Math 411 Homework 4 Due Wednesday, March 30

1. Prove that if α is continuous on $[a, b]$ and f is monotonic, then $f \in \mathcal{R}(\alpha)$.

2. Let V be a Vitali set (a subset of $[0, 1]$ containing exactly one element from each of the equivalence classes under $a \sim b$ if $a - b \in \mathbb{Q}$). Prove that V is unmeasurable.

3. Prove that in any σ-algebra, $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

4. Prove that the Borel σ-algebra of $\mathbb{R}^1$ (the smallest σ-algebra containing all of the open sets) is the same as the smallest σ-algebra containing all of the open intervals.

5. Let $\lambda^*(E) = \inf\{\sum_{n=1}^{\infty}(b_n - a_n) \mid E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\}$. Prove that if $A \subset B$, then $\lambda^*(A) \leq \lambda^*(B)$ and that λ^* is countably sub-additive.

6. **Lebesgue measure** Let $\mathcal{L}$ be the Lebesgue σ-algebra

a) Prove that all sets of Lebesgue outer measure zero are in $\mathcal{L}$.

b) Prove that the Cantor (middle 1/3) set is in $\mathcal{L}$.

7. **Riemann Lebesgue theorem** Let $f : [a, b] \to \mathbb{R}$ bounded and let $A \subset [a, b]$ be the set of discontinuities of f and $M = \sup\{f(x) \mid x \in [a, b]$ and $m = \inf\{f(x) \mid x \in [a, b]\}$.

a) Define $\operatorname{osc}(f; c) = \lim\limits_{h \to 0^+} \sup\{|f(z) - f(y)| \mid y, z \in (c-h, c+h) \cap [a, b]\}$. Prove that $\operatorname{osc}(f; c) = 0$ if and only if f is continuous at $x = c$.

b) Let $A_s = \{x \in [a, b] \mid \operatorname{osc}(f, x) \geq s\}$. Prove that $A_s \subset [a, b]$ is compact. Hint: notice that the definition of the osc is basically the smallest ϵ for which continuity fails.

c) Fix $\epsilon > 0$. Assume that $\lambda^*(A) = 0$ where λ^* is the Lebesgue outer measure. Show there exists a finite collection of intervals $\{I_1, I_2, ..., I_n\}$ with $I_i = (a_i, b_i)$ such that $A_{\frac{\epsilon}{2(b-a)}} \subset \bigcup_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} b_i - a_i < \frac{\epsilon}{2(M-m)}$.

d) Prove that there exists a finite cover of $[a, b] - \bigcup I_i$ by sets of the form $B(x, \delta_x)$ such that for all $y, z \in B(x, \delta_x)$, $|f(y) - f(z)| < \frac{\epsilon}{2(b-a)}$.

e) Combining parts c) and d), we get an open cover of $[a, b]$. Prove that there exists a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that each $[x_{i-1}, x_i]$ is contained in one of the open sets of our cover.

f) For the subintervals contained in an I_i, use M and m for each of the M_i and m_i to show that together, these subintervals contribute $< \epsilon/2$ to the total of $U(P, f) - L(P, f)$.

g) Use your inequality from part d) to show that the subintervals contained in the open sets of the form $B(x, \delta_x)$ together also contribute $< \epsilon/2$ to the total of $U(P, f) - L(P, f)$.

h) Conclude that if $\lambda^*(A) = 0$, then $f \in \mathcal{R}[a, b]$.

i) Let $A_{\frac{1}{k}}$ defined as in b), so $A = \bigcup_{k \in \mathbb{N}} A_{\frac{1}{k}}$, fix $\epsilon > 0$ and $k \in \mathbb{N}$. Assume that $f \in \mathcal{R}[a, b]$. Choose P partition of $[a, b]$ with $U(P, f) - L(P, f) < \frac{\epsilon}{2k}$. Show that the contribution to $U(P, f) - L(P, f)$ from the subintervals of P whose intersection with $A_{\frac{1}{k}}$ is greater than or equal to $\frac{1}{k}$ times the sum of their Δx_is and therefore $A_{\frac{1}{k}}$ can be covered by intervals whose total length is less than $\epsilon/2$.

j) Conclude that $f \in \mathcal{R}[a, b]$ implies that the measure of A is zero.