485 REPRESENTATION THEORY MIDTERM

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April 14, 2017 **Name:** KEY

By printing my name I pledge to uphold the honor code.

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Unless otherwise stated, assume that G is a group of order N with k distinct conjugacy classes, X is a set with a G action, F is a field, and V is a n-dimensional vector space over F with a G action and χ its character and $\mathcal{C}$ is the set of class functions on G with coefficients in F.

I. True/False, circle T or F as appropriate. After you have finished the rest of the quiz, please explain your answer by citing specific theorems/definitions/computations/etc.

1. a) **T** F If $F = \mathbb{C}$ or $\mathbb{R}$, there always exists a positive definite inner product on V. ~~STANDARD~~ FORM DEFINED BY ANY BASIS & I IS POS DEF.

b) **T** F Complex conjugation of a representation is a representation. $\overline{x}\cdot\overline{y} = \overline{xy}$

c) T **F** If $F = \mathbb{C}$ and G is a finite group, then for all $g \in G$, $\chi(g)$ is an N-th root of unity. ALL EIGENVALUES ARE Nth ROOTS OF 1, BUT $\chi(g)$ IS SUM OF EIGENVALUES

d) T **F** The character table for G is a $N \times N$ matrix. IT'S ~~AN~~ $k\times k$ MATRIX.

e) **T** F All one-dimensional representations are irreducible. CAN'T BE WRITTEN AS A $\oplus$ OF LOWER DIMENSIONAL REPRESENTATIONS

f) **T** F A representation of a finite group is completely determined by its character. THAT'S OUR HUGE THEOREM

g) T **F** A group is completely determined by its character table. D_4 & $\mathbb{H}_8$ HAVE SAME CHAR. TABLE

h) **T** F U_n is a compact continuous group with a Haar measure.

i) T **F** For U_1, the the irreducible characters generate the subset of $\mathcal{C}$ consisting of continuous class functions. IRRED CHARACTERS ARE $\chi_n(z)=z^n$ SO ~~[illegible]~~ IF $\chi = \sum a_n \chi_n \Rightarrow \chi(0)$ & $\chi(\pi)$ ARE RELATED (NOT TRUE FOR AN ARBITRARY CONT. CLASS FCTN)

j) T **F** $SU_2 \cong S^3$ puts a group structure on S^3 that treats all 4 coordinates identically. $\rightarrow$ $(1,0,0,0)$ ACTS AS IDENTITY WHILE $(0,1,0,0),(0,0,1,0),(0,0,0,1)$ DON'T.

II. Definitions/Fill in the blank Please define/state the following.

1. A *representation* of G on V is defined to be what? Please give the diagram that expresses the representation both as linear transformations and as matrices (R_gs and ρ_gs).

$$\rho: \begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr, "R"'] & GL(V) \arrow[d, "\cong"] \\ & GL_n\mathbb{C} \end{tikzcd}$$

BASIS FREE VERSION

CHOOSE A BASIS

(s.t. ρ & R ARE GP HOMOMORPHISM)

2. What is a compact continuous group?

GP s.t. $G \hookrightarrow F^N$ CLOSED & BOUNDED & DEFINED BY CONTINUOUS FCTNS. (EQUIVALENT TO MULT & INVERSES BEING CONTINUOUS)

3. Please state our Huge Theorem.

LET ~~[illegible]~~ G BE A FINITE GP & $\rho_1, \dots, \rho_k$ BE THE EQUIV. CLASS OF IRRED REPS W/ CHARACTERS $\chi_1, \dots, \chi_k$. THEN $\langle \chi_i, \chi_j \rangle = \frac{1}{N}\sum_{g \in G} \overline{\chi_i(g)}\chi_j(g) = \delta_{ij}$ AND THE χ_i SPAN THE SET OF CLASS FCTNS ON G.

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4. What does it mean for a linear transformation $T : V \to V'$ to be G-invariant and why is that a not-so-great definition? ~~[illegible]~~ NEED A REP. $\rho : G \to GL(V)$ AND $\rho' : G \to GL(V')$. THEN T IS G-INVARIANT IF ~~[illegible]~~ $T(\rho_g(\vec{v})) = \rho'_g T(\vec{v})$ $\forall \vec{v} \in V$. NOT A GREAT DEF SINCE THERE ARE 2 DIFF. MEANINGS TO G-ACTION IN THIS CASE

5. Please state Schur's lemma. How is it used in the proof of our Huge Theorem?

a) IF T IS A G-INV. LIN. TRANSF AND $\rho \neq \rho' \Rightarrow T = \vec{0}$

b) IF $\rho = \rho'$ AND T IS G-INVARIANT, THEN T IS MULT BY A SCALAR. WE USE a) TO SHOW $\langle \chi_i, \chi_j \rangle = 0$ IF $i \neq j$ AND b) TO SHOW $\langle \chi_i, \chi_i \rangle = 1$

III. Short answer

1. What is the difference between G acting on X and G acting on V. Give one nice consequence of the later type of action.

G ACTING ON X ONLY NEEDS ~~[illegible]~~ $\rho(1) = id$ AND $\rho(g)\rho(h) = \rho(gh)$
G ACTING ON V ALSO NEEDS TO BE COMPATIBLE W/ ~~[illegible]~~ VECTOR SPACE STRUCTURE, SO $k \rho_g(\vec{v}) = \rho_g(k\vec{v})$ AND $\rho_g(\vec{v} + \vec{w}) = \rho_g(\vec{v}) + \rho_g(\vec{w})$

2. What was the point of finding a G-invariant positive definite hermitian form?

POINT WAS TO TURN AN ARBITRARY REPRESENTATION INTO A UNITARY REPRESENTATION. CONSEQ. IS $\rho : G \to GL(V)$ IS A GP. HOMOR! (SINCE ANY POS. DEF. HERM. FORM IS CONJUGATE TO THE STANDARD ONE & U_n PRESERVES THE STANDARD ONE.

3. Extend the action of Σ_3 on an equilateral triangle to a representation of Σ_3 on $\mathbb{R}^2$ (hint: your matrices will be a lot nicer if you choose a basis whose angles are $2\pi/3$ angle apart).

SEE BIG QUIZ

4. Go through the process of constructing a positive definite inner product for your representation of Σ_3 above and use this to get a unitary representation.

SEE BIG QUIZ

5. How many one dimensional representations of Σ_3 are there? Please construct them!

SEE BIG QUIZ

6. Demonstrate explicitly the orthonormal basis portion of our huge new theorem for the representations of Σ_3 you have constructed above.

SEE BIG QUIZ

7. Please write down the character table for Σ_3.

SEE BIG QUIZ.

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8. The table below is a partial character table of a finite group, in which $\zeta = \frac{1}{2}(-1+\sqrt{3}i)$ and $\gamma = \frac{1}{2}(-1+\sqrt{7}i)$. There are no missing conjugacy classes and their sizes are as listed.

	(1)	(3)	(3)	(7)	(7)
χ_1	1	1	1	ζ	$\overline{\zeta}$
χ_2	3	γ	$\overline{\gamma}$	0	0
χ_3	3	$\overline{\gamma}$	γ	0	0

3 a) Determine the order of the group and the number and dimensions of the irreducible representations.

$21 = 1^2 + 3^2 + 3^2 + \dim\rho_4^{\,2} + \dim\rho_5^{\,2}$

$|G| = 21$ 5 IRRED REPS. SO BOTH MISSING REPS ARE 1 dim

3 b) Determine the remaining characters (hint: there's a short-cut!)

ρ_4 = TRIVIAL REP SO $\chi_4(g) = 1$, $\rho_5 = \overline{\rho_1}$ SO $\chi_5(g) = \overline{\chi_1(g)}$

2 c) What can you say about the group G?

NO NORMAL SUBGPS?

9. Why is Schur's lemma called a lemma despite the fact that it is a key part in our Huge Theorem? BECAUSE IT'S PROOF IS REALLY EASY!

5 $T: V \to V'$ WHERE V, V' ARE IRRED REPS & G COMMUTES BETWEEN 2 G-ACTIONS, SO ker T AND Im T ARE G-INVARIANT SUBSPACES, SO MUST BE EITHER 0 OR V OR 0 OR V' RESPECTIVELY

10. How would you go about turning an arbitrary linear transformation $T: V \to V'$ into a G-invariant one? AVERAGE OVER G-ACTION!

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$$\tilde{T} = \frac{1}{N}\sum_{g\in G} \rho_g'^{-1}(T(\rho_g))$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho_g"'] & V' \arrow[d, "\rho_g'"] \\ V \arrow[r] & V' \end{tikzcd}$$

11. Why are all the eigenvalues of $\rho(g)$ Nth roots on unity? What is the version of this for compact continuous groups and why does it still hold?

5 SINCE $|G| = N$, $\forall g \in G$, $g^N = 1$. IF $\rho(g) \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ i.e. λ_i EIGENVALUES OF G THEN $\rho(g)^N = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda_1^N & & \\ & \ddots & \\ & & \lambda_n^N \end{pmatrix} \Rightarrow \lambda_i^N = 1$. IF G IS A CPT GROUP, THEN $\rho(G)$ MUST BE CPT AS WELL SO $\lambda_i = re^{i\theta}$ $\lambda_i^l \in$ CPT SET $\forall l \Rightarrow r = 1$ (IF $r<1$, λ_i^l ISN'T A CLOSED SET, IF $r>1$, λ_i^l ISN'T BOUNDED).

12. Define the regular representation and its decomposition into irreducible subrepresentations and why our Huge Theorem implies this.

5 REGULAR REPRESENTATION IS ACTION OF G ON V(G) (VS. GEN BY G). EACH OF $g \in G \to$ PERMUTATION MATRIX GIVEN BY ~~ORDER~~ PERM $g_1, \cdots, g_N \to gg_1, \cdots, gg_N$ (WHICH IS FIXED POINT FREE) THEREFORE $\chi_{reg}(1) = N$, $\chi_{reg}(g) = 0$ IF $g \neq 1$ AND $\chi_{reg} = \langle \chi_1, \chi_{reg}\rangle \chi_1 + \cdots + \langle \chi_k, \chi_{reg}\rangle \chi_k$.

AND $\langle \chi_i, \chi_{reg}\rangle = \frac{1}{N}\sum \overline{\chi_i(g)}\chi_{reg}(g) = \frac{1}{N}\chi_i(1) N = \dim \rho_i$

13. What role does Mashke's theorem play in the proof of our Huge Theorem?

5 MASHKE'S THE. SAYS ANY REP. CAN BE WRITTEN AS A SUM OF IRRED. REPS. WE NEED THIS TO GET $\rho = \oplus a_i\rho_i$ $\Rightarrow \chi = a_1\chi_1 + \cdots + a_k\chi_k$ FOR SOME a_k'S. THEN O.N. BASIS SAYS a_k'S ARE UNIQUE.

14. Let G be a compact continuous group and ρ a one-dimensional representation. Let $\rho(g) = re^{i\theta}$. What must r be and why?

5 SEE #11. $r = 1$. IF $r < 1$, THEN $\rho(g^l) = r^l e^{il\theta}$ WON'T ~~BE~~ CLOSED (0 WILL BE A LIMIT POINT & $0 \notin GL_1(\mathbb{C})$) IF $r > 1$, $\rho(g^l) = r^l e^{il\theta}$ WON'T BE BOUNDED.

IV. Proofs

1. Prove that any finite subgroup of $SL_2\mathbb{R}$ is cyclic.

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HW PROBLEM.

ANY FINITE SUBGROUP HAS A UNITARY REPRESENTATION (FOUND BY PRODUCING A G-INVARIANT HERMITIAN FORM & DOING BASIS CHANGE TO MAKE THE FORM ~~[illegible]~~ = DOT PRODUCT)

$U_2 = \{A \in GL_2\mathbb{C} \mid A^* = A^{-1}\}$

IF $A \in GL_2\mathbb{R} \Rightarrow A^t = A^{-1} \Rightarrow G \cong$ SUBGROUP OF $SO_2\mathbb{R} \cong S^1$

ANY FINITE SUBGP OF S^1 IS CYCLIC.

2. A key step in the proof of our Huge Theorem is where we pass from a linear transformation being the zero linear transformation to a statement about traces of matrices. How does that step work?

WE ~~SUMMED~~ SUMMED UP ALL THE LINEAR TRANSF. PRODUCED ~~[illegible]~~ BY AVERAGING THE ELEMENTARY MATRICES

3. Illustrate why the proof of the portion of our huge theorem that *doesn't* extend to compact continuous groups fails. Illustrate why it is a good thing we didn't try to extend this result to compact continuous groups.

TO SHOW THAT χ_i'S SPAN $\mathcal{C}$, WE USE THE EXISTANCE OF THE REGULAR REPRESENTATION

FOR INFINITE GROUPS, THE 'REGULAR REP' WOULD BE UNCOUNTABLY INFINITE DIMENSIONAL, SO MAKES NO SENSE.

HOWEVER, EVEN FOR THE SIMPLEST CPT CONT GP U_1, THE IRRED REPS <u>DON'T</u> SPAN THE SET OF CLASS FCTNS. (ONLY COUNTABLY MANY IRRED REPS, UNCOUNTABLY MANY CONJ. CLASSES IN S^1 SINCE IT'S ABELIAN SO EVERY ELEMENT HAS ITS OWN CONJ. CLASS.)

4. Outline our construction of the set of all irreducible representations of U_1. Be sure to include why you have a complete list.

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SHOWED ALL GP HOMOR $\varphi: \mathbb{R}^1 \to \mathbb{R}^1$ ARE MULT. BY A CONSTANT.

U_1 IS ABELIAN, SO ALL IRRED REPS ARE 1-dim

U_1 IS CPT $\Rightarrow$ ALL IRRED REPS ARE IN MAX CPT SUBGP OF $GL_1\mathbb{C} = \mathbb{C}^*$ WHICH IS S^1

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SO ALL REPS OF ~~S~~ U^1 ARE GP HOMOR.

$$S^1 \xrightarrow{f} S^1$$

DEFINED $\exp: \mathbb{R}^1 \to S^1$ GP. HOMOR.

$\exp(x) = e^{ix}$

GIVEN $\rho: S^1 \to S^1$

GET $\varphi: \mathbb{R}^1 \to S^1 \to S^1 \dashrightarrow \mathbb{R}^1$ PATH LIFTING. s.t. $\tilde{\varphi}(\pi) = \tilde{\varphi}(0)$

MULT BY C

MUST BE $\varphi(x) = nx$ $n \in \mathbb{Z}$

5. What are the group structures on S^1 and S^3? How are they related to representations?

S^1's GROUP STRUCTURE IS ADDITION OF ANGLES.

ANY 1 dim REP OF A COMPACT ~~[illegible]~~ GP. LANDS IN S^1

S^3 HAS GROUP STRUCTURE GIVEN BY

$$(x_1, x_2, x_3, x_4) \longrightarrow \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

AND USING MATRIX MULT THERE.

IT IS $\cong SU_2$

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6. If we have a group G with a subgroup H (G is not necessarily finite), how is ρ also representation of H? Is its decomposition into a sum of irreducible H-representations related to its its decomposition into a sum of irreducible G-representations? Why might this be useful?

$\rho: G \longrightarrow GL(V)$ INDUCES $\rho|_H : H \longrightarrow GL(V)$

$\rho|_H(h) = \rho(h)$ WHICH IS A REP OF H.

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DEPENDING HOW WELL H IS CHOSEN (FOR EX, IF $H \cap$ EVERY CONJ. CLASS $\neq \emptyset$) THE DECOMPOSITION OF $\rho|_H$ INTO IRRED H-REPS COULD BE SAME AS DECOMP OF ρ INTO IRRED G-REPS.

IT MIGHT BE ESPECIALLY HELPFUL IF SAY H IS ABELIAN, SO IRRED H-REPS ARE PARTICULARLY SIMPLE.