411 REAL ANALYSIS II WEEKLY QUIZ 6

April 8, 2016 **Name:** ______________________

By printing my name I pledge to uphold the honor code.

No books, notes, internet, etc. This should take about 30 minutes.

REMBER TO SIGN IN!

1. Pointwise vs Uniform Convergence. Let $f_n : X \to Y$ for $n \in \mathbb{N}$ and Y a complete metric space.

a) Please define pointwise and uniform convergence for the sequence $\{f_n\}_{n=1}^{\infty}$.

$f_n \to f$ POINTWISE IF $\forall x \in X$ IT HOLDS THAT $\forall \varepsilon > 0\ \exists N$ st. $d_Y(f_n(x), f(x)) < \varepsilon$ $\forall n \geq N$.

$f_n \to f$ UNIFORMLY IF $\forall \varepsilon > 0\ \exists N$ st. $d_Y(f_n(x), f(x)) < \varepsilon$ $\forall n \geq N, \forall x \in X$.

(ALSO EQUIVALENT TO CAUCHY CONDITION)

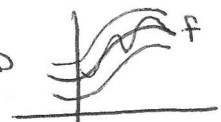
b) What is properties of $f_n : X \to Y$ if Y is a complete metric space are preserved under pointwise continuity? (Yes, there is at least one!!) What statement would we need to prove to show that uniform convergence preserves continuity? Approximately why should this hold?

IF $f_n \to f$ POINTWISE, THEN $f : X \to Y$ IS A WELL DEFINED FUNCTION!
$f_n : X \to Y$ FUNCTIONS

TO SHOW UNIF. CONV. PRESERVES CONTINUITY, YOU NEED TO SHOW

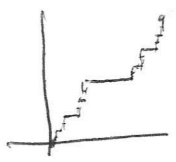
$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

SHOULD HOLD 'CAUSE RIBBON CONDITION COMBINED W/ CONTINUITY OF f_n'S.



c) Please outline at least one application of part b) to measure theory.

WE CAN USE UNIFORM CONVERGENCE TO SHOW THE CANTOR FUNCTION IS CONTINUOUS AND USE CANTOR FUNCTION COMBINED W/ VITALI SET TO FIND A DEFINITELY LEBESGUE MEASURABLE PROBABLY NOT BOREL MEAS. SET. (INVERSE IMAGE OF $\{\text{VITALI} - P/_6\}$ UNDER CANTOR FUNCTION.



2. $C_Y(X)$

a) What is $C_Y(X)$? What are our main examples of Y and why?

$C_Y(X)$ IS THE SET OF CONTINUOUS BOUNDED FUNCTIONS $f : X \to Y$

BOTH X & Y MUST BE METRIC SPACES & TO DEFINE A METRIC ON $C_Y(X)$ Y MUST BE A NORMED SPACE.

SO EX ARE $Y = \mathbb{R}^n$, $Y = \mathbb{C}^n$, $Y = \mathbb{Q}$ UNDER p-ADIC NORM.

b) Please define the sup norm on $\mathcal{C}_Y(X)$ where Y is a normed space. What is the induced metric on the space $\mathcal{C}_Y(X)$?

$\|f\| \stackrel{def}{=} \sup\{|f(x)| \mid x \in X\}$

NORM ON Y

METRIC ON

$\mathcal{C}_Y(X)$ IS

$d_{\mathcal{C}_Y(X)}(f,g) = \|f-g\|$

c) What does it mean for a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}_Y(X)$ to be Cauchy? Why?

IF $\forall \varepsilon > 0, \exists N$ st. $\|f_n - f_m\| < \varepsilon \quad \forall n,m \geq N$

OR $\sup\{|f_n(x) - f_m(x)| \mid x \in X\} < \varepsilon$.

$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon' < \varepsilon \quad \forall x \in X$.

SO $\forall \varepsilon > 0, \exists \varepsilon' < \varepsilon$ st. $|f_n(x) - f_m(x)|_Y < \varepsilon' \quad \forall x \in X$.

THIS IS EQUIVALENT TO $f_n \to f$ UNIF. IF Y IS COMPLETE.

d) How would we go about showing that $\mathcal{C}_{\mathbb{R}}(X)$ is a complete metric space?

SINCE $\mathbb{R}$ IS COMPLETE

CAUCHY SEQ IN $\mathcal{C}_{\mathbb{R}}(X) \Rightarrow \forall \varepsilon > 0 \quad |f_n(x) - f_m(x)| < \varepsilon$

$\Rightarrow \forall x \quad \{f_n(x)\}$ CAUCHY SEQ. IN $\mathbb{R}$ COMPLETE, SO

CONV. CALL THAT LIMIT $f(x)$.

SHOW THE POINTWISE LIMIT IS THE

UNIFORM LIMIT $\Rightarrow \{f_n\} \to f$ IN $\mathcal{C}_{\mathbb{R}}(X)$

e) What does it mean for a subset $F \subset \mathcal{C}_Y(X)$ to be pointwise or uniformly bounded? What was the purpose of those two definitions?

$\mathcal{F}$ IS POINTWISE BOUNDED IF $\exists \varphi : X \to \mathbb{R}$ st.

$|f(x)|_Y < \varphi(x) \quad \forall x \in X$.

$\mathcal{F}$ IS UNIF. BOUNDED IF $\exists M \in \mathbb{R}$ st.

$|f(x)|_Y < M \quad \forall x \in X$.