Math 411 Homework 5 Due Friday, April 22

(Big Quiz 2 on Friday, April 22 also)

1. FTC Let $f : [a, b] \to \mathbb{R}$ continuous.

a) Prove that $\int_a^x f(t)\,dt = 0$ for all $x \in [a, b]$ implies that $f(x) = 0$ for all $x \in [a, b]$.

b) Prove that $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for all $x \in [a.b]$ implies that $f(x) = 0$ for all $x \in [a, b]$.

2. Integral definition of $\ln x$ and e

a) Use the definintion of $\ln x = \int_1^x \frac{1}{t}\,dt$ to prove that $\ln(x/y) = \ln x - \ln y$ for $x, y > 0$.

b) Use the definition of e as the number such that $\int_1^e \frac{1}{t}\,dt = 1$ to prove that $e^{a-b} = e^a/e^b$ and $e^{ab} = (e^a)^b$.

c) Use the definition from part b) to prove that $\frac{d}{dx}e^x = e^x$.

3. Show that the sequence $f_n(x) = \frac{x}{n}$ converges uniformly on $[0, M]$ for any M, but only pointwise on $[0, \infty)$.

4. The sup norm Let X and Y be metric spaces. Define $\mathcal{C}_Y(X) = \{f : X \to Y \mid f \text{ is continuous and bounded }\}$. If Y is a normed space with norm $|\cdot|_Y$, then define the sup norm on $\mathcal{C}_Y(X)$ to be $\|f\| = \sup\{|f(x)|_Y \mid x \in X\}$.
Prove that $\mathcal{C}_Y(X)$ is a metric space under the metric induced by the sup norm ($d_{\mathcal{C}_Y(X)}(f, g) = \|f - g\|$). What does $B(f, \epsilon)$ look like?

5. Let f be a bounded continuous function on $[0, 1]$ and define $\|f\|_n = \left(\int_0^1 |f(x)|^n\,dx\right)^{1/n}$. Prove that $\lim_{n\to\infty} \|f\|_n = \|f\|$.

6. Prove that the uniform limit of bounded functions is bounded and that the sequence itself is uniformly bounded.

7. Pointwise convergence with no uniformly convergent subsequence

a) Let $f_n(x) = \frac{1}{nx+1}$. Prove that $f_n \to 0$ pointwise on $[0, 1]$.

b) Prove that the fact that $f_n(\frac{1}{n}) = \frac{1}{2}$ for all $n \in \mathbb{N}$ implies that there is no subsequence converging uniformly on $[0, 1]$.

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ continuous and let $f_n(x) = f(nx)$ for $n \in \mathbb{N}$. Prove that $\{f_n\}$ equicontinuous implies that $f(x) = c$ for some constant c.

9. Let $f_n : [a, b] \to \mathbb{R}$ such that $\{f_n\}$ is uniformly bounded and $f_n \in \mathcal{R}([a, b])$. Let $F_n(x) = \int_a^x f_n(t)\,dt$ for $x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ converging uniformly on $[a, b]$.

10. Weierstrass Theorem Let $f : [0, 1] \to \mathbb{R}$ and assume that $\int_0^1 f(x)x^n\,dx = 0$ for all $n \in \mathbb{N}$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.