Math 411 Homework 5 Due Friday, April 22

(Big Quiz 2 on Friday, April 22 also)

1. FTC Let $f : [a,b] \to \mathbb{R}$ continuous.

a) Prove that $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$ implies that $f(x) = 0$ for all $x \in [a,b]$.

FTC part 1 implies that $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, so by differentiating both sides of the above equation, we get $f(x) = 0$.

b) Prove that $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for all $x \in [a.b]$ implies that $f(x) = 0$ for all $x \in [a,b]$.

Note that $\int_x^b f(t)\,dt = -\int_b^x f(t)\,dt$, so differentiate both sides of the equation using FTC part 1 gives $\frac{d}{dx}\int_a^x f(t)\,dt = f(x) = -f(x) = \frac{d}{dt} - \int_b^x f(t)\,dt$. Since the only real number y with $y = -y$ is 0 and this holds for all $x \in [a,b]$, we get $f(x) = 0$

2. Integral definition of $\ln x$ and e

a) Use the definition of $\ln x = \int_1^x \frac{1}{t}\,dt$ to prove that $\ln(x/y) = \ln x - \ln y$ for $x, y > 0$.

This is a change of variables argument. $\ln(x/y)$ is equal to by definition $\displaystyle\int_1^{\frac{x}{y}} \frac{1}{t}\,dt$ while the right hand side $\ln x - \ln y = \int_1^x \frac{1}{t}\,dt - \int_1^y \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_y^1 \frac{1}{t}\,dt = \int_y^x \frac{1}{t}\,dt$. Let $u = \frac{x}{y}$. So what I'm looking for is a change of variables $s(t)$ such that $s(1) = y$ and $s(x/y) = x$. The obvious choice is $s = yt$, so $ds = ydt$ and

$$\ln(x/y) = \int_1^{\frac{x}{y}} \frac{1}{t}\,dt = \int_y^x \frac{1}{\frac{s}{y}}\frac{1}{y}\,ds = \int_y^x \frac{1}{s}\,ds = \ln x - \ln y.$$

b) Use the definition of e as the number such that $\int_1^e \frac{1}{t}\,dt = 1$ to prove that $e^{a-b} = e^a/e^b$ and $e^{ab} = (e^a)^b$.

Using the definition from a), this means that e is defined to be the number such that $\ln(e) = 1$. By part a), we know that $\ln(e^a/e^b) = \ln(e^a) - \ln(e^b) = a\ln(e) - b\ln(e) = (a-b)\ln(e) = \ln(e^{a-b})$ if only we show that $\ln(x^r) = r\ln(x)$. However, that is just another change of variables since if we let $t = s^r$, then $dt = rs^{r-1}ds$ which makes

$$\ln(x^r) = \int_1^{x^r} \frac{1}{t}\,dt = \int_1^x \frac{1}{s^r} rs^{r-1} ds = r\int_1^x \frac{1}{s}\,ds = r\ln(x)$$

Therefore, $\ln(e^a/e^b) = \ln(e^{a-b})$, however, since $\frac{1}{t}$ is a strictly positive function, $\ln(x)$ is a strictly increasing function (FTCI), which means that $\ln(x) = \ln(y)$ implies that $x = y$.

Similarly, $\ln(e^{ab}) = ab\ln(e) = b\ln(e^a) = \ln((e^a)^b)$ implies that $e^{ab} = (e^a)^b$.

c) Use the definition from part b) to prove that $\frac{d}{dx}e^x = e^x$.

By our previous work, notice that $\ln(e^x) = x\ln(e) = x$. Also, we know by FTC1 that $\frac{d}{dx}\ln x = \frac{1}{x}$. Differentiating both sides of the above equation now gives $\frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\frac{d}{dx}e^x = 1$. Solving for $\frac{d}{dx}e^x$ gives us $\frac{d}{dx}e^x = e^x$.

3. Show that the sequence $f_n(x) = \frac{x}{n}$ converges uniformly on $[0, M]$ for any M, but only pointwise on $[0, \infty)$.

Notice that $\lim_{n\to\infty} f_n(x) = 0$, so the limit of the functions is the zero function. Given M ($M > 0$ is implied by the problem as $[0, M] = \varnothing$ if $M < 0$ and $[0, 0]$ is a single point where $f_n(0) = 0$ for all n), show that for all $\epsilon > 0$, there exists N such that $|f_n(x) - 0| \leq \epsilon$ for all $x \in [0, M]$. Fix $\epsilon > 0$ and let $N = \frac{M}{\epsilon}$, then for all $n \geq N$, $f_n(x) = \frac{x}{n} \leq \frac{x}{\frac{M}{\epsilon}} = \frac{x\epsilon}{M}$ since $x \in [0, M]$, $f_n(x) \leq \frac{M\epsilon}{\epsilon M} \leq \epsilon$ which is the first thing we wanted to show.
On $[0, \infty)$, $f_n \to 0$ pointwise since for any $x \geq 0$, given $\epsilon > 0$, let $N > x/epsilon$. Then for all $n \geq N$, $|f_n(x)| = |\frac{x}{n}| < |\frac{x}{\frac{x}{\epsilon}}| < \epsilon$ (for pointwise convergence, N is allowed to depend on x). To show that convergence fails to be uniform, we need to show that there exists $\epsilon > 0$ such that for all N, there exists $x \in [0, \infty)$ such that $|f_n(x)| \geq \epsilon$ for some $n \geq N$. Let $\epsilon = 1$. Then for any N, let $x = \frac{1}{N}$. Notice that $|f_n(x)| = |\frac{x}{N}| = |\frac{\frac{1}{N}}{N}| = 1$ which is what we wanted to prove.

4. **The sup norm** Let X and Y be metric spaces. Define $\mathcal{C}_Y(X) = \{f : X \to Y \mid f \text{ is continuous and bounded }\}$. If Y is a normed space with norm $|\cdot|_Y$, then define the sup norm on $\mathcal{C}_Y(X)$ to be $\|f\| = \sup\{|f(x)|_Y \mid x \in X\}$.
Prove that $\mathcal{C}_Y(X)$ is a metric space under the metric induced by the sup norm ($d_{\mathcal{C}_Y(X)}(f, g) = \|f - g\|$). What does $B(f, \epsilon)$ look like?

We need to show
1. $d_{\mathcal{C}_Y(X)} : \mathcal{C}_Y(X) \times \mathcal{C}_Y(X) \to \mathbb{R}^{\geq 0}$ and $d(f, g) = 0 \Rightarrow f = g$.
2. $d(f, g) = d(g, f)$
3. the triangle inequality

1. Since $|-|_Y$ is a norm on Y, it is a map $|-|_Y : Y \to \mathbb{R}^{\geq 0}$, so $|f(x) - g(x)| \in \mathbb{R}^{\geq 0}$ for all $x \in X$. Therefore, $d(f, g) = \|f - g\| = \sup\{|f(x) - g(x)|_Y\} \in \mathbb{R}^{\geq 0}$ and $d_{\mathcal{C}_Y(X)} : \mathcal{C}_Y(X) \times \mathcal{C}_Y(X) \to \mathbb{R}^{\geq 0}$. Assume that $d(f, g) = \|f - g\| = 0$. Namely, we are assuming that $\sup\{|f(x) - g(x)|_Y\} = 0$, and therefore that $|f(x) - g(x)|_Y = 0$ for all $x \in X$. Since $|-|_Y$ is a norm, this implies that $f(x) = g(x)$ for all $x \in X$ and therefore that $f = g$ as functions from X to Y.
2. Since $|-|_Y$ is a norm, $|f(x) - g(x)|_Y = |g(x) - f(x)|_Y$ and therefore $\|f - g\| = \|g - f\|$.
3. Since $|-|_Y$ is a norm, it satisfies the triangle inequality. Namely, if $f, g, h : X \to Y$, then $|f(x) - h(x)|_U \leq |f(x) - g(x)|_Y + |g(x) - h(x)|_Y$. Therefore, $d(f, h) = \|f - h\| = \sup\{|f(x) - h(x)|_Y\} \leq \sup\{|t(x) - g(x)|_Y\} + \sup\{|g(x) - h(x)|_Y\} = \|f - g\| + \|g - h\| = d(f, g) + d(g, h)$ as we wanted.

$B(f, \epsilon) \subset \mathcal{C}_Y(X)$ looks like an epsilon ribbon around the function f.

5. Let f be a bounded continuous function on $[0, 1]$ and define $\|f\|_n = \left(\int_0^1 |f(x)|^n \, dx\right)^{1/n}$. Prove that $\lim_{n\to\infty} \|f\|_n = \|f\|$.

The key to this problem is the fact that $0 < \lambda \leq 1$ implies that $\lim_{n\to\infty} \sqrt[n]{\lambda} = 1$ (this is the inverse function version of the fact that the pointwise limit of $y = x^n$ on $[0, 1)$ is $f(x) = 0$).
Since f is a continuous function on a compact set, f achieves both a max and a min. Let M be the larger of the absolute values of these two quantities. Then $|f(x)| \leq M$ for all $x \in [0, 1]$ and $\|f\| = \sup\{|f(x)| \mid x \in [0, 1]\} = M$ (since both the max and the min are achieved on $[0, 1]$, there exists $x_0 \in [0, 1]$ with $|f(x_0)| = M$). In addition, our inequality implies that $\int_0^1 |f(x)|^n \, dx \leq \int_0^1 M^n \, dx = M^n(1 - 0) = M^n$, so $\|f\|_n \leq M$ for all $n \in \mathbb{N}$. Since sequential limits preserve inequalities, this implies that $\lim_{n\to\infty} \|f\|_n \leq M$.

To show that $\lim_{n\to\infty} \|f\|_n = M$, show that for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $M - \epsilon < \|f\|_n \leq M$ for all $n \geq N$ (i.e. $M - \epsilon < \left(\int_0^1 |f(x)|^n \, dx\right)^{\frac{1}{n}} \leq M$). For now, assume that all those things are non-neg (the only part in doubt is the $M - \epsilon$) and raising to the nth power is an increasing function on positive numbers, so this is $(M-\epsilon)^n < \int_0^1 |f(x)|^n \, dx \leq M^n$.

Assume $M > 0$ (otherwise $f = 0$ on $[0,1]$ and all of the norms involved are 0, so proposition holds). Given $0 < \epsilon < \epsilon_1 < M$, the fact that $|f(x_0)| = M$ implies that there exists $\delta > 0$ such that $|f(x)| \geq M - \epsilon_1$ for $x \in [x_0 - \delta, x_0 + \delta]$. Then $\left(\int_0^1 |f(x)|^n \, dx\right)^{\frac{1}{n}} \geq \left(\int_{x_0-\delta}^{x_0+\delta} |f(x)|^n \, dx\right)^{\frac{1}{n}} \geq \left(\int_{x_0-\delta}^{x_0+\delta} (M-\epsilon_1)^n \, dx\right)^{\frac{1}{n}} = (M - \epsilon_1)\sqrt[n]{2\delta}$. Since $0 < 2\delta \leq 1$, we know that $\lim_{n\to\infty} \sqrt[n]{2\delta} = 1$ (this is a *pointwise* limit), so $\lim_{n\to\infty} \|f\|_n \geq M - \epsilon_1 > M - \epsilon$, which implies that there exists N such that $\|f\|_n > M - \epsilon$ for all $n \geq N$.

6. Prove that the uniform limit of bounded functions is bounded and that the sequence itself is uniformly bounded.

For both parts, let $f_n : X \to Y$ and assume that for all $n \in \mathbb{N}$, there exists M_n such that $|f_n(x)| < M_n$ (i.e. f_n is bounded). Let $f_n \to f$ (with $f : X \to Y$) uniformly. The first part is prove that there exists M such that $|f(x)| < M$ for all $x \in X$. The second part is prove that there exists $\widetilde{M}$ such that $|f_n(x)| < \widetilde{M}$ and $|f(x)| < \widetilde{M}$ for all $x \in X$.

Since $f_n \to f$ uniformly, for all $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and for all $x \in X$. If $\epsilon = 1$, using the Cauchy condition, we get N such that $|f_n(x) - f_N(x)| < 1$ for all $n \geq N$ and all $x \in X$. In other words, for all $x \in X$ and for all $n \geq N$, $d(f_N, f_n) < 1$, or $f_n \in B_{\mathcal{C}_Y(X)}(f_N, 1)$ for all $n \geq N$. This implies that $f(x) = \lim_{n\to\infty} f_n(x) \in \overline{B_{\mathcal{C}_Y(X)}(f_N, 1)}$. That is, $d(f_N, f) \leq 1$, and since $|f_N(x)|$ is bounded above by M_N, $|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M_N$ for all $x \in X$ which is the first part of what we wanted to show.

For the second part, simply let $\widetilde{M} = \max\{M_1, M_2, ..., M_{N-1}, M_N + 1\}$.

7. Pointwise convergence with no uniformly convergent subsequence

a) Let $f_n(x) = \frac{1}{nx+1}$. Prove that $f_n \to 0$ pointwise on $(0,1]$ (this was a typo, the interval shouldn't have included 0.

Fix $\epsilon > 0$ and fix $0 < x \leq 1$. Then any $N > \frac{\frac{1}{\epsilon} - 1}{x}$ will give us $|\frac{1}{nx+1}| < \epsilon$.

b) Prove that the fact that $f_n(\frac{1}{n}) = \frac{1}{2}$ for all $n \in \mathbb{N}$ implies that there is no subsequence converging uniformly on $(0,1]$.

Assume there is a uniformly convergent subsequence. This means that for all $\epsilon > 0$, there exist K such that for all $k \geq K$, $|f_{n_k}(x) - 0| < \epsilon$ for all $x \in (0,1]$. Assume that this is the case and let $\epsilon = \frac{1}{2}$ and K as given. Notice that $f_{n_k}(\frac{1}{n_k}) = \frac{1}{n_k \frac{1}{n_k} + 1} = \frac{1}{2}$. Since $n_k \in \mathbb{N}$, $\frac{1}{n_k} \in (0,1]$, however, $|f_{n_k}(\frac{1}{n_k})| = \frac{1}{2} \geq \epsilon$ which is a contradiction.

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ continuous and let $f_n(x) = f(nx)$ for $n \in \mathbb{N}$. Prove that $\{f_n\}$ equicontinuous implies that $f(x) = c$ for some constant c.

Proof by contradiction. Assume that there exist $a, b \in \mathbb{R}$ such that $f(a) \neq f(b)$ and assume that $\{f_n\}$ is equicontinuous (i.e. for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$. Let $|f(a) - f(b)| = \epsilon$. Equicontinuity gives a corresponding $\delta > 0$. The idea is that the points $\frac{a}{n}$ and the points $\frac{b}{n}$ both approach 0, so will eventually be less than δ apart (for large enough n). However, our assumption $f_n(x) = f(nx)$ implies that $f_n(\frac{a}{n}) = f(n\frac{a}{n}) = f(a)$ and $f_n(\frac{b}{n}) = f(b)$, so their y coordinates will still have distance ϵ from each other. This will contradict equicontinuity and we will have shown that f is necessarily constant.

Since $a \in \mathbb{R}$, $\lim_{n\to\infty} \frac{a}{n} = 0$, so there exists N_1 such that $|\frac{a}{n}|, \delta/2$ for all $n \geq N_1$ and similarly there exists N_2 such that $|\frac{b}{n}| < \delta/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then $|\frac{a}{N} - \frac{b}{N}| < \delta$ by design but $f_N(\frac{a}{N}) = f(N\frac{a}{N}) = f(a)$ and $f_N(\frac{b}{N}) = f(b)$, so $|f_N(\frac{a}{N}) - f_N(\frac{b}{N})| = |f(a) - f(b)| = \epsilon$ which is not $< \epsilon$ and so we have a contradiction.

9. Let $f_n : [a, b] \to \mathbb{R}$ such that $\{f_n\}$ is uniformly bounded and $f_n \in \mathcal{R}([a, b])$. Let $F_n(x) = \int_a^x f_n(t)\ dt$ for $x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ converging uniformly on $[a, b]$.

We need to show that $\{F_n\}$ is pointwise bounded and equicontinuous. If this is the case, the Arzelá-Ascoli theorem guarantees a convergent subsequence in $\mathcal{C}_\mathbb{R}([a, b])$ and convergence in the function space is uniform convergence of the subsequence.

Since $\{f_n\}$ is uniformly bounded, there exists M such that $|f_n(x)| < M$ for all $x \in [a, b]$. This implies that $|F_n(x)| = |\int_a^x f_n(t)\ dt| \leq \int_a^x |f_n(t)|\ dt \leq \int_a^b |f_n(t)|\ dt \leq \int_a^b M\ dt = M(b - a)$ so the set $\{F_n(x)\}$ is pointwise bounded by the constant function $\phi(x) = M$.

To show that the set is equicontinuous, we need to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|F_n(x) - F_n(y)| = |\int_a^x f_n(t)dt - \int_a^y f_n(t)dt| = |\int_y^x f(t)dt| < \epsilon$. Without loss of generality, assume that $y < x$. Notice that $|\int_y^x f_n(t)dt| \leq \int_y^x |f_n(t)|\ dt \leq \int_y^x M\ dt = M(x - y)$, so $\delta = \epsilon/M$ will work.

10. **Weierstrass Theorem** Let $f : [0, 1] \to \mathbb{R}$ continuous and assume that $\int_0^1 f(x)x^n\ dx = 0$ for all $n \in \mathbb{Z}^{\geq 0}$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

The hint was use the Weierstrass Theorem to show that $\int_0^1 (f(x))^2\ dx = 0$.

By the Weierstrass Approximation Theorem, there exists a sequence of polynomials $p_n(x)$ that converge uniformly to $f(x)$ on $[0, 1]$. Let $p_n(x) = a_0(n) + a_1(n)x + \cdots + a_n(n)x^n$. Since $\int_0^1 f(x)x^k\ dx = 0$ for all $k = 0, 1, ...,$ for any $n = 0, 1, ...$ the coefficient of x^k in $p_n(x)$, $a_k(n) \in \mathbb{R}$, so $\int_0^1 f(x)a_k(n)x^k = 0$ and by summing these up, we get that $\int_0^1 f(x)p_n(x)\ dx = 0$. Recall that if the limit is uniform and all of our functions are integrable, then $\int_a^b \lim_{n\to\infty} f_n(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx$, so we need to show that $f(x)p_n(x) \to (f(x))^2$ uniformly and we will have achieved the hint.

Since $p_n(x) \to f(x)$ uniformly, for every $\widetilde{\epsilon} > 0$, there exists $\widetilde{N}$ such that $|p_n(x) - f(x)| < \widetilde{\epsilon}$ for every $n \geq \widetilde{N}$ and for every $x \in [0, 1]$. We need to show that for all $\epsilon > 0$, there exists N such that $|f(x)p_n(x) - f(x)f(x)| = |f(x)| \cdot |p_n(x) - f(x)|$. Since f is continuous on a closed interval, f is bounded. Let $|f(x)| < M$. Then let $\widetilde{\epsilon} = \epsilon/M$ and $N = \widetilde{N}$, and we have, for all $n \geq N$, $|f(x)p_n(x) - f(x)f(x)| = |f(x)| \cdot |p_n(x) - f(x)| < M|p_n(x) - f(x)| < M\frac{\epsilon}{M} = \epsilon$ and the convergence is uniform.

Therefore, $\lim_{n\to\infty} \int_0^1 p_n(x)f(x)\ dx = 0 = \int_0^1 \lim_{n\to\infty} p_n(x)f(x)\ dx = \int_0^1 (f(x))^2\ dx$. However, $(f(x))^2 \geq 0$, so for all partitions P of $[0, 1]$, $M_i, m_i \geq 0$. If there exists $x \in [0, 1]$ with $f^2(x) > 0$, then since f^2 is a product of continuous functions, so continuous, there exists a partition P with $x \in [x_{i-1}, x_i]$ and $m_i > 0$. This implies that $\int_0^1 f^2(x)\ dx \geq L(P, f^2) \geq m_i\Delta x_i > 0$ which is a contradiction. Therefore, $f^2(x) = 0$ and $f(x) = 0$.