485 REPRESENTATION THEORY WEEKLY QUIZ 5

March 20, 2017 **Name:** KEY

By printing my name I pledge to uphold the honor code.

No books, notes, internet, etc. This should take about a half hour. 13½

REMBER TO SIGN IN!

Unless otherwise specified, assume that G is a group of order N and $\rho : G \to GL(V)$ is a representation of G and χ is its character.

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1. Definitions. Please state the following:

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a) The character χ of the representation ρ $\chi : G \to F$ DEFINED BY $\chi(g) = tr(R_g)$ — WELL DEFINED AS TRACE IS INDEP OF CHOICE OF BASIS.

b) $\mathcal{C}$ the set of class functions on G

$\{f : G \to F \mid f(g) = f(hgh^{-1})\ \forall h \in G\}$ (CONSTANT ON CONJUGACY CLASSES)

c) The Hermitian form on $\mathcal{C}$ (what does the field F need to be for this to make sense?) $F = \mathbb{C}$ OR $\mathbb{R}$ (OR CPLX CONJUGATION DNE).

$$\langle \varphi, \psi \rangle = \frac{1}{N} \sum_{g \in G} \overline{\varphi(g)}\, \psi(g)$$

d) What are two ways of thinking of $V(G)$? V(G) IS A VECTOR SPACE THAT HAS ELEMENTS OF G FOR A BASIS. IT CAN ALSO BE THOUGHT OF AS $\mathbb{C}[G]$ POLYS W/ COEFF. IN $\mathbb{C}$ (NOTE THAT ALL SUCH POLY ARE OF THE FORM $\sum a_i g_i$ SINCE $g_i^2 = g_j$ SOME j !!!)

e) The regular representation of G IS GIVEN BY ACTION OF G ON V(G) BY LEFT MULTIPLICATION. THIS PERMUTES THE BASIS ELEMENTS!

f) $Z(G)$ (the center of a group) THE SET OF ELEMENTS THAT COMMUTE W/ EVERYTHING. THIS IS A SUBGP!

g) $T : V \to V'$ is G-invariant linear transformation if ...

$\forall g \in G\ \forall \vec{v} \in V,\ T(g\vec{v}) = gT(\vec{v})$ (TECHNICALLY, THIS DEPENDS ON A REP $\rho : G \to GL(V)$ & $\rho' : G \to GL(V')$ & SHOULD READ $T(\rho(g)\vec{v}) = \rho'(g)T(\vec{v})$)

h) Write out what it would mean for a linear transformation to be G invariant in terms of matrices.

IN TERMS OF MATRICES, LET β BASIS FOR V, β' BASIS FOR V' AND T = MULT. BY A. THIS SAYS $AR_g = R'_g A\ \forall g \in G$.

2. Theorems. Please state the following:

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a) What is special about characters of 1 dimensional representations?

FOR A 1-dim REP, $tr(A) = A \in F$, SO $\chi(\rho) = \rho$ AND THEREFORE χ IS A GROUP HOMOM $G \to F$.

1

b) Please state our Huge New Theorem

IF G IS A GROUP OF ORDER N W/ k CONJUGACY CLASSES, THEN THERE ARE k ISOM CLASSES OF IRRED REPS & THEIR CHARACTERS FORM AN ORTHONORMAL BASIS FOR $\mathcal{C}$. (TECHNICALLY, k IRRED REPS COMES FOR FREE W/ O.N. BASIS & SINCE $\mathcal{C} \cong F^k$. (1 ELEMENT OF F FOR EACH CONJ. CLASS)

c) As a consequence of our Huge New Theorem, what special property does the regular representation have?

THE REGULAR REP CONTAINS ~~EACH IRRED REP~~ $\dim \rho_i$ ~~[illegible]~~ COPIES OF ~~EACH~~ IRRED REP ρ_i. 8

d) What is special about characters of Abelian groups?

SINCE $k=N$ FOR AN AB. GP AND $N = \dim \rho_{reg}$, ALL IRRED REPS FOR AN AB. GP ARE 1 dim. $\Rightarrow$ CHARACTERS ARE GP. HOMOMS.

e) Please state Schur's lemma

IF ρ_i, ρ_j ARE IRRED REPS AND $T: V \to V'$ A "G-INVARIANT" LIN. TRANSF (WE NOW CALL IT A "LIN. TRANSF THAT ~~[illegible]~~ IS COMPATIBLE W/ G-ACTIONS"), THEN EITHER $T = \vec{0}$ OR T IS AN ISOM. & IF $T: V \to V$ ~~[illegible]~~ ITS MULT. BY A SCALAR.

3. Explanations/computations

a) Why is the character of a representation well defined?

TRACE IS INDEP. OF CHOICE OF BASIS. FOR $A \in GL_n F$, $tr(A) = \sum$ EIGENVALUES OF A & EIGENVALUES DO NOT DEPEND ON CHOICE OF BASIS.

b) Every character of a representation is a class function because ....

$tr(AB) = tr(BA)$ SO $tr(ABA^{-1}) = tr(A^{-1}AB) = tr(B)$ SO trace IS CONSTANT ON ~~[illegible]~~ CONJUGACY CLASSES.

c) $\chi(1) = \dim \rho$ because $1 \longrightarrow I : V \longrightarrow V$ (IDENTITY MATRIX) AND $I = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}_{\dim V}$ so $tr\, I = n = \dim \rho = \dim V$.

d) Why might we care about $Z(G)$ in light of our Huge New Theorem?

EVERY ELEMENT OF $Z(G)$ IS IN ITS OWN CONJUGACY CLASS, SO FOR EACH $g \in Z(G)$ $\chi(g)$ NEEDS TO BE COMPUTED INDIVIDUALLY.

e) Give two separate reasons that all irreducible representations of Abelian groups are one dimensional.

I GAVE 1 REASON IN 2d ABOVE. ANOTHER IS THAT IF $\rho: G \to GL(V)$ IS AN IRRED REP OF AN AB. GP G, $g \in G$ W/ EIGENVALUE λ, THEN V_λ EIGENSPACE IS G-INVARIANT SO MUST BE ALL OF V. THIS APPLIES TO ALL $g \in G$, SO $g = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}$ SO UNLESS $\dim V = 1$, THIS ISN'T IRRED!

f) Any linear transformation $T: V \to V'$ can be used to create a G-invariant linear transformation $\widetilde{T} = \frac{1}{N}\sum_{g \in G} R'_{g^{-1}} A R_g = \frac{1}{N}\sum \rho'_{g^{-1}}(T(\rho_g))$ because... AS IN 1h ABOVE, $A = R'_{g^{-1}} A R_g$ IS CONDITION FOR G-INVARIANCE, AND IF $h \in G$, SHOW ~~[illegible]~~ $\widetilde{T}(h\vec{v}) = h\widetilde{T}(\vec{v})$ AKA $h^{-1}\widetilde{T}(h(\vec{v})) = \widetilde{T}$ BY ABSORBING ρ_h's & ρ'_h's & USING ρ IS A GP. HOMOM!!!

REMEMBER TO PUT YOUR COMPLETED QUIZ IN THE CORRECT ENVELOPE AND SIGN OUT!