Math 485 Homework 3

due Friday, February 10

1. Apply the Graham-Schmidt procedure to the basis $(1,1,1)^t, (1,0,1)^t, (0,1,1)^t$, when the form is ordinary dot product.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form $\langle \overrightarrow{v}, \overrightarrow{w} \rangle_A$.

3. Let $\langle\ ,\ \rangle$ be a symmetric bilinear form on a vector space V over a field F. The function $q : V \to F$ defined by $q(\overrightarrow{v}) = \langle \overrightarrow{v}, \overrightarrow{v} \rangle$ is called a *quadradic form* associated to the bilinear form. Show how to recover the bilinear form from q, if $1 + 1 \neq 0$ in the field, by expanding $q(\overrightarrow{v} + \overrightarrow{w})$.

4. Is $\langle \overrightarrow{v}, \overrightarrow{w} \rangle = v_1 w_1 + i v_1 w_2 - i v_2 w_1 + i v_2 w_2$ on $\mathbb{C}^2$ a hermitian form?

5. Let A, B be positive definite hermitian matrices. Determine which of the following matrices are positive definite hermitian: $A^2, A^{-1}, AB, A + B$.

6. Prove that a hermitian form on a complex vector space has an orthonormal basis if and only if it is positive definite.

7. Let $\langle\ ,\ \rangle$ be a hermitian form on a complex vector space V, and let $\{\ ,\ \}$ be the real part of the complex number $\langle\ ,\ \rangle$. Prove that if V is regarded as a real vector space, then $\{\ ,\ \}$ is a symmetric bilinear form on V, and that if $\langle\ ,\ \rangle$ is positive definite, then so is $\{\ ,\ \}$. What can you say about the imaginary part?

8. Determine whether or not the following rules define hermitian forms on the space $\mathbb{C}^{n \times n}$.

a) $A, B \to \text{trace}(A^* B)$

b) $A, B \to \text{trace}(\overline{A} B)$

9. Let P be a unitary matrix, and let $\overrightarrow{v}_1, \overrightarrow{v}_2$ be eigenvectors for P with eigenvalues λ_1, λ_2 such that $\lambda_1 \neq \lambda_2$. Prove that $\overrightarrow{v}_1, \overrightarrow{v}_2$ are orthogonal with respect to the standard hermitian product on $\mathbb{C}^n$.

10. Let P be the vector space of polynomials in x of degree $\leq n$ with coefficients in $\mathbb{C}$.

a) Show that

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta} d\theta$$

is a positive definite hermitian form on P.

b) Find an orthonormal basis for this form.

11. Let A be any complex matrix. Prove that $\det(I + A^*A) \neq 0$.