Math 411 Homework 4 Due Wednesday, March 30

1. Prove that if α is continuous on $[a,b]$ and f is monotonic, then $f \in \mathcal{R}(\alpha)$.

Usually we would need to show that for all $\epsilon > 0$, there exists P partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. However, in this case, since α is no longer required to be increasing, it is perfectly possible that $\Delta\alpha_i \le 0$, so it is no longer necessarily true that $U(P,f,\alpha) \ge L(P,f,\alpha)$. However, we can still show that $-\epsilon < U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ which is what we will do. That is, we want to show that for some $P = \{x_0, x_1, ..., x_n\}$, it holds that $-\epsilon < \sum_{k=1}^n (M_i - m_i)\Delta\alpha_i < \epsilon$ where $M_i = \sup\{f(x) \mid \ x_{i-1} \le x \le x_i\}$ and $m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

Fix $\epsilon > 0$. Since f is assumed to be monotonic, WLOG assume that f is increasing (if not, replace f by $-f$). Therefore, since f is increasing, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for all i. Also, since α is continuous on a closed bounded interval, α is uniformly continuous on $[a,b]$, so for every $\widetilde{\epsilon} > 0$, there exists a $\widetilde{\delta} > 0$ such that $|x - y| < \widetilde{\delta}$ with $x, y \in [a,b]$ implies that $|\alpha(x) - \alpha(y)| < \widetilde{\epsilon}$ or $-\widetilde{\epsilon} < \alpha(x) - \alpha(y) < \widetilde{\epsilon}$.

Recall, we are looking to find $P = \{x_0, x_1, ..., x_n\}$ such that

$$\sum_{k=1}^n \left(f(x_i) - f(x_{i-1})\right)\left(\alpha(x_i) - \alpha(x_{i-1})\right) < \epsilon.$$

The plan is to bound $\alpha(x_i) - \alpha(x_{i-1})$ with a suitable bound that is the same for all of the subintervals and then use the fact that the sequence without the $\Delta\alpha_i$s is a telescoping sequence.

Let $\widetilde{\epsilon} = \frac{\epsilon}{f(b)-f(a)}$ which gives us a $\widetilde{\delta}$ a above. Choose P such that $\Delta x_i < \widetilde{\delta}$ for all $i = 1, ..., n$. Then $-\epsilon/(f(b)-f(a)) < \alpha(x_i) - \alpha(x_{i-1}) < \epsilon/(f(b)-f(a))$ for all $i = 1, ..., n$, so

$$\frac{-\epsilon}{f(b)-f(a)}(f(x_i)-f(x_{i-1})) < (f(x_i)-f(x_{i-1}))(\alpha(x_i)-\alpha(x_{i-1})) < (f(x_i)-f(x_{i-1}))\frac{\epsilon}{f(b)-f(a))}.$$

Therefore,

$$\frac{-\epsilon}{f(b)-f(a)}\sum_{k=1}^n f(x_i)-f(x_{i-1}) < U(P,f,\alpha)-L(P,f,\alpha) < \frac{\epsilon}{f(b)-f(a)}\sum_{k=1}^n f(x_i)-f(x_{i-1})$$

However, $\sum_{k=1}^n f(x_i) - f(x_{i-1}) = f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1}) = f(b) - f(a)$ is a telescoping sum, so

$$-\epsilon < U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

which is what we wanted to show.

2. Let V be a Vitali set (a subset of $[0,1]$ containing exactly one element from each of the equivalence classes under $a \sim b$ if $a - b \in \mathbb{Q}$). Prove that V is unmeasurable.

Let $\alpha \in \mathbb{R}$ and $\overline{\alpha}$ be the equivalence class of α under the specified relation. Then V contains exactly one element in $\overline{\alpha} \cap [0,1]$ by definition. Call that single element $0 \le a \le 1$ and let $Q = \mathbb{Q} \cap [-1,1]$.

Claim: $a + Q := \{a + q \mid q \in Q\} \supset \overline{\alpha} \cap [0,1]$.

Let $b \in \overline{\alpha} \cap [0,1]$. Then $b = \alpha + \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ with $0 \leq b \leq 1$. Since $a \in \overline{\alpha} \cap [0,1]$ as well, there exist $p', q' \in \mathbb{Z}$ with $a = \alpha + \frac{p'}{q'}$ and $0 \leq a \leq 1$. In other words $-1 \leq -\alpha - \frac{p'}{q'} \leq 0$ so $-1 \leq \frac{p}{q} - \frac{p'}{q'} \leq 1$ and $b = \alpha + \frac{p}{q} = a - \frac{p'}{q'} + \frac{p}{q} = a + q$ for $q = \frac{p}{q} - \frac{p'}{q'}$.

Using this claim, we see that $V + Q = \{a + q \mid a \in V, q \in Q\}$ contains all of $[0,1]$ (since it contains all of the $\overline{\alpha} \cap [0,1]$ f*every* equivalence class). In addition, $V + Q \subset [-1,2]$ since $V \subset [0,1]$. Therefore, we have $[0,1] \subset V + Q \subset [-1,3]$.

Let μ be a measure on $\Sigma \subset \mathcal{P}(\mathbb{R})$ such that $\mu([a,b]) = b - a$ (so, in particular, μ is countably additive and translation invariant). Notice that $V + Q = \bigcup_{q \in Q} V + q$,

$$1 = \mu([0,1] \leq \sum_{q \in Q} \mu(V + q) = \sum_{q \in Q} \mu(V) \leq \mu([-1,2]) = 3$$

Now, if $\mu(V)$ exists, $\mu(V) \in \mathbb{R}^{\geq 0}$ by the last inequality, so let $\mu(V) = \epsilon$. If $\epsilon = 0$, the inequality is $1 \leq 0 \leq 3$ which is clearly false, and if $\epsilon > 0$, the inequality is $1 \leq \infty \leq 3$ which is also clearly false. Therefore, $\mu(V)$ can not exist.

3. Prove that in any σ-algebra Σ with a measure $\mu : \Sigma \to \mathbb{R}^{\geq 0} \cup \{\infty\}$, that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

Since μ is a measure, it must be countably additive, and, in particular, finitely additive. Namely, if $\{E_i\} \subset \Sigma$ with $E_i \cap E_j = \varnothing$ for $i \neq j$, then $\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$. Notice that $A = (A - B) \cup (A \cap B)$, similarly for B and $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ are all unions of disjoint subsets. Therefore, finite additivity gives us $\mu(A) = \mu(A - B) + \mu(A \cap B)$, $\mu(B) = \mu(B - A) + \mu(A \cap B$ and $\mu(A \cup B) = \mu(A \cap B^c) + \mu(B \cap A^c) + \mu(A \cap B)$. Combining these gives $\mu(A \cup B) = \mu(A) - \mu(A \cap B) + \mu(B) - \mu(A \cap B) + \mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cap B)$ which is what we wanted to show.

4. Prove that the Borel σ-algebra of $\mathbb{R}^1$ (the smallest σ-algebra containing all of the open sets) is the same as the smallest σ-algebra containing all of the open intervals.

The difference between the smallest σ-algebra containing all of the open sets and the smallest σ-algebra containing all of the open intervals is that open sets can be written as *arbitrary* unions of open intervals and only *countable* unions of elements are gaurenteed to be in a σ-algebra. Therefore, it is enough to show that an artitrary union of open intervals in $\mathbb{R}$ can always be written as a countable union of intervals in $\mathbb{R}$.

Let $E = \bigcup_{\alpha \in I} (a_\alpha, b_\alpha)$ be an arbitrary union of intervals in $\mathbb{R}$ with $a_\alpha < b_\alpha$ (so none of the intervals are empty). Since $\varnothing \neq (a_\alpha, b_\alpha) \subset \mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\alpha \in I$, there exists $a_\alpha < q_\alpha < b_\alpha$ with $q_\alpha \in \mathbb{Q}$.

Claim: these q_αs can be chosen so that $\alpha \neq \beta \Rightarrow q_\alpha \neq q_\beta$.

In particular, we will show that if there exists $\beta \in I$ with $(a_\beta, b_\beta) \cap \mathbb{Q} \subset \{q_\alpha \mid \alpha \in I, \alpha \neq \beta\}$, then $\bigcup_{\alpha \in I} (a_\alpha, b_\alpha) = \bigcup_{\alpha \in I, \alpha \neq \beta} (a_\alpha, b_\alpha)$. Let $J = I - \beta$, so $E = \cup_{\alpha \in J} (a_\alpha, b_\alpha)$.

Once we prove this claim, it follows that J was in fact countable since we will have defined a bijection between J and a subset of $\mathbb{Q}$ which is countable, so we can conclude that E can be written as a union of countably many open intervals so must be in the σ-algebra generated by the open intervals.

Proof of Claim

Assume that we have such a β, so for all $q \in \mathbb{Q}$ such that $a_\beta < q < b_\beta$ there exists $\alpha \in I$ with $q = q_\alpha$ (which, in particular, implies that $a_\alpha < q < b_\alpha$). Since $a_\alpha < q_\alpha = q < b_\alpha$, there must exist $\epsilon_\alpha > 0$ such that $a_\alpha < q_\alpha - \epsilon_\alpha < q < q_\alpha + \epsilon_\alpha < b_\alpha$ and this holds for every $q \in (a_\beta, b_\beta) \cap \mathbb{Q}$. Let $\epsilon_\gamma = 0$ if $q \neq q_\gamma$, so $(q_\gamma - \epsilon_\gamma, q_\gamma + \epsilon_\gamma) = \varnothing$. Notice that this implies that $(a_\beta, b_\beta) \subset \bigcup_{\alpha \in I, \alpha \neq \beta} (q_\alpha - \epsilon_\alpha, q_\alpha + \epsilon_\alpha) \subset \bigcup_{\alpha \in I, \alpha \neq \beta} (a_\alpha, b_\alpha)$ which proves our claim.

5. Let $\lambda^*(E) = \inf\{\sum_{n=1}^{\infty}(b_n - a_n) \mid E \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\}$. Prove that if $A \subset B$, then $\lambda^*(A) \leq \lambda^*(B)$ and that λ^* is countably sub-additive.

Assume that $A \subset B$. Then, $\{\{(a_n, b_n)\}_{n=1}^{\infty} \mid A \subset \bigcup_{n=1}^{\infty}(a_n, b_n)\} \supset \{\{(a_m, b_m)\}_{m=1}^{\infty} \mid B \subset \bigcup_{m=1}^{\infty}(a_m, b_m)\}$ and since the infemium of a subset is bigger than or equal to the infemium of a superset, we get that $\lambda^*(A) \leq \lambda^*(B)$.

Let $\{E_i\}_{i=1}^{\infty}$ be a collection of disjoint subsets of the real number line. Prove that $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(E_i)$ (this is countable subadditivity). Let $1 \leq i < \infty$. Then $\lambda^*(E_i) = \inf\{\sum_{j=1}^{\infty} b_{ij} - a_{ij} \mid E_i \subset \bigcup_{j=1}^{\infty}(a_{ij}, b_{ij})\}$. Therefore, by the definition of an infemium, for all $\epsilon_i > 0$, there exists $\{(a_{ij}, b_{ij})\}_{j=1}^{\infty}$ such that $E_i \subset \bigcup_{j=1}^{\infty}(a_{ij}, b_{ij})\}$ but $\lambda^*(E_i) \leq \sum_{j=1}^{\infty} b_{ij} - a_{ij} < \lambda^*(E_i) + \epsilon_i$.

Fix $\epsilon > 0$ and let $\epsilon_i = \frac{\epsilon}{2^i}$. Notice that $\{\{(a_{ij}, b_{ij})\}_{j=1}^{\infty}\}_{i=1}^{\infty}$ is a collection of open intervals covering $\bigcap_{i=1}^{\infty} E_i$, and therefore, $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} - a_{ij} < \sum_{i=1}^{\infty}(\lambda^*(E_i) + \frac{\epsilon}{2^i}) = (\sum_{i=1}^{\infty} \lambda^*(E_i)) + \epsilon$. In other words, $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) < (\sum_{i=1}^{\infty} \lambda^*(E_i)) + \epsilon$ and this holds for all $\epsilon > 0$. Therefore, $\lambda^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \lambda^*(E_i)$ which is what we wanted to show.. (Note, disjoint wasn't actually used for countable subadditivity, but if we want countable additivity, we will definitely need it!)

6. **Lebesgue measure** Let $\mathcal{L}$ be the Lebesgue σ-algebra

a) Prove that all sets of Lebesgue outer measure zero are in $\mathcal{L}$.

Let $E \subset \mathbb{R}$ such that $\lambda^*(E) = 0$. Show that for all $A \subset \mathbb{R}$, $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$. We will be using both parts of the previous problem. Notice that $A \cap E \subset E$ which implies that $\lambda^*(A \cap E) \leq \lambda^*(E) = 0$, so $\lambda^*(A \cap E) = 0$. Similarily, $A \cap E^c \subset A$, so $\lambda^*(A \cap E^c) \leq \lambda^*(A)$. In addition, $A \cap E$ and $A \cap E^c$ are disjoint sets whose union is A, so by subadditivity, $\lambda^*(A) \leq \lambda^*(A \cap E) + \lambda^*(A \cap E^c) = 0 + \lambda^*(A \cap E^c)$. This combined with the previous observation tells us that they are, in fact, equal.

b) Prove that the Cantor (middle 1/3) set is in $\mathcal{L}$.

In a previous homework, we showed that the sum of the removed intervals from the Cantor middle third set has total length 1 and therefore, the length of the Cantor set is 0. We can now make a much more formal arguement that says that let C_n be the nth iterate of the Cantor set. Then for all $\epsilon > 0$, there exists N such that $\lambda^*(C_n) < \epsilon$ for all $n \geq N$. Since $C = \bigcap_{n \in \mathbb{N}} C_n$, we have that $\lambda^*(C) < \epsilon$ for all $\epsilon > 0$ and therefore, $\lambda^*(C) = 0$. To do this, we just need to choose N such that $\frac{1}{2}\frac{\frac{2}{3}^N - \frac{2}{3}}{-\frac{1}{3}} > 1 - \epsilon$.

7. **Riemann Lebesgue theorem** Let $f : [a, b] \to \mathbb{R}$ bounded and let $A \subset [a, b]$ be the set of discontinuities of f and $M = \sup\{f(x) \mid x \in [a, b]$ and $m = \inf\{f(x) \mid x \in [a, b]\}$.

a) Define $\operatorname{osc}(f; c) = \lim_{h \to 0^+} \sup\{|f(z) - f(y)| \mid y, z \in (c - h, c + h) \cap [a, b]\}$. Prove that $\operatorname{osc}(f; c) = 0$ if and only if f is continuous at $x = c$.

This is basically just the definition of continuity.

$\Rightarrow$ Assume that $\lim_{h\to 0^+} \sup\{|f(z)-f(y)| \mid y,z \in (c-h,c+h)\cap[a,b]\} = 0$. Since $|f(z)-f(y)| \geq 0$, $\lim = 0$ implies that for all $\widetilde{\epsilon} > 0$, there exists a $\widetilde{\delta} > 0$ such that $0 \leq \sup\{|f(z)-f(y)|\} < \widetilde{\epsilon}$ for all $0 < h < \widetilde{\delta}$. This is continuity itself because for any fixed $\epsilon > 0$, let $\widetilde{\epsilon} = \epsilon$ and let $\delta = \widetilde{\delta}$. Then for all $y', z' \in (c-\delta, c+\delta)$ for which f is defined, $|f(z)-f(y) \leq \sup\{|f(z)-f(y)|\} < \epsilon$ and since $c \in (c-\delta,c+\delta)$ and $f(c)$ is defined, we have $|f(c)-f(y)| < \epsilon$ for all $y \in [a,b]$ such that $y \in B(c,\delta)$ which is continuity.

$\Leftarrow$ Assume that for all $\widetilde{\epsilon} > 0$, there exists $\widetilde{\delta} > 0$ such that $|x-c| < \widetilde{\delta}$ implies that $|f(c)-f(x)| < \widetilde{\epsilon}$. The limit is zero if for all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < h < \delta$ implies that $\sup\{|f(z)-f(y)|\} < \epsilon$ for all $y,z \in B(c,\delta)\cap[a,b]$, so we will need to show that. To show that the supremium is less than ϵ, we will need to show that for all $y', z' \in (c-h,c+h)\cap[a,b]$, $|f(y')-f(z')| < \epsilon' < \epsilon$.

b) Let $A_s = \{x \in [a,b] \mid \operatorname{osc}(f,x) \geq s\}$. Prove that $A_s \subset [a,b]$ is compact. Hint: notice that the definition of the osc is basically the smallest ϵ for which continuity fails.

We need to show that A_s is closed (it is clearly bounded). Namely, let y be a limit point of A_s, prove that $\operatorname{osc}(f;y) \geq s$.

I think Codie was right, I'm way over thinking this. Fix $h > 0$ and $\epsilon > 0$. Our goal is to show that there exists a pair $z, w \in (y-h,y+h)$ such that $|f(z)-f(w)$ is greater than $s-\epsilon$. This would imply that the supremium of all these values was at least s and since this would hold for all $h > 0$, this would imply that the limit $h \to 0^+$ of the supremiums would be at least s (so $y \in A_s$).

Given this fixed $h > 0$, since y is a limit point of A_s, there exists $x \in (y-h,y+h)$ with $\operatorname{osc}(f;x) = t \geq s$. This means that for any, $\epsilon' > 0$, there exists $\delta' > 0$ such that $t-\epsilon' < \sup\{|f(z)-f(w)|\ z,w \in (x-h',x+h')\} < t+\epsilon'$ for all $0 < h/ < \delta'$ (definition of a limit). In particular, we can choose h' such that $(x-h',x+h') \subset (y-h,y+h)$. Since the set we are talking about is a supremium (aka least upper bound), we know that for every $\epsilon'' > 0$, there exists a pair $z,w \in (x-h',x+h')$ such that $\sup - \epsilon'' < |f(z)-f(w)| < \sup$. Combining these two strings of inequalities with the fact that $t \geq s$, we get $s-\epsilon'-\epsilon'' \leq t-\epsilon'-\epsilon'' < \sup -\epsilon'' < |f(z)-f(w)| < \sup < t+\epsilon'$.

Notice that both $\epsilon' > 0$ and $\epsilon'' > 0$ were arbitrary, so we can choose them so that $s-\epsilon \leq s-\epsilon'-\epsilon''$ and get a pair of points $z,w \in (x-h',x+h') \subset (y-h,y+h)$ such that $s-\epsilon < |f(z)-f(w)|$ which is what we wanted to show.

c) Fix $\epsilon > 0$. Assume that $\lambda^*(A) = 0$ where λ^* is the Lebesgue outer measure. Show there exists a finite collection of intervals $\{I_1, I_2, ..., I_n\}$ with $I_i = (a_i,b_i)$ such that $A_{\frac{\epsilon}{2(b-a)}} \subset \bigcup_{i=1}^n I_i$ and $\sum_{i=1}^n b_i - a_i < \frac{\epsilon}{2(M-m)}$.

Fix $\epsilon > 0$.

Since $\lambda^*(A) = \inf\{\sum_{n=1}^\infty (b_n-a_n) \mid E \subset \bigcup_{n=1}^\infty (a_n,b_n)\} = 0$, for all $\epsilon' > 0$, there exists a collection of open intervals $\{I_n\}_{n=1}^\infty$ such that $A \subset \bigcup_{n=1}^\infty I_n$ and $\sum_{n=1}^\infty \ell(I_n) < \epsilon'$, where $\ell(I_k)$ is the length of that open interval (this is since λ^* is the infemium, so ϵ' is not a lower bound).

Let $\epsilon' = \frac{\epsilon}{2(M-m)}$ where M and m are defined as in the statement of the problem.

Since $A_{\frac{\epsilon}{2(b-a)}} \subset A \subset \bigcup_{i=1}^\infty I_n$, the collection $\{I_n\}$ form an open cover of $A_{\frac{\epsilon}{2(b-a)}}$, and since A_s is compact for all s, $\{I_n\}$ has a finite subcover. Relabel so that this finite subcover is $\{I_1, I_2, ..., I_n\}$ with $I_k = (a_k,b_k)$. Then $A_{\frac{\epsilon}{2(b-a)}} \subset \bigcup_{i=1}^n I_i$ and $\sum_{i=1}^n \ell(I_i) = \sum_{i=1}^n b_i - a_i \leq \sum_{i=1}^\infty \ell(I_i) < \epsilon' = \frac{\epsilon}{2(M-m)}$, which is what we wanted to show.

d) Prove that there exists a finite cover of $[a, b] - \bigcup I_i$ by sets of the form $B(x, \delta_x)$ such that for all $y, z \in B(x, \delta_x)$, $|f(y) - f(z)| < \frac{\epsilon}{2(b-a)}$.

Let $x \in [a, b] - \bigcup I_i$. Therefore, $\text{osc}(f; x) < \frac{\epsilon}{2(b-a)}$ since $\{I_i\}_{i=1}^n$ cover the set where this is not true. Therefore, $\text{osc}(f; x) = \lim_{h \to 0^+} \sup\{|f(z) - f(y)| \mid y, z \in (c - h, c + h) \cap [a, b]\} = t < \frac{\epsilon}{2(b-a)}$, so for all $\epsilon' > 0$, there exists $\delta' > 0$ such that $t = \epsilon' < \sup\{|f(z) - f(w)| \mid z, w \in (x - h, x + h)\} < t + \epsilon'$ for all $0 < h < \delta'$. Since $t < \frac{\epsilon}{2(b-a)}$ and ϵ' was arbitrary, we can choose ϵ' so that $t + \epsilon' < \frac{\epsilon}{2(b-a)}$. For this choice of ϵ', we get a δ' and choose $\delta_x > 0$ such that $\delta_x < \delta'$ (notice that all of this depended on which $x \in [a, b] - \bigcup I_i$ we are looking at). Therefore, for all $y, z \in (x - \delta_x, x + \delta_x) = B(x, \delta_x)$, $|f(y) - f(z) \leq \sup\{|f(z) - f(w)| \mid z, w \in (x - \delta_x, x + \delta_x)\} < \frac{\epsilon}{2(b-a)}$ which is what we wanted to show.

e) Combining parts c) and d), we get an open cover of $[a, b]$. Prove that there exists a partition $P = \{x_0, x_1, ..., x_p\}$ of $[a, b]$ such that each $[x_{i-1}, x_i]$ is contained in one of the open sets of our cover. (I changed the size of our partition to p because I had already used n in the statement of part c, and did not mean to imply they were the same. In fact, they can't be the same!!)

In the previous part, we produced $\{B(x, \delta_x)\}_{x \in [a,b] - \bigcup I_i}$ which is an open cover of $[a, b] - \bigcup I_i$. Since each I_i is an open interval, and the union of open sets is open, $[a, b] - \bigcup I_i = [a, b] \cap (\bigcup I_i)^c$ is a closed and bounded subset of $\mathbb{R}^1$, so is compact. Therefore, we have a finite subcover, call it $\{B(\overline{x_1}, \delta_{x_1}, ..., B(\overline{x_m}, \delta_{x_m}\}$ which covers $[a, b] - \bigcup I_i$ and combined with $\{I_1, ..., I_n\}$ which covers $\bigcup I_i$, we have a finite open cover of $[a, b]$ (it has $n + m$ open sets in it) which consists entirely of intervals. Call this open cover $\mathcal{U} = \{U_i\}_{i=1}^{n+m}$.

Since we have a set of open intervals which cover all of $[a, b]$, the intervals must overlap (if $U_i \cap U_j = \varnothing$ for all $i \neq j$, then either $U_i = [a, b]$, or $\inf U_i$ and $\sup U_i$ are not in the open cover).

Assume that (a_i, b_i) and (a_j, b_j) are two such overlapping intervals with $a_i < a < b_i$, $a_j < b_i$, and $a_j < b < b_j$ (see Kirkwood for an illustration). This is the start of an induction arguement that I'm going to be lazy with and only explain the induction step. The notation gets horrible if I don't just start with $[a, b]$ covered with those two intervals (we would have to construct a sequence of partitions, each one with one more pair of overlapping sets taken care of). Then $[a, \frac{1}{2}(a_j + b_i)]$ and $[\frac{1}{2}(a_j + b_i), b]$ form a partition where $[a, \frac{1}{2}(a_j + b_i)]$ is in (a_i, b_i) and $[\frac{1}{2}(a_j + b_i), b]$ is in (a_j, b_j).

The size of our partition p will depend on the arrangement of the overlaps between our sets.

f) For the subintervals contained in an I_i, use M and m for each of the M_i and m_i to show that together, these subintervals contribute $< \epsilon/2$ to the total of $U(P, f) - L(P, f)$.

Let $[x_{i-1}, x_i] \subset I_k$ for some $1 \leq i \leq p$ and some $1 \leq k \leq n$. Then the contribution from that interval to the total of $U(P, f) - L(P, f)$ is $(M_i - m_i)(x_i - x_{i-1}) < (M - m)(b_k - a_k)$, and if two such $[x_{i=-}, x_i], [x_i, x_{i+1}]$ are both in the same I_k, then they contribute $(M_i - m_i)(x_i - x_{i-1}) + (M_{i+1}, m_{i+1})(x_{i+1} - x_i) < (M - m)(x_{i+1} - x_{i-1}) < (M - m)(b_k - a_k)$.

Let Let $i_1, i_2, ..., i_q$ be the indicies of intervals contained in the $I_1, ..., I_n$. Then the total $\Delta x_{i_1} + \Delta x_{i_2} + \cdots + \Delta x_{i_q} < (b_1 - a_1) + \cdots + (b_n - a_n) < \frac{\epsilon}{2(M-m)}$ by part c, so the total contribution to $U(P, f) - L(P, f)$ is $\sum_{j=1}^q (M_{i_j} - m_{i_j})\Delta x_{i_j} < \sum_{j=1}^q (M - m)\Delta x_{i_j} = (M - m) \sum_{j=1}^q \Delta x_{i_j} < (M - m)\frac{\epsilon}{2(M-m)} = \epsilon/2$

g) Use your inequality from part d) to show that the subintervals contained in the open sets of the form $B(x, \delta_x)$ together also contribute $< \epsilon/2$ to the total of $U(P, f) - L(P, f)$.

We proved in part d) that for all $y, z \in B(\overline{x_j}, \delta_{x_j})$, it holds that $|f(y) - f(z)| < \frac{\epsilon}{2(b-a)}$. This implies that if $[x_{i-1}, x_i] \subset B(\overline{x_j}, \delta_{x_j})$, then $|M_i - m_i| < \frac{\epsilon}{2(b-a)}$ as well.

Therefore, the contribution of that interval is $(M_i - m_i)(x_i - x_{i-1})$ and if $i_1, ..., i_r$ are a list of such intervals, then the total contribution amoung all of them to $U(P, f) - L(P, f)$ is $\sum_{k=1}^{r}(M_{i_k} - m_{i_k})\Delta x_{i_k} < \sum_{k=1}^{r} \frac{\epsilon}{2(b-a)} \Delta x_{i_k} = \frac{\epsilon}{2(b-a)} \sum_{k=1}^{r} \Delta x_{i_k}$.

Since the total $\sum_{i=1}^{p} \Delta x_i = b - a$, it must hold that $\sum_{k=1}^{r} \Delta x_{i_k} \leq b - a$, so the total contribution of such intervals is $\sum_{k=1}^{r}(M_{i_k} - m_{i_k})\Delta x_{i_k} < \frac{\epsilon}{2(b-a)}(b - a) = \epsilon/2$

h) Conclude that if $\lambda^*(A) = 0$, then $f \in \mathcal{R}[a, b]$.

Notice that $U(P, f) - L(P, f) = \sum_{i=1}^{p}(M_i - m_i)\Delta x_i = \sum_{j=1}^{q}(M_{i_j} - m_{i_j})\Delta x_{i_j} + \sum_{k=1}^{r}(M_{i_k} - m_{i_k})\Delta x_{i_k}$ where $p = q + r$ because each sub-interval belongs to one fo these two sets. By part f), $\sum_{j=1}^{q}(M_{i_j} - m_{i_j})\Delta x_{i_j} < \epsilon/2$ and by part g), $\sum_{k=1}^{r}(M_{i_k} - m_{i_k})\Delta x_{i_k} < \epsilon/2$ and therefore, in total $U(P, f) - L(P, f) < \epsilon$. Since ϵ was arbitrary, by the awesome condition for integrability, $f \in \mathcal{R}[a, b]$.

i) Let $A_{\frac{1}{k}}$ defined as in b), so $A = \bigcup_{k \in \mathbb{N}} A_{\frac{1}{k}}$, fix $\epsilon > 0$ and $k \in \mathbb{N}$. Assume that $f \in \mathcal{R}[a, b]$. Choose P partition of $[a, b]$ with $U(P, f) - L(P, f) < \frac{\epsilon}{2k}$. Show that the contribution to $U(P, f) - L(P, f)$ from the subintervals of P whose intersection with $A_{\frac{1}{k}}$ is greater than or equal to $\frac{1}{k}$ times the sum of their Δx_is and therefore $A_{\frac{1}{k}}$ can be covered by intervals whose total length is less than $\epsilon/2$.

It's on p. 144 of Kirkwood!! I'm seriously about to kill this homework. I'm sorry it was so long!!!

j) Conclude that $f \in \mathcal{R}[a, b]$ implies that the measure of A is zero.