# 411 REAL ANALYSIS II WEEKLY QUIZ 5

March 30-31, 2016

**Name:** KEY

By printing my name I pledge to uphold the honor code.

No books, notes, internet, etc. This should take about 50 minutes.

REMBER TO SIGN IN!

1. Measure theory

a) What are the three properties that define a $\sigma$-algebra and what is the purpose of a $\sigma$-algebra? Please give 4 examples of $\sigma$-algebras.

$\Sigma$ $\sigma$-ALGEBRA, THEN $\Sigma \subset \mathcal{P}(X)$

① $\phi, X \in \Sigma$

② IF $A \in \Sigma \Rightarrow A^c \in \Sigma$

③ IF $\{A_n\} \subset \Sigma \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Sigma$

4 EXAMPLES

$\mathcal{B}$ BOREL $\sigma$-ALG

$\mathcal{L}$ LEBESGUE $\sigma$-ALG.

$\mathcal{P}(X)$ DISCRETE $\sigma$-ALG.

$\phi, X$ TRIVIAL $\sigma$-ALG.

$\Sigma$ ALGEBRAS ARE THERE TO PROVIDE A NOTION OF A MEASURABLE SET. IE $\mu$ MEASURE $\mu : \Sigma \to \mathbb{R}^{\geq 0} \cup \{\infty\}$

b) Please define (on $\mathbb{R}^1$) the Lebesgue outer measure $\lambda^* : ? \to \mathbb{R}^{\geq 0} \cup \{\infty\}$. What is "?"

$$\lambda^* : \mathcal{P}(X) \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

IE IT ~~AS~~ IS DEFINED ON <u>ANY</u> SUBSET.

$$\lambda^*(A) = \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) \,\middle|\, A \subset \bigcup_{n=1}^{\infty} (a_n, b_n) \right\}$$

OPEN INTERVAL IN $\mathbb{R}^1$

c) How do we go from the Lebesgue outer measure to the Lebesgue measure on $\mathbb{R}^1$. Please be as specific as possible.

WE RESTRICT OURSELVES TO $\mathcal{L}$ LEBESGUE $\sigma$-ALG.

WHERE $\mathcal{L} = \{ E \in \mathcal{P}(X) \mid \forall A \in \mathcal{P}(X)\ \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c) \}$

ON $\mathcal{L}$ THE LEBESGUE OUTER MEASURE IS AN ACTUAL MEASURE (IE IT IS COUNTABLY ADDITIVE RATHER THAN JUST COUNTABLE SUB-ADDITIVE LIKE IT IS ON $\mathcal{P}(X)$)

2. Riemann-Lebesgue theorem

a) Please state the Riemann-Lebesgue theorem.

$f : [a,b] \longrightarrow \mathbb{R}$ st. $f$ IS CONTINUOUS ON $[a,b] - A$ AND

$\lambda(A) = 0 \Longleftrightarrow f \in \mathcal{R}[a,b]$.

(THIS IS IF AND ONLY IF)

b) What are the main ideas in its proof?

WE RESTRICT OURSELVES TO ONLY THE REALLY BAD DISCONTINUITIES. THAT SET IS COMPACT, ~~SO HAS A TINY AND~~ MEASURE ZERO SO HAS A TINY OPEN COVER. AWAY FROM THAT OPEN COVER, THE DIFFERENCE BETWEEN $M_i$ AND $m_i$ IS RELATIVELY SMALL.

3. FTC

a) Please state both parts of the Fundamental Theorem of Calculus.

FTC 1 $\quad f\in\mathcal{R}[a,b] \Rightarrow \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

FTC 2 $\quad$ IF $F'(x)=f(x)$, $f\in\mathcal{R}[a,b]$ THEN

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

b) What are the ideas behind the proofs of your theorems from a)?

PROOF OF 1 $\quad$ USE $\frac{d}{dx}\int_a$ DEF OF DERIV. $\lim_{y\to x}\frac{\int_a^x f(t)\,dt-\int_a^y f(t)\,dt}{x-y} = \lim_{y\to x}\frac{\int_y^x f(t)\,dt}{x-y}$.

ESTIMATE SIZE OF THIS, IT'S ~~f(x)~~ BETWEEN $M(x-y)$ AND $m(x-y) \to f(x)$

PROOF OF 2

USE MVT ON EACH SUB-INTERVAL IN PARTITION $P$ w/ $U(P,f)-L(P,f)<\varepsilon$

USE ESTIMATE $\left|\int_a^b f\,dx - \sum f(t_i)\Delta x_i\right| < \varepsilon$.

TELESCOPING SERIES.

4. Sequences of functions

a) Define two different notions of convergence of sequences of functions.

POINTWISE CONVERGENCE: $f_n: X\to Y$ $\quad f: X\to Y$

$f_n\to f$ POINTWISE IF $\forall x\in X$ IT HOLDS THAT $\forall\varepsilon>0\ \exists N$ st. $d_Y(f_n(x),f(x))<\varepsilon\ \forall n\geq N$ (EACH POINT IN X PRODUCES A CAUCHY SEQ IN Y)

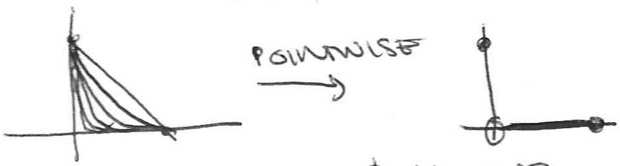
UNIFORM CONV.

$f_n\to f$ UNIFORMLY IF $\forall\varepsilon>0, \exists N$ st. $d_Y(f_n(x),f(x))<\varepsilon\ \forall n\geq N, \forall x\in$

b) Illustrate using words and picture which of the two notions from part a) is preferred and why. Namely, give an example where the less good notion goes wrong and illustrate why the better notion has no such issues.

POINTWISE CONVERGENCE

$(1-x^2)^n$ ON $[0,1]$.




SINCE AS $x\to 0^+$, $N\to\infty$.

UNIFORM CONV. INVOLVES

$f_n(x)\in(f(x)-\varepsilon, f(x)+\varepsilon)\ \forall x\in$

ie