Math 485 Homework 6

1. Computing with small examples:

a) Let T be the group of rotations of the regular tetrahedron. Compute the characters for the the operation of T on F the set of faces of the regular tetrahedron and on E the set of edges of the regular tetrahedron.

b) Let ρ be the permutation representation associated to the operation of D_3 on itself by conjugation. Decompose the character of ρ into irreducible characters.

2. Show that the standard representation of the symmetric group Σ_n by permutation matrices is the sum of a trivial representation and an irreducible representation.

3. Here is the character table for the group $G = PSL_2(\mathbb{F}_7)$, with $\gamma = \frac{1}{2}(-1+\sqrt{7}i)$ and $\gamma' = \overline{\gamma}$.

	(1)	(21)	(24)	(24)	(42)	(56)
	1	a	b	c	d	e
χ_1	1	1	1	1	1	1
χ_2	3	-1	γ	γ'	1	0
χ_3	3	-1	γ'	γ	1	0
χ_4	6	2	-1	-1	0	0
χ_5	7	-1	0	0	-1	1
χ_6	8	0	1	1	0	-1

a) Identify, so far as possible, the conjugacy classes of the elements $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 2 & \\ & 4 \end{pmatrix}$, and find matrices which represent the remaining conjugacy classes.

b) G operates on the set of one-dimensional subspaces of $\mathbb{F}_7 \times \mathbb{F}_7$. Decompose the associate character into irreducible characters.

4. Let G be a cyclic group of order n, generated by an element x, and let $\zeta = e^{2\pi i/n}$.

a) Prove that the irreducible representations are $\rho_0, \rho_1, ..., \rho_{n-1}$ where ρ_k is the representation that takes $x \in G$ to ζ^k.

b) Identify the character group of G.

5. Representations of D_3.

a) Let $\rho = \rho'$ be the two-dimensional representation of the dihedral group D_3 acting as symmetries of a regular triangle, and let $A = \begin{pmatrix} 1 & 1 \\ & \end{pmatrix}$. Use the averaging process to produce a G-invariant transformation from left multiplication by A.

b) Show that $R_r = \begin{pmatrix} 1 & 1 & -1 \\ & & 1 \\ 1 & & -1 \end{pmatrix}, R_v = \begin{pmatrix} & -1 & -1 \\ -1 & & 1 \\ & & -1 \end{pmatrix}$ defines a representation of D_3, where r is the rotation and v is the vertical flip in D_3.

c) Let ρ_2 be the sign representation of D_3 thought of as a 1×1 matrix. Let T be the linear transformation $\mathbb{C}^1 \to \mathbb{C}^3$ whose matrix is $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t$. Use the averaging method to produce a G-invariant linear transformation from T using the action on $\mathbb{C}^1$ by ρ_2 and the action on $\mathbb{C}^3$ by the representation R defined in part (b).

6. Prove that the functions χ_n (the character of the representation T acting on V_n form a basis for the vector space spanned by $\{\cos n\theta\}$.

7. Computations involving infinitecimals

a) Prove the formula $\det(I + A\epsilon) = 1 + \operatorname{trace} A\epsilon$. Use this to conclude that $\mathfrak{sl}_2\mathbb{C} = \{X \in M_{n,n}\mathbb{C} | \operatorname{trace} X = 0\}$.

b) Compute $(I + \epsilon X)^*(I + \epsilon X)$ and use this to find $\mathfrak{su}_n\mathbb{C}$, the Lie algebra of $SU_n\mathbb{C}$.

8. Lie bracket

a) Show that $(PQP^{-1}Q^{-1})^{-1} = QPQ^{-1}P^{-1}$ implies $[A, B] = -[B, A]$, i.e. the bracket is skew-symmetric.

b) Show that the associativity of multiplication in G implies the Jacobi identity $0 = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$.

9. $SL_2\mathbb{C}$

a) Show that $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ form a basis for $\mathfrak{sl}_2\mathbb{C}$ (use **9a)**).

b) Prove that $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

c) Let ρ_n be the representation of $GL_2\mathbb{C}$ consisting of degree n homogeneous polynomials in x and y with coefficients in $\mathbb{C}$. Show that $d\rho(e)x^iy^j = jx^{i+1}y^{j-1}$, $d\rho(f)x^iy^j = ix^{i-1}y^{j+1}$ and $d\rho(h)x^iy^j = (i - j)x^iy^j$.

d) Use your computations from **c)** to find $E, F, G \in M_{n+1,n+1}\mathbb{C}$ satisfying the Lie bracket computations in part **b)**.

e) Let V be the vectorspace for $d\rho_n$ and $V_\lambda = \{v \in V | h\vec{v} = \lambda v\}$. Compute all of the V_λ using **c)**.

f) Show that if $v \in V_\lambda$, then $ev \in V_{\lambda+2}$ and $fv \in V_{\lambda-2}$.

g) If $ev = 0$, show that $\langle v, fv, f^2v, ...\rangle$ is a $\mathfrak{sl}_2$ invariant subspace of V.

h) Conclude that $d\rho_n$ is not an irreducible representation of $\mathfrak{sl}_2$ and hence ρ_n is not an irreducible representation of $SL_2\mathbb{C}$.