411 MIDTERM

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March 16-17, 2016 **Name:** KEY

By printing my name I pledge to uphold the honor code.

I. True/False, circle T or F as appropriate. Then explain your answer by citing specific theorems/definitions/computations/etc. as time permits.

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1. a) T **F** The infemium and the supremium of a set are always limit points.

THEY COULD BE ISOLATED POINTS

b) T **F** The continuous image of a convergent sequence converges.

NEED UNIFORM CONTINUITY FOR THIS!
(OR CONTINUITY AT THE LIMIT POINT)

c) **T** F The proof that convex implies connected requires the LUB property.

$\underset{\alpha}{\bullet}$ $\underset{\beta}{\bullet}$ $z = \text{lub}\{x \in [a,b] \mid x \in \alpha\}$ WAS AN IMPORTANT PART.

d) **T** F Most of Calculus comes down to the fact that $[a, b]$ is compact and connected. THESE GIVE US EVT & IVT & HENCE MVT & FTC.

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e) **T** F $f(x)$ monotonic implies at least one of left or right limits exist at every point in the domain. $x<y \Rightarrow f(x) \leq f(y)$ MONOTONIC INC. (WLOG)

$x_0 \in$ DOMAIN.
$f(x_0)$ IS AN UPPER BOUND FOR $\{A \mid A = f(x)\ x < x_0\}$ SO $\exists$ LUB. $\lim_{x \to x_0^-} f(x) = \text{LUB}$ $\forall \varepsilon > 0, \exists \delta > 0$ ST. $x \in (x_0 - \delta, x_0) \Rightarrow f(x) \in (\text{LUB}-\varepsilon, \text{LUB}+\varepsilon)$
LUB $- \varepsilon$ NOT LUB $\Rightarrow \exists y$ W/ LUB $- \varepsilon < f(y) \leq$ LUB $\boxed{x_0 - y = \delta}$.

f) **T** F $f'(x)$ satisfies the Intermediate Value Theorem.

IT MAY NOT BE CONTINUOUS, BUT IT DEFINITELY SATISFIES IVT.

g) T **F** $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} f(n, m) = \lim\limits_{m\to\infty} \lim\limits_{n\to\infty} f(n, m)$.

NOT IF CONVERGENCE IS POINTWISE.
(THERE WAS A HW PROBLEM W/ THIS)

h) **T** F It is immediate from the definitions that $L(P, f, \alpha) \leq U(P, f, \alpha)$ for any P, f, α.

SINCE $L(P,f,\alpha) = \sum m_i \Delta \alpha_i$ AND $U(P,f,\alpha) = \sum M_i \Delta \alpha_i$
WHERE $m_i \leq M_i$

i) T **F** It is immediate from the definitions that $\underline{\int_a^b} f(x) d\alpha \leq \overline{\int_a^b} f(x) d\alpha$.

SINCE $\underline{\int_a^b} f d\alpha$ IS <u>SUP</u> OF $L(P,f,\alpha)$'S AND $\overline{\int_a^b} f d\alpha$ IS THE <u>INF</u> OF THE $U(P,f,\alpha)$S
IT'S NOT ~~LIKE~~ JUST PREVIOUS QUESTION

II. Definitions Please define/state the following.

1. What are our two different definitions of a *continuous function* (assume $f : X \to Y$ where X and Y are metric spaces)? Give examples of a theorem that is easier to prove with one and one that is easier to prove with the other.

$f: X \to Y$ CONT IF $\forall x \in X, \forall \varepsilon > 0\ \exists \delta > 0$ st. $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.
EASIER TO PROVE SUM OF CONT. FCTNS IS CONT W/ THIS

$f: X \to Y$ CONT IF $\forall V \overset{\circ}{\subset} Y$, $f^{-1}(V) \overset{\circ}{\subset} X$. EASIER TO PROVE IVT & EVT W/ THIS.

2. A set K in a metric space X is *compact* if ... (please give the topological definition that we usually use in proofs) $\forall \{V_\alpha\}$ OPEN COVER OF K $(V_\alpha \overset{\circ}{\subset} X$ AND $K \subset \bigcup_\alpha V_\alpha)$
$\exists$ FINITE SUBCOVER $(\exists V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}$ st. $K \subset \bigcup_{i=1}^n V_{\alpha_i})$

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3. A is *connected* if what holds? What method of proof is usually helpful for proving connectedness and why is it so effective?

A IS CONNECTED IF ∄ A SEPARATION OF A. THAT IS, ∄ α, β st. $\alpha \cup \beta = A$ AND $\bar{\alpha} \cap \beta = \alpha \cap \bar{\beta} = \phi$. CONTRAPOSITIVE OR CONTRADICTION ARE IS GOOD IDEA 'CAUSE DEFINITION INVOLVES ∄ WHICH IS HARD TO WORK W/.

4. What is the idea behind uniform continuity? Name one thing that it is useful for.

$f: X \to Y$ IS UNIF. CONT IF $\forall \varepsilon > 0\ \exists \delta > 0$ st. $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.
ie ε/δ PAIR DOESN'T DEPEND WHERE IN DOMAIN X THE x,y ARE LOCATED.
UNIF. CONT. IMAGE OF A CON. SEQ. CONVERGES (IN A COMPLETE MS)
(ie UNIF. ~~IMAGE OF~~ CONT IMAGE OF A CAUCHY SEQ. IS CAUCHY) CAN EXTEND UNIF. CONT. FCNS TO THE CLOSURE OF DOMAIN

5. Please state the general form of the Extreme Value Theorem. What are some things we have proven using EVT in Math 411?

THE CONTINUOUS IMAGE OF A COMPACT SET IS COMPACT.
(THEREFORE, IF $f: [a,b] \to \mathbb{R}$, $f([a,b])$ CPT $\Rightarrow$ $\sup f([a,b])$ EXISTS AND IS IN SET)
USED IT TO PROVE MVT, TAYLOR'S TH, L'HOPITAL'S RULE. WE WILL USE IT TO PROVE FTC.

6. By defintion a function is *Riemann-Stiltjes integrable* on $[a, b]$ with respect to an increasing function α on $[a, b]$ if what holds? (Again, please define all of your terms.)

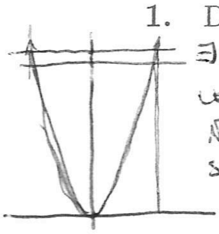
IF $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \Rightarrow f \in \mathcal{R}([a,b], \alpha)$ WHERE $\underline{\int_a^b} f\,d\alpha = \sup\{L(P,f,\alpha) \mid P$ PARTITION OF $[a,b]\}$
WHERE $L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ AND $P = \{x_0 = a < x_1 < \dots < x_n = b\}$ AND $\overline{\int_a^b} f\,d\alpha = \inf\{U(P,f,\alpha) \mid P$ PARTITION OF $[a,b]\}$
$m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$
$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$
$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$ $M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$.

7. What is our useful condition for showing a function is Riemann/Riemann-Stiltjes integrable?

$f \in \mathcal{R}(\alpha)$ IF $\forall \varepsilon > 0$, $\exists P$ PARTITION OF $[a,b]$ SUCH THAT
$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

III. Short answer Please answer the following using a sentence or two.

1. Demonstrate that $y = x^2$ isn't uniformly continuous on $\mathbb{R}$.



$\exists \varepsilon > 0$ st. $\forall \delta > 0\ \exists x,y$ w/ $|x - y| < \delta$ BUT $|f(x) - f(y)| \ge \varepsilon$.
LET $\varepsilon = 1$ THEN PAIRS $x,y \in \mathbb{R}$ w/ $|f(x) - f(y)| \ge \varepsilon$ ARE PAIRS LIKE
$\sqrt{100}, \sqrt{101}$ AND $\sqrt{1000}, \sqrt{1001}$ AND $\sqrt{1{,}000{,}000}, \sqrt{1{,}000{,}001}$
SHOW $\forall \delta > 0\ \exists N$ st. $\sqrt{N+1} - \sqrt{N} < \delta$. $\frac{(\sqrt{N+1} - \sqrt{N})(\sqrt{N+1} + \sqrt{N})}{(\sqrt{N+1} + \sqrt{N})} = \frac{1}{\sqrt{N+1} + \sqrt{N}} < \frac{1}{2\sqrt{N}}$ CHOOSE N st. $\left(\frac{1}{2\delta}\right)^2 < N$.
THEREFORE, THIS HOLDS $\forall \delta > 0$ AND $f(x) = x^2$ IS NOT UNIF. CONT ON $\mathbb{R}$.

2. Prove that every differentiable function is continuous.

ASSUME $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ EXISTS. $\forall x \in \text{Domain}(f)$, SHOW $\lim_{t \to x} (f(t) - f(x)) = 0$.

$$\lim_{t \to x} f(t) - f(x) = \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x}\right) \cdot (t - x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \lim_{t \to x} t - x = f'(x) \cdot 0 = 0$$

OK SINCE $t \neq x$ OK SINCE BOTH EXIST

3. Use your definition of a Riemann integral above to explain what a Riemann integral mean geometrically and why?

$U(P,f) = \sum M_i \Delta x_i$ WHERE $\Delta x_i = x_i - x_{i-1}$ AND $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$
SO $M_i \Delta x_i$ IS AREA OF A RECTANGLE THAT IS LARGER THAN AREA UNDER CURVE $y = f(x)$ ON $[x_{i-1}, x_i]$, SO $U(P,f)$ IS AN OVERESTIMATE OF AREA UNDER CURVE.
SIMILARLY $m_i \Delta x_i$ IS AN UNDERESTIMATE (SINCE $m_i \le f(x) \mid x \in [x_{i-1}, x_i]$) OF AREA UNDER $f(x)$ ON $[x_{i-1}, x_i]$; SO $L(P,f)$ IS AN UNDERESTIMATE (IN RECTANGLES) OF AREA UNDER $f(x)$ ON $[a,b]$.

$\overline{\int_a^b} f\,dx$ IS GLB OF OVER-ESTIMATES
$\underline{\int_a^b} f\,dx$ IS LUB OF UNDERESTIMATES
IF THEY'RE THE SAME, WE GOT ACTUAL AREA UNDER CURVE!

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4. Why does the lower Riemann-Stiltjes integral (that you defined above) exist for any bounded function f on $[a,b]$ with respect to any increasing function α?

IF f IS BOUNDED ON $[a,b] \Rightarrow$ ~~~~ FOR ALL $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ PARTITION, f IS BOUNDED ON ~~~~ $[x_{i-1}, x_i] \Rightarrow m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ EXISTS $\forall i = 1, \ldots, n$ $\Rightarrow L(P, f, \alpha)$ EXISTS $\forall P$. ALSO, SINCE f IS BOUNDED $\exists\ M = \sup\{f(x) \mid x \in [a,b]\} \Rightarrow$

$L(P,f,\alpha) = \sum m_i \Delta\alpha_i \le \sum M \Delta\alpha_i = M(\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) \cdots + \alpha(x_n) - \alpha(x_{n-1})) = M(b-a)$, SO $\{L(P,f,\alpha)\}$ IS BOUNDED ABOVE (BY $M(b-a)$) SO HAS A LUB $= \underline{\int_a^b} f\,d\alpha$.

5. Let P^* be any refinement of a partition P of $[a,b]$. How would we show that $U(P^*, f, \alpha) \le U(P, f, \alpha)$ for any f bounded on $[a,b]$ and any α increasing on $[a,b]$. Be as explicit as possible in a few sentences.

1st WE WOULD DO THIS FOR $P^* = P \cup \{c\}$ (ADDING 1 ELEMENT TO P, ANY REFINEMENT P^* CAN BE MADE ADDING ONE ELEMENT AT A TIME).

$\exists\, i$ w/ $c \in [x_{i-1}, x_i]$

THEN $U(P^*, f, \alpha) = \sum_{k=1}^{i-1} M_k \Delta\alpha_k + M_{c_1}(\alpha(c) - \alpha(x_{i-1})) + M_{c_2}(\alpha(x_i) - \alpha(c)) + \sum_{k=i+1}^{n} M_k \Delta\alpha_k \le \sum_{k=1}^{i-1} M_k \Delta\alpha_k + M_i(\cancel{\alpha(c)} - \alpha(x_{i-1})) + M_i(\alpha(x_i) - \cancel{\alpha(c)}) + \ldots = U(P,f,\alpha)$

$U(P,f,\alpha) = \sum M_k \Delta\alpha_k + M_i(\alpha(x_i) - \alpha(x_{i-1})) + \sum M_k \Delta\alpha_k$.

ALIKE WITH $M_{c_1} \le M_i$ AND $M_{c_2} \le M_i$ SO

6. Show there exist non-Riemann integrable functions by providing a specific example and demonstrating that your definition from **II.6** fails to hold.

LET $f(x) = \begin{cases} 1 & \text{IF } x \in \mathbb{Q} \\ 0 & \text{IF } x \notin \mathbb{Q} \end{cases}$ THEN ~~~~ ON $[0,1]$, $P = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$ ANY PARTITION $m_i = 0$, $M_i = 1$ $\forall\ i = 1, \ldots, n$.

SO $U(P,f) = 1\Delta x_1 + 1\Delta x_2 + \cdots + 1\Delta x_n = 1(b-a) = 1$

$L(P,f) = 0\Delta x_1 + 0\Delta x_2 + \cdots + 0\Delta x_n = 0$

SO $\underline{\int_0^1} f\,dx = 0$ $\overline{\int_0^1} f\,dx = 1$

THEY ARE UNEQUAL.

7. Let X be a set. Is $\mathcal{P}(X)$ a σ-algebra and why? Why is it a poor choice for a σ-algebra?

σ-ALGEBRA ON X MUST HAVE: ① $\phi, X \in \Sigma$ ← ϕ & X ARE BOTH SUBSETS

② IF $A \in \Sigma$, $A^c \in \Sigma$ — IF $A \subset X$, $A^c \subset X$ SO $A \in \mathcal{P}(X) \Rightarrow A^c \in \mathcal{P}(X)$

③ IF $\{A_i\} \subset \Sigma \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$ — UNION OF SUBSETS IS A SUBSET

IT IS A POOR CHOICE FOR A Σ ALG SINCE IT CONTAINS UNMEASURABLE SETS LIKE THE VITALI SETS.

III. Proofs Please prove or explain the following as appropriate.

1. Prove that the uniformly continuous image of a Cauchy seq is Cauchy.

ASSUME $\{a_n\}$ IS CAUCHY ($\forall \hat{\varepsilon} > 0\ \exists\ \hat{N}$ st. $d_X(a_n, a_m) < \hat{\varepsilon}$ $\forall n, m \ge \hat{N}$)

ASSUME $f : X \to Y$ UNIF. CONT ($\forall \hat{\varepsilon} > 0\ \exists\ \hat{\delta} > 0$ st. $d_X(x,y) < \hat{\delta} \Rightarrow d_Y(f(x), f(y)) < \hat{\varepsilon}$).

SHOW $\{f(a_n)\}$ CAUCHY (SHOW $\forall \varepsilon > 0$, $\exists N$ st. $d_Y(f(a_n), f(a_m)) < \varepsilon$ $\forall n, m \ge N$)

FIX $\varepsilon > 0$, LET $\hat{\varepsilon} = \varepsilon$. THEN $\exists\ \hat{\delta}$ st. $d_X(x,y) < \hat{\delta} \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

LET $\tilde{\varepsilon} = \hat{\delta}$, THEN $\exists\ \hat{N}$ st. $d_X(a_n, a_m) < \hat{\delta}$ $\forall n, m \ge \hat{N}$.

LET $N = \hat{N}$. THEN $\forall n, m \ge N$, $d_X(a_n, a_m) < \hat{\delta} \Rightarrow d_Y(f(a_n), f(a_m)) < \hat{\varepsilon} = \varepsilon$.

WHICH IS WHAT WE WANTED TO SHOW.

2. Let $f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b) and $c \in (a,b)$ is a local maximium. Give the technical definition of a local maximium and prove that $f'(c) = 0$.

c IS A LOCAL MAX IF $\exists\ \delta > 0$ st. $\forall x \in (c - \delta, c + \delta)$, $f(x) \le f(c)$.

DO SIDED LIMITS.

3. Let $f : [a, b] \to \mathbb{R}$ be a constant function. Prove that $f \in \mathcal{R}(\alpha)$ for all α increasing on $[a, b]$ using your definition from section **II**. 30

CAN COMPUTE $\underline{\int_a^b} f d\alpha$ AND $\overline{\int_a^b} f d\alpha$ DIRECTLY.

$m_i = M_i = c$ (WHERE $f(x) = c$)

WE SEE THAT (TELESCOPING SEQ)

$$U(P, f, \alpha) = L(P, f, \alpha) = c(\alpha(b) - \alpha(a))$$

SO $\overline{\int_a^b} f d\alpha = \inf\{U(P, f, \alpha) | P\} = c(\alpha(b) - \alpha(a)) = \sup\{L(P, f, \alpha) | P\} = \underline{\int_a^b} f d\alpha$

4. Let $f \in \mathcal{R}(\alpha)[a, b]$. Prove that your useful criterion for showing that a function is Riemann-Stiltjes integrable from section **II** holds.

ASSUME $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$, SHOW $\forall \varepsilon > 0$ $\exists P$ st. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

SINCE $\overline{\int_a^b} f d\alpha = \inf\{U(P, f, \alpha) | P$ PARTITION OF $[a, b]\}$ (BY DEF OF inf)

$\forall \varepsilon > 0$, $\exists P_1$ PARTITION st. $\int_a^b f d\alpha + \frac{\varepsilon}{2} > U(P_1, f, \alpha) \geq \overline{\int_a^b} f d\alpha$.

SIMILARLY $\exists P_2$ st. $\underline{\int_a^b} f d\alpha - \frac{\varepsilon}{2} < L(P_2, f, \alpha) \leq \underline{\int_a^b} f d\alpha$.

P = MUTUAL REFINEMENT OF P_1 & P_2.

5. Please construct an un-measurable set and demonstrate why is it unmeasurable.

VITALI SETS, IT'S ON SOLUTIONS TO HW 4

SET UP EQUIV. RELATION

$\bar{a} = \{b | b \sim a\}$ PARTITION $\mathbb{R}$.

CREATE V USING AXIOM OF CHOICE st. V CONTAINS EXACTLY ONE ELEMENT FROM EACH OF THE $\bar{a} \cap [0, 1]$'s.