# Math 485 Homework 2

due Friday, January 27

**1.** A subgroup of a group of motions is *discrete* if it does not contain any arbitrarily small rotations or translations. That is, there exists some $\varepsilon > 0$ such that for any translation by a vector $\overrightarrow{v}$ in the subgroup, $|\overrightarrow{v}| > \epsilon$ and for every rotation (about a point $\mathbf{p}$ through an angle $\theta$) in the subgroup, $\theta > \varepsilon$.
Let $M$ be the group of symmetries of a plane (namely, $M$ consists of all of the translations, rotations, reflections and glide reflections) and let $G$ be a discrete subgroup of $M$.

**a)** Prove that the stabilizer $G_p$ of a point is finite.

**b)** Prove that the orbit $O_p$ of a point $p$ is a discrete set. That is, prove that there exists $\varepsilon > 0$ such that the distance between any two points in the orbit is $> \varepsilon$.

**c)** Let $B, B'$ be two bounded regions in the plane. Prove that there are only finitely many elements $g \in G$ such that $gB \cap B' \neq \varnothing$.

**2.** Let $X = M_{m,n}(\mathbb{R})$ be the set of all $m$ by $n$ matricies with coefficients in $\mathbb{R}$ and let $G = GL_m(\mathbb{R}) \times GL_n(\mathbb{R})$.

**a)** Prove that $(P, Q), A \to PAQ^{-1}$ defines an action of $G$ on $X$.

**b)** Describe the decomposition of $X$ into $G$-orbits.

**c)** Assume that $m \leq n$. What is the stabalizer of the matrix $[I \mid \mathbf{0}]$ (the matrix that consists of $I_m$ the identity matrix followed by $n$ zeroes)?

**3.** Let $GL_2(F)$ act on $M_2(F)$ by conjugation.

**a)** For $F = \mathbb{R}$, describe the orbit and stabilizer of the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$?

**b)** If $F = \mathbb{F}_3$ (that is, $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$ treated as the field it is), compute $|O_A|$.

**4.** We say a group $G$ acting on a set $X$ acts *doubly transitively* on $X$ if for all $(x_1, x_2)$ and $(y_1, y_2)$ in $X \times X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists $g \in G$ with $gx_1 = y_1$ and $gx_2 = y_2$.

**a)** Prove that the action of $\Sigma_n$ on $\{1, 2, ..., n\}$ is doubly transitive.

**b)** Let $X$ be a set with $|X| > 2$. Prove that $G$ acts doubly transitively on $X$ if and only if the stabalizer of $x \in X$ acts transitively on $X - \{x\}$.

**5.** We say that a $G$-set $X$ is *primitive* if the only non-empty subsets $B$ such that $B$ and $gB$ are either equal or disjoint for all $g \in G$ are the whole set $X$ and the single point subsets. Prove that a doubly transitive $G$-set is always primative.

**6.** Let $G$ be the subgroup of $\Sigma_5$ generated by the 5−cycle (12345) and let $G$ act on $\{1, 2, 3, 4, 5\}$ in the usual way. Show that this action is primative but not doubly transitive.

**7.** BONUS: Let $X$ be a $G$-set and let $[X]$ denote the isomorphism class of the set $X$ as a $G$-set.

**a)** If $G$ is a finite group, let $S(G)$ denote the set of isomorphism classes of finite $G$-sets. Show that we can define a sum and product operation on $S(G)$ by $[X]+[Y] = [X \cup Y]$ and $[X][Y] = [X \times Y]$. (Show that these operations respect the $G$-action.)

**b)** Let $B(G)$ be the set obtained from $S(G)$ by adjoining formal additive inverses (here the additive identity is $[\varnothing]$). Show that $B(G)$ is a commutative ring under the operations in part **a)**. ($B(G)$ is the *Burnside ring* of $G$.)

**c)** Show that any element of $B(G)$ can be written uniquely as a $\mathbb{Z}$ linear combination of isomorphism classes of transitive $G$-sets.

**d)** Let $H$ be any subgroup of $G$. Show that there is a unique ring homomorphism $\phi_H : B(G) \to \mathbb{Z}$ which sends $[X]$ to the number of elements of $X$ fixed by $H$.

**e)** Show that the intersection of $\ker \phi_H$ for all $H$ subgroup of $G$ is zero.

**f)** Show that any non-zero ring homomorphism $\phi : B(G) \to \mathbb{Z}$ is equal to $\phi_H$ for some $H$.