Math 411 Homework 6 Due Wednesday, May 4

(Due with the Final Exam)

1. Stone-Weierstrass Theorem Demonstrate that the set $\mathbb{Q}[x]$ (polynials with coefficients in $\mathbb{Q}$) satisfies the conditions for the (real) Stone-Weierstrass Theorem on $[a,b] \subset \mathbb{R}$. Optional hint: one method involves showing that the Algebra generated by the linear functions is all of $\mathbb{Q}[x]$.

2. Let $\sum a_k$ be a series that converges absolutely. Prove that $\sum a_k$ converges.

3. Rearrangements of absolutely covergent series In this problem, we will prove that if $\sum a_n$ converges absolutely and $\sum b_n$ is any rearrangement of $\sum a_n$, then $\sum b_n = \sum a_n$. Fix $\epsilon > 0$.

a) Let $\sum a_n = L$. Show that there exists N such that $\left|\sum_{n=1}^{N} a_n - L\right| < \epsilon$ and $\sum_{k=n}^{m} |a_k| < \epsilon$ for all $n, m \geq N$.

b) If s_n is the sequence of partial sums for $\sum a_n$ and t_n is the sequence of partial sums for $\sum b_n$, show there exists M such that $|t_m - s_N| < \epsilon$ for all $m \geq M$.

c) Conclude that $|t_m - L| < 2\epsilon$ for all $m \geq M$ and therefore, $\lim_{m\to\infty} t_m = L$.

4. Series of functions We did not quite get to the definition of a convergent series of functions in class, but it follows directly from the definition of a series of numbers, namely, via its sequence of partial sums. Let $\{f_n\}$ be a sequence of functions with $f_n : K \to \mathbb{R}$.

a) Weierstrass M-test Assume that for all $n \in \mathbb{N}$ there exists $M_n \in \mathbb{R}$ with $\|f_n(x)\| \leq M_n$. Prove that if the series $\sum M_n$ converges, then the series $\sum f_n$ converges uniformly on E. Hint: use the Cauchy criterion for series convergence.

5. Power Series Now that we have defined series, it is straightforward to define a power series, namely $\sum a_k x^k$ as a function $f : [a,b] \to \mathbb{R}$ which takes $x \in [a,b]$ to the real number $\sum a_k x^k$ if this real number exists. Prove that if $\sum a_k x^k$ converges for all $x \in (-R, R)$, then $\sum k a_k x^{k-1}$ also converges for all $x \in (-R, R)$. What (if anything) does this have to do with derivatives of power series?

6. Taylor Series Long ago, we defined the Taylor polynomial for a function. Now that we have a good definition of series and series convergence, we can simply take the limit of the Taylor polynomials to get the Taylor series. Prove that $f(x) = \sum a_n x^n$ implies that $a_n = \frac{f^{(n)}(0)}{n!}$ using the previous problem.