485 REPRESENTATION THEORY BIG QUIZ

March 1, 2017 **Name:** KEY

By printing my name I pledge to uphold the honor code.

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Unless otherwise stated, assume that G is a group, X is a set with a G action, F is a field, and V is a n-dimensional vector space over F with a G action.

I. True/False, circle T or F as appropriate. After you have finished the rest of the quiz, please explain your answer by citing specific theorems/definitions/computations/etc.

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1. a) T (F) The action of G on a set X is faithful if and only if all of the stabilizers are the identity subgroup.
FAITHFUL ⟺ ∩(STABILIZERS) = {1}

b) (T) F The size of an orbit always divides the size of the group.
SIZE OF ORBIT IS $\frac{|G|}{|\text{STABILIZER}|}$ & STABILIZER IS A SUBGP

c) T (F) GL_nF is a vector space over F.
NOT CLOSED UNDER VECTOR ADDITION — THOUGH M_nF IS A VS.

d) T (F) There always exists a positive definite inner product on V.
F MUST BE ℝ OR HAVE A NORM ON IT, OTHERWISE THERE'S NO WAY TO SAY >0.

e) (T) F Given a representation $\rho : G \to GL(V)$ then $\rho(g)$ is uniquely determined, but R_g is not. R_g DEPENDS ON CHOICE OF BASIS, BUT $\rho(g) \in GL(V)$ IS A LIN. TRANSF.

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f) (T) F The permutation representation isn't technically a representation.
IT'S AN ACTION OF G ON ITSELF, & G IS NOT ~~[illegible]~~ A V.S.

g) T (F) SL_2 is a compact continuous group.
IT'S NOT COMPACT (THOUGH IT IS CONTINUOUS)
EX: $\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \in SL_2\ \forall \alpha$, SO ISN'T IN ANY CLOSED RECTANGLE (CLOSED ≠ NOT BOUNDED)

h) T (F) To produce a Haar measure for $GL_2\mathbb{R}$, we need merely divide the ordinary measure on $\mathbb{R}^4$ by the determinate.
IT'S BY $\det^2$ (FROM HW)

i) (T) F Every finite subgroup of $SL_2\mathbb{R}$ is cyclic.
DEFINE A G-INV. INNER PRODUCT. THIS GIVES AN O.N. BASIS & WRT THAT BASIS, $R_g \in SO(2) \cong S^1$

j) T (F) Taking the trace of a matrix is really stupid thing to do because out of n^2 entries of a matrix, it only depends on n of them.
TRACE IS INDEPENDENT OF CHOICE OF BASIS!

II. Definitions/Fill in the blank Please define/state the following.

1. A form is *positive definite* if what holds? Why do we care about positive definite forms? $\forall \vec{v} \in V,\ \vec{v} \neq \vec{0}$, $\langle \vec{v}, \vec{v} \rangle > 0$. POSITIVE DEFINITE ⟺ ∃ O.N. BASIS!

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2. A *representation* of G on V is defined to be what? Please give the diagram that expresses the representation both as linear transformations and as matrices (R_gs and ρ_gs). ACTION OF G ON THE VS V COMPATIBLE W/ VECTOR ADDITION & SCALAR MULT. ⇒ MAP $\rho : G \to GL(V)$
ANY CHOICE OF BASIS FOR V GIVES $GL(V) \cong GL_nF$
R_g IS IMAGE OF $\rho(g)$ UNDER THIS ≅.

$g \mapsto R_g$, $GL(V) \cong GL_nF$

$(1,1)=0$

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3. What is a *Haar measure*? Why do we care about them?

IT'S A TRANSLATION INVARIANT MEASURE. IF $S \subset X$ IS A SUB-SET THEN $\mu(S) = \mu(a+S)$ $\forall a$. IT'S WHAT WE USE FOR INFINITE (CPT) GROUPS INSTEAD OF $\sum_{g \in G}$ IT'S $\int_G d\mu$

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4. Huge new theorem

a) Define the *character* χ of a representation.

$\chi: G \to F$ FOR $\rho: G \to GL(V)$ $\chi(g) = tr(\rho_g)$

b) A *class function* from G to F is what?

IT'S A FUNCTION THAT IS CONSTANT ON CONJUGACY CLASSES.

c) How are representations related to class functions? THE CHARACTER OF A REP. IS A CLASS FCTN.

d) Define our hermitian form on the set of class functions. Is it positive definite? ← YES!!

$\varphi, \psi \in \mathcal{C}$, $\langle \varphi, \psi \rangle = \left(\sum_{g \in G} \overline{\varphi(g)} \psi(g)\right) \frac{1}{N}$

e) Please state our new huge theorem.

THE SET OF CHARACTERS OF ~~[illegible]~~ (ISOM CLASSES OF) IRRED REPS FORM AN O.N. BASIS FOR $\mathcal{C}$!

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5. Basis change preserves which of the following things? (Please circle the things that are preserved. If something is not preserved, write how change of basis effects the quantity in question)

PAP^{-1}

a matrix, a linear transformation, the eigenvalues of a matrix, the determinate of a matrix, the trace of a matrix, a Hermitian form $\langle\ ,\ \rangle$, the hermitian form $\langle\ ,\ \rangle_A$

$\langle\ ,\ \rangle_{PAP^{-1}}$

III. Short answer

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1. Give four different ways that a finite group G can act on itself (as a set). Which of your actions are faithful?

G CAN ACT TRIVIALLY WHERE $g, h \to h$ (NOT FAITHFUL)

G CAN ACT BY LEFT MULT, RIGHT MULT, CONJUGATION } ALL FAITHFUL AND ACT TRIVIALLY BY $g, h \to 1$ (ALSO NOT FAITHFUL)

(CONJUGATION) ← NOT NECESSARILY!! (SAY IF G IS ABELIAN)

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2. Extend the action of Σ_3 on an equilateral triangle to a representation of Σ_3 on $\mathbb{R}^2$ (hint: your matrices will be a lot nicer if you choose a basis whose angles are $2\pi/3$ angle apart):

$r \to \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $r^2 \to \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$\sigma \to \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ $d_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $d_2 \to \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$

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3. Go through the process of constructing a positive definite inner product for your representation of Σ_3 above and use this to get a unitary representation.

$\langle \vec{v}, \vec{w} \rangle = \frac{1}{6} \sum_{g \in G} \langle g\vec{v}, g\vec{w} \rangle = \frac{1}{6} \sum_{g \in G} \vec{v}^t g^t g \vec{w} = \vec{v}^t \left(\frac{1}{6} \sum_{g \in G} g^t g\right) \vec{w}$

MATRIX IS $\frac{1}{6}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\right) = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix}$ SO G-INV. FORM IS $\langle \vec{v}, \vec{w} \rangle = \vec{v}^t \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix} \vec{w}$

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4. Let $F = \mathbb{R}$ or $\mathbb{C}$. Why does the existence of a G-invariant subspace in V give us a pair of lower dimensional representations?

IF W IS G-INVARIANT, THEN THE FACT THAT ~~DOT PRODUCT~~ HERMITIAN PRODUCT IS POSITIVE DEFINITE TELLS US THAT $V = W \oplus W^\perp$ WHERE $W^\perp = \{\vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0\ \forall \vec{w} \in W\}$ AND $W^\perp$ IS ALSO G-INVARIANT, SO ∃ CHOICE OF BASIS FOR V FOR WHICH $R_g = \begin{pmatrix} R_g' & 0 \\ 0 & R_g'' \end{pmatrix}$ WHERE $\{R_g''\}$ AND $\{R_g'\}$ ARE REPS OF G ON $W^\perp$ & W.

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5. How many one dimensional representations of Σ_3 are there? Please construct them!

~~GENER~~ $\Sigma_3 = \langle r, \sigma \rangle$, SO GENERALLY, TO GET A REAL 1-dim REP WE LOOK AT GENERATORS $\to \pm 1$, SO THERE COULD BE UP TO 4 OF THEM. HOWEVER $r^3 = 1$, SO $r \to -1$ ISN'T A REP, SO WE HAVE TRIV. REP & $\sigma \to -1$, $r \to 1$

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6. Demonstrate explicitly the orthonormal basis portion of our huge new theorem for the representations of Σ_3 you have constructed above.

LET ρ_1 = TRIV. REP, ρ_2 = NON-TRIV. 1dim REP (SIGN REP) ρ_3 = 2dim REP. (WHICH IS IT 'CAUSE Σ_3 HAS 3 CONJ. CLASSES) $\langle \chi_1, \chi_1 \rangle = \frac{1}{6} \sum_{g \in G} \chi_1(g)^2 = \frac{1}{6} 6 = 1$ AS ρ_2 ARE THEIR OWN CHAR., BUT $\langle \chi_2, \chi_2 \rangle = \frac{1}{6} \sum \chi_2(g)^2 = 1$

χ_3: $1 \to 2$, $r^2 \to -1$, $d_1 \to 0$ $\langle \chi_3, \chi_3 \rangle = \frac{1}{6} \sum \chi_3(g)^2 = \frac{1}{6}(4+1+1) = 1$

$\langle \chi_1, \chi_2 \rangle = \frac{1}{6}(1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot -1 + 1 \cdot -1 + 1 \cdot -1) = 0$

$\langle \chi_1, \chi_3 \rangle = \frac{1}{6}(1 \cdot 2 + 1 \cdot -1 + 1 \cdot -1 + 0 + 0 + 0) = 0$

$\langle \chi_2, \chi_3 \rangle = \frac{1}{6}(1 \cdot 2 + 1 \cdot -1 + 1 \cdot -1 + 0 + 0 + 0) = 0$ YAY!