x)dx \le U(P_0, f) < L(P_0, f) + \epsilon/4$ and

$U(P_c, f) - \epsilon/4 < L(P_c, f) \le \int_c^1 f(x)dx \le U(P_c, f) < L(P_c, f) + \epsilon/4$,

Multiplying the second inequality by negative one gives

$-L(P_c, f) - \epsilon/4 < -\int_c^1 f(x)dx < -U(P_c, f) + \epsilon/4$, and adding the first and last inequalities gives

$U(P_0, f) - L(P_c, f) - \epsilon/2 < \int_0^1 f(x)dx - \int_c^1 f(x)dx < L(P_0, f) - U(P_c, f) + \epsilon/2$

Notice that $U(P_0, f) - L(P_c, f) = M_1\Delta x_1 + \cdots + M_i\Delta x_i + (M_{i+1} - m_{i+1})\Delta x_{i+1} + \cdots + (M_n - m_n)\Delta x_n = U(P_{0c}, f) + (U(P_c, f) - L(P_c, f)) > -\epsilon/4 - \epsilon/4$.

Similarly, $L(P_0, f) - U(P_c, f) = m_1\Delta x_1 + \cdots + m_i\Delta x_i + (m_{i+1} - M_{i+1})\Delta x_{i+1} + \cdots + (m_n - M_n)\Delta x_n = L(P_{0c}, f) - (U(P_c, f) - L(P_c, f)) < \epsilon/4 + \epsilon/4$.

Putting all of these together gives

$-\epsilon < \int_0^1 f(x)dx - \int_c^1 f(x)dx < \epsilon$ which was what we wanted to show.