Physical properties of hyperbolic space in relation to the history of clothing and armor

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ABSTRACT

All of the ‘problem areas’ in human anatomy, the ones that make fitting clothes to humans a challenge, are hyperbolic in nature. Hence, why the structure of hyperbolic space is a key to the history of clothing and armor. This paper outlines which aspects of hyperbolic space are most relevant, and demonstrates a variety of historical and modern solutions to the challenge of producing hyperbolic space out of flat material. Solutions discussed include :drapery, flexible materials (knits and co-knits), piecing, and multiple versions of moulding. Kinetic as well as static models are considered. Embedded in the paper are seven optional experiments, to aid the reader in visualizing some of the issues encountered while trying to solve the clothing problem.

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Hyperbolic space, saddle surface, clothes, historical clothes, armor

1. Introduction

This paper grew out of a presentation I gave at the 2010 JMM in San Francisco in the Mathematics and Art MAA Special Session. However, the real impetus of the project was Halloween of 2006, when I tried to make Hoplite armor. The particular armor I wanted to make is called a ‘bell corslet’: a two piece bronze breastplate with an outward slanted bottom decorated with lions or dragons or fake pectoral muscles. I did manage to locate sheet bronze (from a plumbing supply company) and a pointy (welding) hammer for the details, but that took most of the week, so Halloween itself found me in my backyard with a $4' \times 10'$ foot sheet of bronze, various hammers, metal shears, a MAP gas torch, and a small pile of rocks. For anyone wishing to make the experiment, anvils exist for exactly this reason.

During my long and not very productive afternoon of hitting a piece of sheet metal repeatedly with a rubber mallet, I came to realize that a good portion of the difficulty I was having is that the breastbone (and many other places on the human body) are saddle surfaces[1]. (This, I would speculate, is one reason high fashion models are more boyish looking than other glamorized female figures. They are less hyperbolic, and so clothing can be fitted to them more rapidly.) And the only way to form a saddle surface out of metal is to stretch it!

This insight proves generalizable across the history of costume: flat materials cannot be crafted to hug the human body's saddle surfaces, unless that material is transformed in such a way as to alter its surface area. This paper pursues that insight, analyzing costume to show how human hands and minds have crafted an array of practical solutions to what is, at its heart, a geometrical problem.

This paper also contains a series of optional experiments. To do all of the experiments you will need
0. Paper, scissors, tape.
1. 1 moulded potato chip (Pringle or its generic counterpart).
2. 1 vase, wider at the top and bottom than in the middle (see figure 6).
3. Various knit fabrics (t-shirts, underwear, socks).
4. Woven fabric.
5. Two pins (safety pin preferable).
6. One small piece of flexible sheet metal, such as aluminum flashing. The sheet be damaged in the experiment.
7. 3-6 Cadbury brand Mini-Eggs (solid chocolate egg shaped Easter candy poured in one piece rather than two). These too will be damaged in the experiment, but may still be eaten.

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2. The problem with people...

Is we are too hyperbolic. This is why fitting clothing to even a standing human figure, ignoring the requirement of motion (which no-one is suggesting we should do, but first things first), is difficult. Even clothing for a mummy would be difficult if it weren't produced by wrapping a sufficiently thin ribbon around the form over and over (something that will work to cover *any* static form but fails rather miserably when motion is required - as illustrated by the embarrassed mummy). A waist, which mummies don't really have, is one of the chief sources of hyperbolically, at least in western eyes. This doesn't mean that a round waistless person (think Homer Simpson) lacks hyperbolic places, it just means that the places that the person is especially hyperbolic have been moved downward.

2.1. Hyperbolic/saddle surfaces

A surface is *hyperbolic* if it is bent up in one direction and down in another, like a saddle for a horse which is bent up in front of and behind the rider, and down on the sides to accommodate the rider's legs. Another classic example of a hyperbolic surface is a potato chip, especially Pringle's brand.[2] Because these are two of the best

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[1] They are called saddle surfaces because, like a saddle, they curve in two opposite directions that meet at right angles.

[2] A Pringle is, of course, made in a mold rather than by frying a thin slice of a potato, however, the *shape* of a Pringle is an idealization of the shape that fried slices of potatoes tend to produce which is due to the internal structure of the starch molecules in a potato (R.Field unpublished note on potatoes and multi-variable

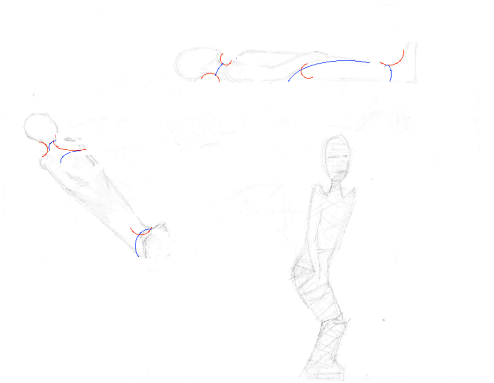

Figure 1. Hyperbolic space on mummies

examples of a hyperbolic surface, they are often called either saddle surfaces or potato chips. Being bent up in one direction, and down in another, causes a saddle surface to have cross sections of three types: upward facing (like a U, which we would get from slicing an actual saddle front to back), downward facing (an upside down U, which we would get by slicing a saddle side to side); both of these are outlined in Figure 2 [left]; and linear slices that lie in between: a simple saddle surface is actually a *ruled* surface and can be made out of strings, as in Figure 2 [right]. If you happen to have a Pringle and a pencil handy, these lines can be found experimentally. In fact, all hyperbolic surfaces contain too much area to lie flat in a plane, and hence must 'pop up' into three dimensional space.

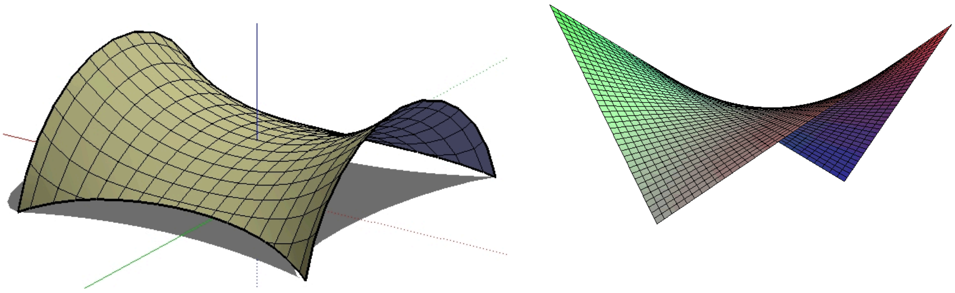

Figure 2. two views of a saddle surface

A saddle surface is what we would get if we (Experiment 0:) cut a circle out of paper, slit the circle and added in an extra pie piece (Figure 3). Be sure to tape down both sides so that there are no sharp angles except at the center. You'll find that the result can't all be flattened at once and will always have a part sticking up due to the material we added.

The too much area to lie flat is also visible if we were to attempt to (Experiment 1:) flatten a Pringle (break it into small flat-ish pieces) and try to put the pieces back together like a puzzle. You'll find that the only way to get the result to lie flat is to have the pieces overlap. Any small portion can be flattened, and any two adjacent

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calculus).

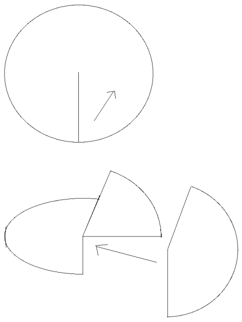

Figure 3. Experiment 0

pieces may fit together with a somewhat flat result, but there is just too much surface area for the whole object to be constructible as anything other than what it started out as, a surface in three dimensions.

The concept of ‘too much area to be flat’ is going to be a recurring theme for this paper! In fact, ‘curvature in two different directions’, along with ‘too much area to lie flat’, are our two identifying characteristics for a hyperbolic or potato chip surface.

More generally, a hyperbolic surface is one built out of potato chip shapes. A potato chip itself is built out of potato chip shapes: even our small pieces from Experiment 1 were ever-so-slightly curved in a potato-chip-y way; but an even better example of a hyperbolic surface is a human. Even idealized, we are horribly hyperbolic (neck, armpits, ankles, groin) and not coincidently, regions of hyperbolicity are the hardest parts of people to fit clothing to.

2.2. The Human Sides of Clothing and Armor

In addition to our annoying hyperbolicity (annoying from the point of view of clothing manufacture), we also move, which makes the situation worse. Because the complexity of the situation that includes movement is enough to fill several books, we have chosen to mostly consider the ‘local’ solution to the problem, namely, how to fit material to a single saddle surface. Hence a good way to think of the main bulk of this paper is: *How to make potato chips out of various materials.*

There are, of course, issues with clothing other than the potato-chip problem, even without considering movement, and one of the most crucial is structural integrity of the static form. Otherwise known as how a garment stays on when standing still. With many modern solutions for the clothing problem, structural integrity comes, more or less, for free with the complexity of the solution (t-shirts, for example, don’t spontaneously fall off). However, for some garments, where the garment hangs, how it holds up its own weight, etc. are crucial to understanding how and why it is a functional garment.

As for the complicated issue of mobility of the garment, we will see that different solutions to the potato-chip problem come with different mobility issues.

2.3. Anatomy of a potato chip

The difficulty of making a potato chip shape varies widely by material, so we will start with an abstract space (namely, a flat square piece of theoretical material which has no thickness and is flexible).

How to make a potato chip starting with a square piece of a flat plane:
pull the square up on two of the opposite sides (keeping the center where it is) and push down on the remaining two sides. This will produce a three dimensional object (the original square piece of a flat plane and four different views of the final object are in Figure 4)

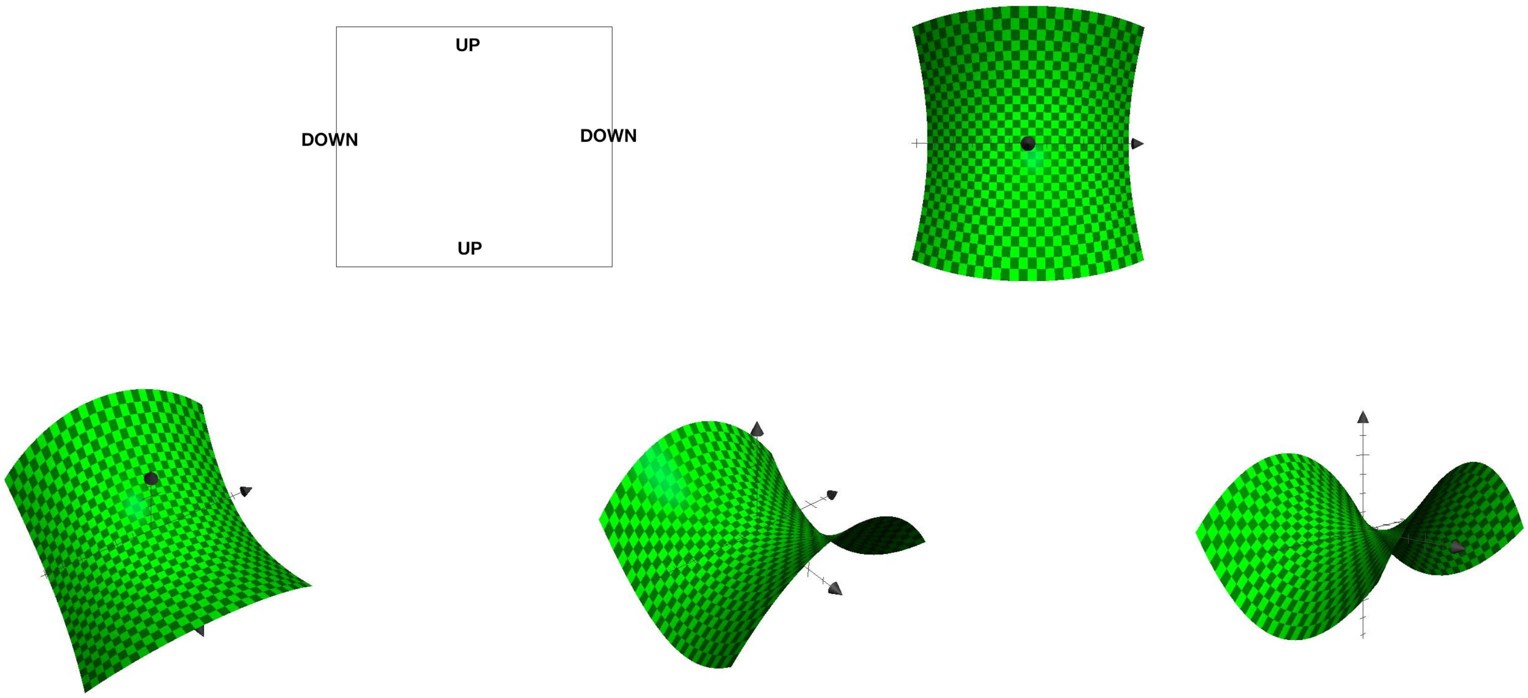

Figure 4. a schematic for and four views of a saddle surface

This process puts strain on the original material. The exact type of strain present will vary, but it comes in two varieties: compression and stretch. The stretching strain will be most prominent in the areas that are pulled further apart than they were before, namely the places on Figure 4a labeled UP and DOWN. This area is labeled red in Figure 5. The compression force will be most prevalent in the center region which changes the least in Figure 4. The more difficult it is to stretch a physical material the more the middle of the material will bunch up. This area is outlined in blue in Figure 5. We will see a perfect example of a stretch free material later when we look at chain mail. A diagram with these areas outlined is as follows.

Our standard model of hyperbolic space will be a vase which is narrower in the middle and has a smooth curve. In fact, any object with a ‘waist’ will do, and possessing such a model will be crucial for several of the Experiments outlined in this paper.

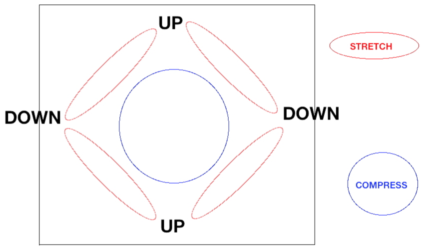

Figure 5. stresses on a flat surface forced to form a saddle

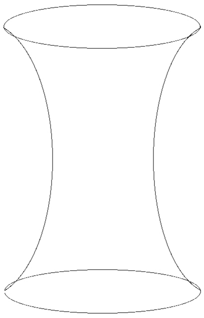

Figure 6. sample saddle surface

3. Solutions employed to solve the potato chip problem

Throughout history various solutions have been employed to make a stable garment that fits a hyperbolic surface.

3.1. Draping

This solution amounts to ignoring and covering up the potato chip problem. In all cases of the draping solution, fabric is loosely distributed over the form in such a way that movement is still possible. The weight of the fabric is distributed to relatively stable areas of the body (such as the waist or shoulders). Because weight and movement are the important issues, not all of these solutions can be demonstrated with a model of a single piece of hyperbolic space like our vase.

For any draping solution, no real attempt is made to form hyperbolic space other than what happens (essentially for free) when a mobile garment is anchored at the waist (like a sari or a kilt).

Some examples of this solution are

- **Toga** A toga (especially in its fully developed form) is actually an extremely complicated garment made of a large half-ellipse of fabric (approximately seven and a half feet by eighteen feet) with a complicated wrapping method. The

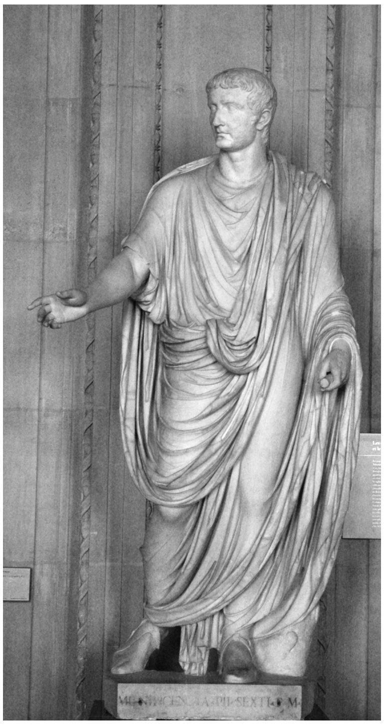

Figure 7. A genuine Roman toga from the 1st century AD (Emperor Tiberius) y Marie-Lan Nguyen - Own work, Public Domain

various folds had names and there were often weights sewn into the hem at the correct points to make the folds lie well. It was impossible to be very active in a toga as at least some of the structural integrity of the garment came from arm placement (the rest came from the weight of the draped fabric which was wound around the torso twice with the excess worn on the back).(Wilcox) All genuine Roman depictions of the toga had a wrinkled appearance (quite unlike neoclassical depictions), so some sort of pleating was probably involved, which, in turn, made movement possible, within some formal constraints.

- **Sari** Saris are anchored by the weight of the excess fabric, which is thrown over one's shoulder. This anchoring method is similar to a toga. However, saris have specific concessions made for ease of movement (such as a kick-pleat built into the front of the garment). It is possible to do practically anything (other than hanging upside down) in a sari, though it can take a bit of practice. A sari is a strip of cloth between four to nine yard long, whose width makes up the skirt and top wrap piece. A sari is worn with a sari-shirt made of the same (or purposely contrasting) material. This shirt is very tightly cropped, with a scooped neck and short sleeves. An additional typical undergarment, with a more structural role to play, is the 'sari-skirt'. The sari-skirt is floor length, usually black, and , crucially, has a tightly cinched drawstring waist. The outer sari can be anchored at the waist by tucking a slight amount of actual sari inside the sari-skirt's cinched waist. The waist is one of the most stable portions of the human body, being both more flexible and softer than the surrounding area. Therefore, an even moderately fitted waist strap can keep position, so anchoring a garment at the waist impedes neither an upright walking motion nor moderate tumbling. The weight of the back panel and friction do an excellent job of keeping the upper portion of the garment stable. Figure 7 shows the process of donning the most basic form of sari, the last picture being a back view of the completed garment. The total effect of a sari will often emphasize the curviness of a figure.

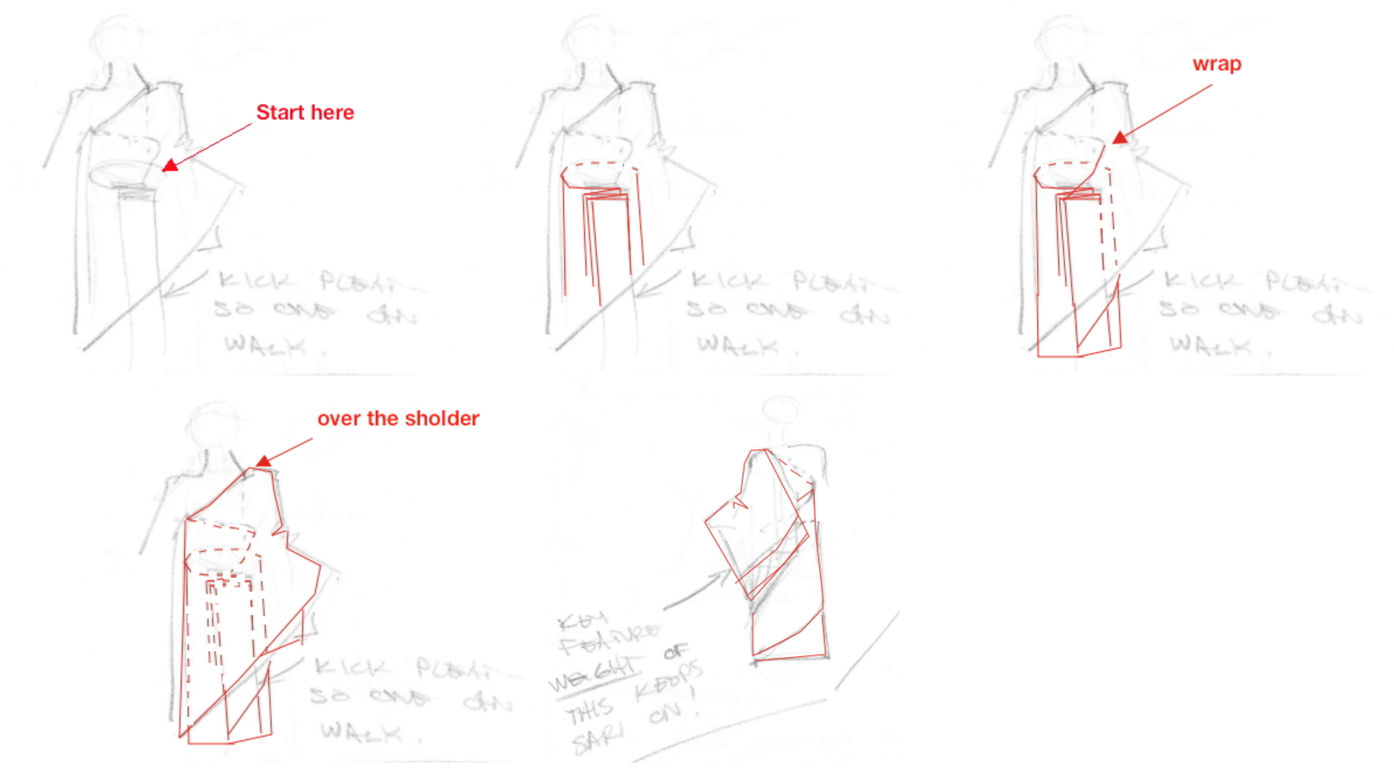

Figure 8. Basic sari diagram

- **Kilt** A kilt, especially in its early form (often called a great kilt or a belted plaid) functioned a both a garment and a sleeping bag. Made of plaid wool, it was worn knee length and pleated all over (for complete freedom of movement) and belted tightly at the waist. In order to wear the garment knee length, half of the fabric was hiked above the waist. The excess was either twisted up and thrown over the shoulder (like a sari or toga) or draped over the shoulders like a baggy cloak (see Figure 8 for a schematic and a picture of 18th century highlanders in kilts). Modern kilts are cut down so that the skirt is the whole width of the fabric. The pleats are sewn down and a flat front panel is added. They still retain the original characteristic of large amounts of fabric for complete freedom of movement (a modern 'traditional' kilt is made of ten yards of fabric).

3.2. Stretchy material

Stretchy material solves the potato chip problem, not by ignoring it (as with the draping solution) ,but by letting the fabric itself expand and contract to take care of the problem. That is, the material we are forming a garment from can be either flat or hyperbolic as the situation calls for. The remaining issues for a garment become topological (i.e. number of holes).

Modern examples of stretchy fabric can be made from stretchy substances such as elastic thread or sheet latex. Stretchy fabrics can also be produced from non-stretchy

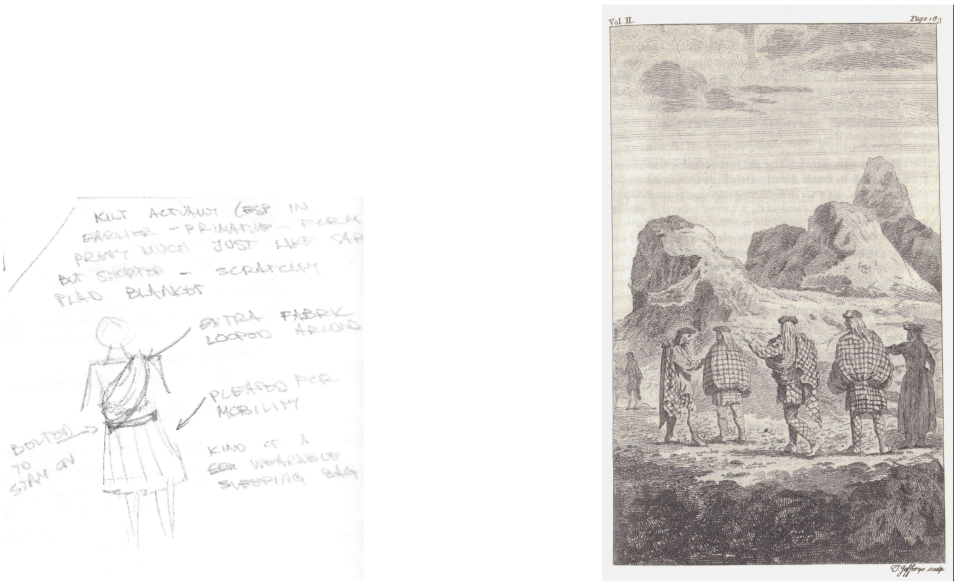

Figure 9. [right] Edward Burt *Letters from a Gentleman in the North of Scotland*, London 1752 (Tarrant)

substances such as nylon, microfilament, or even metal. However, the most common stretchy fabrics are knits.

Knitting requires being able to form a single long thread and twist that thread into interlocking rows, as seen in Figure 9.

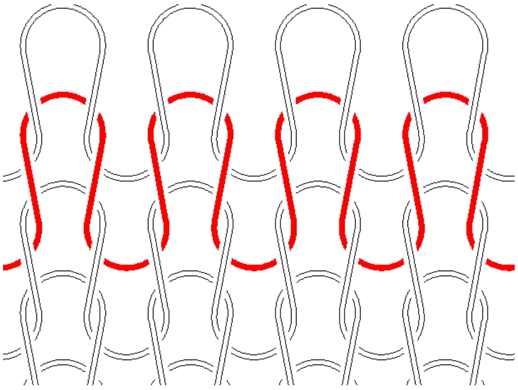

Figure 10. Stockinette, the simplest knitted pattern. Variations on this are mostly based on switching which thread is on top in each intersection (pearl) and by re-ordering the loops (cabling) https://commons.wikimedia.org/w/index.php?curid=114385903

As long as the first row and the last row are tied off properly, the fabric is stable and remarkably flexible. If the material is stretched horizontally, the individual rows will start to straighten, producing a shorter wider rectangle (see Figure 10). The thread that the knit is made from does not need to be elastic for there to be a large amount of stretch: the stretch actually depends on how tightly the fabric is knit. A tighter knit will produce loops that are nearly piecewise linear (see Figure 14 [right]).

Similarly, if the fabric is stretched vertically, the individual loops will stretch, while the spaces between the loops will shrink.

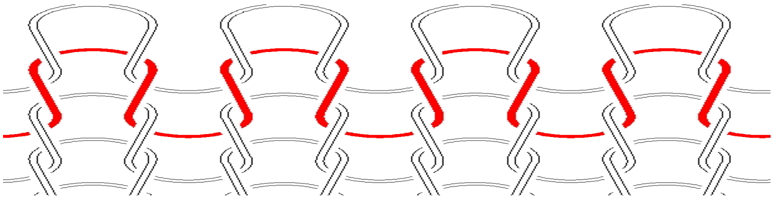

Figure 11. stockinette stretched horizontally

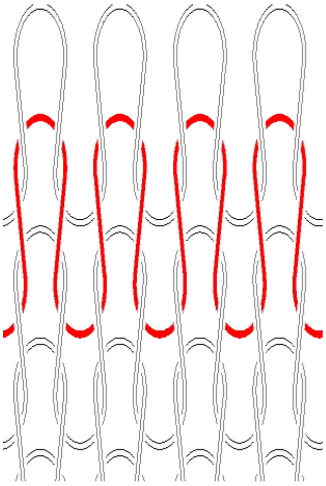

Figure 12. stocknette stretched vertically

It is easy to see for yourself (Experiment 2:) that any knit fabric (t-shirt, underwear, socks) will readily form to your vase shape and the looser knits will do this more easily than the tighter ones.

In fact, although a standard stockinette knit (as pictured) will stretch in both directions, one of them (the horizontal direction from Figure 10) is more natural than the other. Although knits do stretch vertically, as can be seen in Figure 14, the vertically stretched fabric is less efficient than the horizontal stretched fabric, and a tight machine knit has limited vertical stretch. For this reason, the grain of the knit (vertical, horizontal, diagonal) is best chosen to be the direction in which stretch is considered most needed. A t-shirt, for example, allows the chest to expand and contract, because the body of the shirt is made with the knit in a horizontal position. The span of the chest is not treated as a constant. The distance from shoulder to waist, however, is treated as a constant, and so poses which increase or decrease this distance on either side will expose the torso or make the t-shirt bulge with excess fabric.

Knits do not naturally contract. The smallest amount of area that a knit naturally takes up (without wrinkling) is its resting shape. At its most extreme amount of stretch, a standard knit pattern (as in figure 9) takes on the shape of a grid of rectangular boxes and stretched vertically, resembles a house of cards.

3.3. Chainmail

Chainmail, while not being a knit in the strictest sense (it is not formed from a single wire), functions as a knit, but with a contrasting method. While actual knits have a

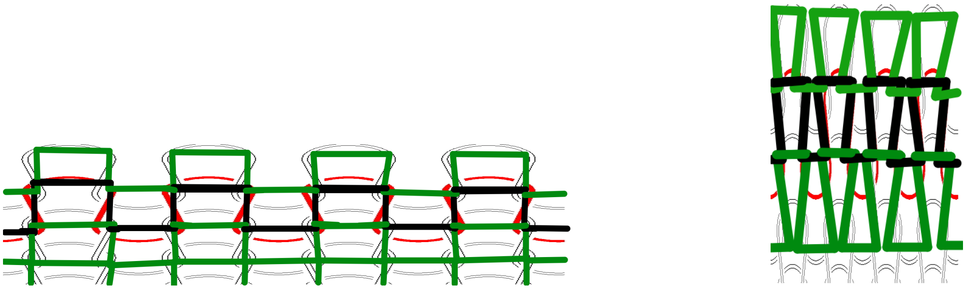

Figure 13. limits of horizontal (left) and vertical (right) stretch for knits

resting shape that is smaller than their stress shape (i.e. they stretch), chainmail has the ability to *shrink* to fit the curvature conditions! In this sense, chainmail can be thought of as a knit whose resting position is fully stretched out. Chainmail achieves this shrinkage by disordering its links. The sample below is of European four in one chainmail both in its resting state and with its center disordered (inside the blue circle). As you can see from the third picture, this ability for disordered links to take up less area (though not less volume) allows chainmail to adapt to saddle surfaces.

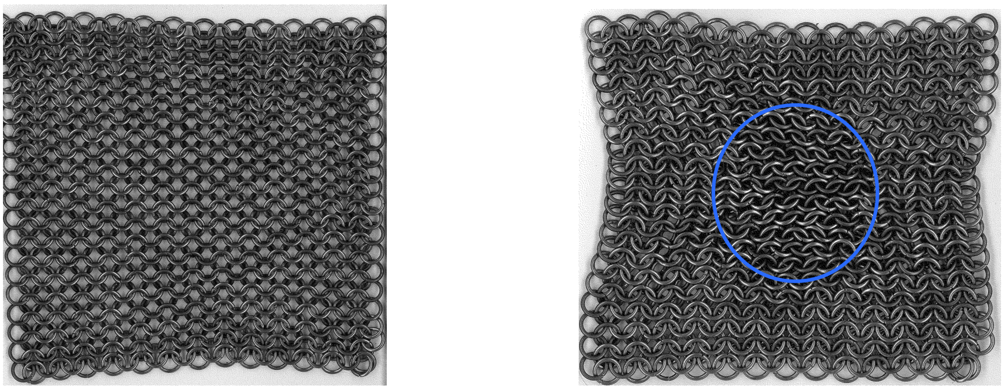

Figure 14. ordered and disordered chainmail links

Figure 15. chainmail functioning as a knit

In fact, mathematical terms, chainmail is a co-knit[3].

3.4. *Woven material*

Most modern woven clothing (not including knits) is pieced. Woven fabric usually requires some sort of piecing to form a potato chip shape. A wide variety of piecing options have gone in and out of fashion over the years.

The problem is that woven fabric doesn't stretch very much.

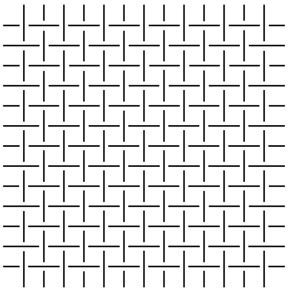

Figure 16. close up of plain woven fabric

The vertical and horizontal threads are called warp and weft depending on which direction the original fabric was woven in (unlike when making a cherry pie - or my illustration) the warp is pulled over and under the weft while the weft is stretched taunt.

As with knits, the individual threads are the least stretchy portion of a fabric. With a straight weave, the individual threads are aligned with the directions we want to stretch it to fit our saddle surface vase (the top and bottom should stretch up while the left and right should stretch down). For this reason, a woven fabric held straight over our vase (Experiment 3:) will tend to bunch up in the middle.

One popular method of piecing involves organizing those bunches of fabric into pleats and sewing in those tucks. We can easily come up with an approximation of this using our woven fabric and two pins. (Experiment 3:) wrap your woven fabric around the vase with the threads parallel or perpendicular to the vase's vertical axis). Pull it tight. With your fingers, organize the wrinkles into two vertical lines, and pin them with your pins. If the vase was the waist of a woman't garment, those two tucks we made would be called 'princess seams', iconic in women's garments of the 1960s. These use much less fabric and are easier to sew than the more complicated piecing options will will discuss below.

We've seen that woven fabric doesn't stretch very much if we are pulling in the direction of the warp and weft. However, if we turn the fabric diagonally, something else happens.

This doesn't look very different than the original situation, but if we stretch this fabric (left to right or up and down), we aren't asking the individual threads to stretch - instead, we are shifting how they relate to each other and this they can accommodate!

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[3] As Carl Droms pointed out, as it is armor, it even 'turns the arrows around'. (Ignore this comment if you dislike either puns or category theory.)

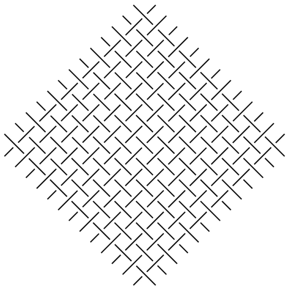

Figure 17. plain weave oriented diagonally (on the bias)

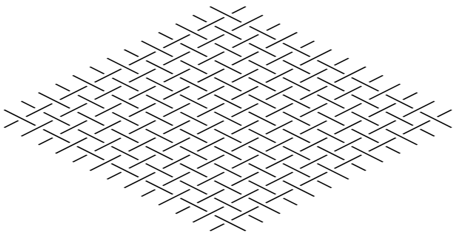

Figure 18. bias oriented plain weave stretched horizontally

Notice how what used to be squares in the original diagonal fabric are now rhombi. We have not changed the length of any of the threads, but nonetheless, we have succeeded in stretching the fabric.

(Experiment 4:) Place your woven fabric so that it is absolutely evenly diagonal (called 'true bias' in dressmakers terms). Now see if you can get the fabric to form to the vase without wrinkling. Your success will depend on how tightly your fabric is woven, but no matter which fabric you use, it should definitely work *better* than it did with your fabric cut on the straight.

This method of forming a potato chip out of woven fabric is called bias cutting and has long been used for difficult places on garments. (Perhaps the most difficult is sleeves, which contain several saddle surfaces and require a lot of movement.) Bias cutting entire garments uses much more fabric than does cutting on the straight (there will almost always be big left over triangles of fabric). It also is far more difficult to sew, as one is trying to sew in a direction the fabric stretches easily (one book recommends hand stitching the fabric to a piece of paper and ripping the paper off when the seam is complete (Talbot)) so it has fallen out of favor.

4. Rigid media - aka plate armor

Rigid material like sheet metal will only curve in one direction naturally.

(Experiment 5:) Take a thin flexible piece of sheet metal (aluminum flashing works well). Notice that it easily curls up into a cylinder, and that this can be done in any direction. However, if we want it to make a potato chip (curvature in two directions at once) we will need to make our cylinder into, essentially, a slice of bent pipe. As anyone who has worked with a pipe bender knows, it is very easily to cause your pipe to kink and collapse, and this is essentially what happens when we try to bend our sheet metal in two directions at once. Take your cylinder and try to make it into a saddle by bending it in the opposite direction. The result will be a flat section of metal bent in your second direction wrinkling into the sections that remain bent in the first direction.

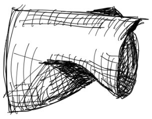

Figure 19. results of experiment 5

So although it doesn't seem like metal belongs in the category of 'stretchy material', forcing the metal into a potato chip shape by stretching is the only solution. An illustration of our plan for creating a metal potato chip is in Figure 20.

The result is stretching and deforming the metal into the necessary potato chip shape (illustrate with my breast bone armor). However, this is time consuming, and even with modern metal stamping equipment (such as the metal presses used by the auto companies to form car doors) has complicated mathematical and engineering issues of its own.

4.1. Casting

One other option we may like to try is casting our potato chip. Similar to the formation of a Pringle, casting metal involves heating it to a liquid state and pouring the liquid

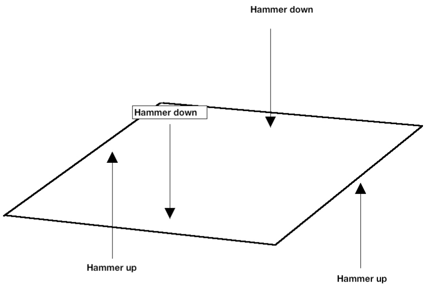

Figure 20. recipe for making a saddle out of sheet metal

into a mold that will then cool and with care, form (more or less) our desired shape. (There is an issue with ‘bounce back’ and shrinkage where the metal doesn’t form exactly the same shape as our mold). In fact, armor can be cast, and many of the more intricate pieces probably were, but practical armor, armor that is designed to resist penetration by an edged weapon, is beaten, not cast.

Perhaps surprisingly, metal is a crystal, and when cast, the crystals form in all directions making the metal weak and susceptible to breaking along the temporary surfaces formed by the rolling liquid. This is similar to delamination, where substances that are formed in sheets (like sedimentary rock, or fingernails) are prone to breakage between the layers. A good model for this is chocolate as chocolate is also melted and cast and forms crystals.

(Experiment 6:) Take your Cadbury mini-eggs (candy coated cast chocolate eggs), hold one between your front teeth vertically and bite down. Some, though probably not all, of your eggs will show evidence of breakage along a smooth curved surface which is created during the casting and cooling process. This is why armor is not cast!

Figure 21. sample results from Experiment 6

4.2. Cylinders

Practical armor is made by first rolling metal flat (and hence aligning the metal crystals to play well with each other) and then bending and stretching the metal into the appropriate shapes.

As we’ve seen in experiment 5 (and we know from using tin cans), sheet metal does an excellent job at making cylinders, and pieces of cylinders and a lot of what forming armor comes down to is picking out what parts of the body make good cylinders. Incidentally, the exact same thing happens with sheet plastic. It forms cylinders readily, but forms traditional seams unhappily, leaving a stiff ridge and perforations in the plastic that will eventually cause it to rip down the seam. For sheet plastic, the key is overlapping the plastic and combining the pieces of cylinders with glue and/or tape. For plate armor, the key is rivets and careful planning.

5. Conclusion

As we have seen, the potato chip problem has no one best solution, and many aspects of the history of clothing and armor can be thought of as a series of attempts to solve

(a) By Konrad Seusenhofer, Innsbruck - Photograph by Sandstein, https://commons.wikimedia.org/w/index.php?curid=12168237

(b) By Rama CC BY-SA 2.0 fr https://commons.wikimedia.org/w/index.php?curid=1647064

Figure 22. Two samples of rivited plate armor at opposite ends of the price spectrum

the problem of how to make potato chips out of various materials. Along the way, we saw how clothes can perscribe a range of acceptable movements, how chainmail is a co-knit, and how the inflexible can be made flexible. In all these senses, clothing is just a series of tricks that can be used to solve a fundamental issue with human anatomy, the fact that we are hyperbolic and like covering ourselves with clothing[4].

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[4] Clothing is often considered to be a practical necessity, and in some places (like the Arctic) it genuinely is. But as can be seen from historic Tierra Del Fuego whose climate is similar to that of Ohio, clothing is not actually necessary in most locations.