485 REPRESENTATION THEORY WEEKLY QUIZ 1

January 23, 2017 **Name:** KEY

By printing my name I pledge to uphold the honor code.

No books, notes, internet, etc. This should take about a half hour.

REMBER TO SIGN IN!

1. Definitions. For this problem, assume G is a group and X is a set.

a) Group G *acting* on the set X.

$G \times X \longrightarrow X$, $(g, x) \longrightarrow gx$ st. $1, x \longrightarrow x \ \forall x \in X$, $(gh)x = g(hx) \ \forall g, h \in G, \forall x \in X$

b) why is the second property called 'associativity', and why is this a slightly misleading name for it?

ITS CALLED ASSOCIATIVITY BECAUSE IT INVOLVES MOVING AROUND PARENTHESIS. ITS SLIGHTLY MISLEADING AS WE'RE TALKING ABOUT 2 DIFFERENT OPERATIONS $(g * h)x = g(hx)$

c) The *orbit* O_x of $x \in X$ is defined as what? Please list some properties of O_x.

$O_x = \{gx \mid g \in G\}$ $|O_x| \leq |G|$, $x \in O_x$

d) The *stabilizer* G_x of $x \in X$ is defined to be what? Please list some properties of G_x.

$G_x = \{g \in G \mid gx = x\}$ G_x IS A SUBGROUP OF G.

$|O_x| = |G/G_x|$

e) The G-action is *faithful* if ...

~~$\exists g \in G$~~ $gx = x \ \forall x \in X \Rightarrow g = 1_G$.

f) The G-action is *transitive* if ...

$\forall x, y \in X \ \exists g \in G$ st. $gx = y$.

2. Theorems: Please state the theorem we covered in class that best fits each discription.

a) G actions on X vs Σ_X THERE IS A NATURAL BIJECTIVE CORRESPONDENCE BETWEEN THE TWO. $G \times X \longrightarrow X$ SAYS $\forall g \in G$, $X \longrightarrow X$, $x \longrightarrow gx$ IS A PERMUTATION OF X

b) Faithfullness in terms of stabailzers

ACTION IS FAITHFUL IF $\bigcap_{x \in X} G_x = \langle 1_G \rangle$

c) Transitivity in terms of orbits

$X = O_x \ \forall x \in X$.

d) All transitive G-sets are isomorphic as G-sets to what?

THE SET OF COSETS OF ITS STABALIZER.

3. Permutation representation: Let $\mathbb{Z}/3\mathbb{Z}$ act on itself. Please write out its *permutation representation*.

WHERE n REPRESENTS $[n] = n + 3\mathbb{Z}$

$0 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

$1 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

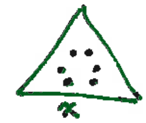
$2 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

4. Let $G = \Sigma_3$ acting on an equilateral triangle

a) What are the sizes of the orbits? Give an example of one of each possible size.

SAMPLE ORBIT 

SIZES ARE 1 

3  OR 6  SAMPLE ORBIT 

b) What are the stabalizers for your example orbits?

STABALIZER OF ORBIT OF SIZE 1 IS ALL OF Σ_3

STABALIZER OF ORBIT OF SIZE 3 IS $\{1, d_2\}$

STABALIZER OF ORBIT OF SIZE 6 IS $\{1\}$

5. Matrix example: let $GL_2(\mathbb{R})$ act on $\mathbb{R}^2$ by left multiplication.

a) Describe the decomposition of $\mathbb{R}^2$ into orbits.

ORBITS ARE $\mathbb{R}^2 - \{0\}$ AND $\{0\}$

SINCE FOR ANY PAIR OF POINTS P_1 & P_2 W/ NEITHER P_1 OR $P_2 = 0$, $\exists$ $A \in GL_2\mathbb{R}$ TAKING P_1 TO P_2

b) What is the stabalizer of $e_1 = (1,0)$ under this action?

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ SO STABALIZER IS $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$ WHERE $d \neq 0$.

(IT'S A SUBGP)

6. Group actions on vector spaces

a) Define a vector space

A VECTOR SPACE OVER A FIELD F IS A SET V WITH 2 OPERATIONS + AND SCALAR MULT. BY ELEMENTS OF F. S.t. IT'S AN ABELIAN GP UNDER VECTOR ADDITION SCALAR MULT DISTRIBUTES OVER VECTOR ADDITION

b) What should a group acting on a vector space look like?

G ACTS ON THE SET V WITH ~~THE~~ ADDITIONAL PROPERTIES THAT SAY HOW THE ACTION INTERACTS W/ + & $\cdot$

NAMELY $\forall\, g \in G$, $\vec{v}, \vec{w} \in V$, $a \in F$

$g(\vec{v} + a\vec{w}) = g\vec{v} + a(g\vec{w})$

c) What would be the difference between $\mathbb{Z}/3\mathbb{Z}$ acting on $\mathbb{R}^3$ as a set and as a vector space? AS A SET, ALL OF THE ORBITS ARE SIZE 1 OR 3, BUT AS A VECTORSPACE, $\bar{0}$ ACTS AS IDENTITY

$\bar{1}$ ACTS AS AN ORDER 3 PERMUTATION THAT TAKES $\langle\vec{v}\rangle$ TO $\langle\bar{1}\vec{v}\rangle$ AS VECTOR SUBSPACES, SO A DECOMPOSITION INTO ORBITS IS MORE LIKE

 etc.

1/2

485 REPRESENTATION THEORY WEEKLY QUIZ 2

February 1, 2017

Name: KEY

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No books, notes, internet, etc. This should take about a half hour.

REMBER TO SIGN IN!

1. Definitions. For this problem, assume that V is a vector space over a field F.

a) What properties does a vector space have?

VECTOR ADDITION (IT'S AN ABELIAN GP WRT THIS) AND SCALAR MULT (ASSOC, DISTRIBUTES OVER ADDITION)

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b) A *basis* for V is ...

ANY SET OF LINEARLY INDEPENDENT VECTORS OF MAXIMUM SIZE

c) Change of basis, change of basis matrix

REWRITING VECTORS AS A LINEAR COMB OF DIFFERENT BASIS ELEMENTS $\vec{v} = a_1\vec{b}_1 + \cdots + a_n\vec{b}_n = c_1\vec{d}_1 + \cdots + c_n\vec{d}_n$. P = CHANGE OF BASIS MATRIX $P\vec{v} = \vec{w}$ VECTOR IN TERMS OF NEW BASIS

$P = (\vec{b}_1 | \cdots | \vec{b}_n)$ $\vec{b}_i$ WRITTEN IN TERMS OF BASIS $\vec{d}_i$

1

d) Bilinear form on V is....

$V \times V \longrightarrow F$ LINEAR IN EACH COORDINATE. DENOTED $\langle \vec{v}, \vec{w} \rangle$

GIVEN IN TERMS OF A MATRIX A $\langle \vec{v}, \vec{w} \rangle_A = \vec{v}^t A \vec{w}$

e) A bilinear (or Hermetian) form on V is positive definite if... (what *must* F be?) $\langle \vec{v}, \vec{v} \rangle > 0$

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f) A Hermetian form of V is ..(what *must* F be?), the standard Hermatian form is .... F MUST BE $\mathbb{R}$ OR $\mathbb{C}$

$\langle \vec{v}, \vec{w} \rangle_A = \vec{v}^* A \vec{w}$ WHERE $*$ MEANS CONJUGATE TRANSPOSE. F MUST BE $\mathbb{C}$ (FOR CONJUGATION TO MAKE SENCE). STANDARD HERM. FORM HAS $A = I$

g) GL_n, U_n ← UNITARY MATRICES $U_n < GL_n\mathbb{C}$ st. $\forall A \in U_n$, $A^{-1} = A^*$

1

GL_n: $n \times n$ MATRICIES W/ NON-ZERO DETERMINATE (INVERTABLE)

h) A group G acting on V satisfies what conditions?

G MUST SATISFY THE CONDITIONS FOR ACTING ON A SET ($G \times V \longrightarrow V$ W/ $(1, \vec{v}) \longrightarrow \vec{v}$ AND $((gh), \vec{v})$ & $(g, h\vec{v})$ MAP TO SAME VECTOR) AS WELL AS RESPECTING VECTOR ADDITION ~~& SCALAR MULT.~~ & $g(\vec{v} + a\vec{w}) = g\vec{v} + ag\vec{w}$.

2. Theorems

a) What are the matricies whose bilinear forms are equivalent to dot product under all possible change of bases? What do we call the condition on the bilinear form that is equivalent to dot product? MATRICES OF THE FORM P^tP FOR ANY $P \in GL_n$. SUCH MATRICES ARE CALLED POSITIVE DEFINITE.

1/2

A IS POS. DEF $\Leftrightarrow$ ALL IT'S TOP LEFT MINORS HAVE POSITIVE DETERMINATE.

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b) Hermetian symmetry corresponds to what condition on the matrix for the transformation? $A = A^*$

c) What sort of change of basis preserves the standard hermetian form? UNITARY ONES

d) ~~Why does the action of G on V produce a subgroup of $GL(V)$?~~

3. Explanations

a) Why is every bilinear form given by a matrix?

SINCE THE FORM IS LINEAR IN EACH COORDINATE, FOR ANY BASIS $\mathcal{B} = \{b_1, \cdots, b_n\}$

IF $\vec{v} = \sum v_i \vec{b_i}$ $\quad \vec{w} = \sum w_i \vec{b_i}$

THEN $\langle \vec{v}, \vec{w} \rangle = \langle \sum v_i \vec{b_i}, \sum w_i \vec{b_i} \rangle$

$= \sum v_i \langle \vec{b_i}, \sum w_i \vec{b_i} \rangle$

$= \sum v_i w_j \langle \vec{b_i}, \vec{b_j} \rangle$

SO FORM IS COMPLETELY DETERMINED BY ITS VALUE ON BASIS ELEMENTS.

b) How does a change of basis effect the matrix for a bilinear form? why? ~~(maybe add to the problems section)~~

IF P IS A CHANGE OF BASIS MATRIX, AND $\langle \vec{v}, \vec{w} \rangle_A$ IS A BILINEAR FORM

WE NEED $\langle P\vec{v}, P\vec{w} \rangle_B = \langle \vec{v}, \vec{w} \rangle_A$ FOR THIS TO BE THE SAME FORM WRT A DIFFERENT BASIS.

IN OTHER WORDS

$\vec{v}^t A \vec{w} = \vec{v}^t P^t B P \vec{w} \quad \forall \vec{v}, \vec{w} \in V$ OR $A = P^t B P$

THIS MEANS OUR NEW MATRIX B IS

$P^{t^{-1}} A P^{-1}$

c) Why do we not use the standard bilinear form if $F = \mathbb{C}$?

BECAUSE IT ISN'T POSITIVE DEFINITE!

IF WE WANT $\langle \vec{v}, \vec{v} \rangle$ TO BE SQUARE OF EUCLIDEAN DISTANCE, WE NEED $\vec{v} = (a_1 + b_1 i)\vec{e}_1 + \cdots + (a_n + b_n i)\vec{e}_n$

TO GIVE US. $\langle \vec{v}, \vec{v} \rangle = a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2$

HOWEVER, $\vec{v}^t \vec{v} = a_1^2 - b_1^2 + 2a_1 b_1 i + \cdots + a_n^2 - b_n^2 + 2a_n b_n i$

d) Why does the action of G on V produce a subgroup of $GL(V)$?

IN ORDER FOR ACTION OF G TO RESPECT VECTOR ADDITION & SCALAR MULT, WE NEED $g(\vec{v} + a\vec{w}) = g\vec{v} + ag\vec{w}$. HOWEVER, THIS IS EXACTLY THE DEFINITION OF A LINEAR TRANSFORMATION & THE FACT THAT $\exists g^{-1} \in G$ PLUS "ASSOCIATIVITY" AND IDENTITY PROPERTIES FROM ACTION OF G ON A SET MEANS THIS LINEAR TRANSF IS INVERTIBLE!

AKA $g \longrightarrow \rho_g \in GL(V)$. SUBGP ALSO COMES FROM

485 REPRESENTATION THEORY WEEKLY QUIZ 3

February 13, 2017 **Name:** KEY

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No books, notes, internet, etc. This should take about a half hour.

REMBER TO SIGN IN!

1. Definitions. For this problem, assume that V is a vector space over $\mathbb{C}$ with $\{\ ,\ \}$ a positive definite hermitian form on V and G is a group acting on V.

a) Please give both a matrix level condition for being unitary and a condition that is basis free (this last will be in terms of $\{\ ,\ \}$).

MATRIX LEVEL CONDITION IS $A^{-1}=A^{*}$ BASIS FREE IS $\{\vec{v},\vec{w}\} = \{A\vec{v},A\vec{w}\}$ FOR THE HERMITIAN FORM $\{\ ,\ \}$

b) A unitary representation is ...

$\rho: G \longrightarrow GL(V)$ s.t. $\rho(G) \subset U_n \subset GL(V)$

c) A form on V is is G-invariant if ...

$\forall\, g\in G,\ \{\vec{v},\vec{w}\} = \{g\vec{v}, g\vec{w}\}$

d) What is the 'averaging trick' for a finite group G acting on a vs V?

WE CAN TURN AN ARBITRARY POS. DEF. HERM FORM INTO A G-INVARIANT ONE BY DEFINING $\langle \vec{v},\vec{w}\rangle = \frac{1}{N}\sum_{g\in G}\{g\vec{v}, g\vec{w}\}$

e) What (roughly speaking) is a *measure* μ on a set X?

IT'S A WAY OF SPECIFYING HOW MUCH SPACE CERTAIN SUBSETS OF X TAKE UP (CAN NOT BE DEFINED ON ALL OF $\mathcal{P}(X)$)

f) If a group G acts on X, what would it mean for μ to be a *Haar measure*?

THE MEASURE MUST BE G INVARIANT. NAMELY IF $S\subset X$, $g\in G \Rightarrow \mu(S)=\mu(gS)$

g) How do we generalize the 'averaging trick' to non-finite groups?

WE MUST $\langle \vec{v},\vec{w}\rangle = \frac{1}{\mu(G)}\int_G \{g\vec{v}, g\vec{w}\}\,d\mu$ FOR μ A HAAR MEASURE

h) A group is *compact* if ....

IT IS CLOSED & BOUNDED AS A SUBSET OF $\mathbb{C}^{n^2}$ OR $\mathbb{R}^{n^2}$ (REALLY F^{n^2})

i) G is continuous if what matrix level condition holds?

IF EACH ENTRY IN THE MATRIX IS GIVEN BY A CONTINUOUS FUNCTION.

j) G is a *topological group* if what holds?

IF $G\times G \longrightarrow G$, $g,h \longrightarrow gh$ AND $G \longrightarrow G$, $g \longrightarrow g^{-1}$ ARE CONTINUOUS FUNCTIONS.

k) Given a positive definite hermitian form on V and W a subspace of V, what is the orthogonal complement $W^{\perp}$ of W?

1/2

$W^{\perp} = \{\vec{v}\in V \mid \{\vec{v},\vec{w}\}=0\ \forall\, w\in W\}$

2. Theorems

a) A unitary representation exists when?

FOR ALL FINITE GROUPS & ALL CPT GROUPS W/ A HAAR MEASURE (AKA ALL TOP. GPS)

b) Please list some corollaries for the theorem in part a)

EVERY REPRESENTATION $\rho : G \longrightarrow GL(V)$ $\rho(G)$ IS CONJUGATE TO A SUBGP OF U_n

ALL R_g'S ARE DIAGONALIZABLE. ~~ALL REAL GROUPS AT~~ ANY FINITE GP ACTING ON A VS$/\mathbb{R}$ HAS AN ORTHOGONAL REP.

c) When does a Haar measure exist?

ANY CPT TOPOLOGICAL GP.

d) Give a connection between continuous groups and topological groups.

IF EACH ENTRY IS GIVEN BY A CONTINUOUS FUNCTION, THEN EACH ENTRY IN MATRIX MULT IS GIVEN BY $\sum f_i g_i$ WHICH IS CONTINUOUS

SIMILAR FOR g^{-1}, SO CONTINUOUS $\Rightarrow$ TOPOLOGICAL

3. Explanations/computations

a) Sketch the idea behind the theorem for 2a)

START W/ ARBITRARY POS. DEF. HERM. FORM, PERFORM AVERAGING TRICK TO GET G-INVARIANT ONE. THEN FIND AN O.N. BASIS FOR THAT FORM.

WRT THAT BASIS, $R_g \in U_n\ \forall g \in G$.

$G \longrightarrow GL(V)$

$\searrow GL_n\mathbb{C}$ $\quad \cong$

b) What is the point of the 'averaging trick' and why does it work?

POINT IS ~~THAT~~ SINCE WE'RE ~~TRANSLATING~~ LOOKING AT IMAGE OF FORM WRT EACH GROUP ELEMENT AND LEFT MULT SIMPLY PERMUTES THE GROUP ELEMENTS AND FINITE ADDITION IS COMMUTATIVE, WE GET A G-INV. FORM.

c) Let $G = \langle A \rangle < GL_2(\mathbb{C})$ where $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. What is the G-invariant form that we get from the 'averaging trick'?

$A^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\quad A^3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

SO $\langle A \rangle = \{I, A, A^2\}$

$\langle \vec{v}, \vec{w} \rangle = \frac{1}{3}(\vec{v}^* I \vec{w} + \vec{v}^* A^* A \vec{w} + \vec{v}^* A^{2*} A \vec{w}) = \frac{1}{3}\vec{v}^*(I + A^*A + A^{2*}A^2)\vec{w}$

$A^* = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ SO $A^*A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$A^{2*}A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ SO MATRIX IS $\frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right) = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$

d) Why is $d\theta$ a Haar measure for the action of SO_2 on S^1?

$SO_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ACTS AS MULT BY $e^{i\theta}$ ON S^1

SO SINCE MULT BY $e^{i\theta}$ TAKES $e^{i\varphi} \to e^{i(\varphi+\theta)}$ SO ACTION IS ADDITIVE AND $d(\varphi+\theta) = d\theta + d\varphi$ PRESERVES ADDITIVITY.

(ALTERNATIVELY, MAP TO $[0, 2\pi)$. THE TRANSLATION INVARIANT HAAR MEASURE IS dx)

REMEMBER TO PUT YOUR COMPLETED QUIZ IN THE CORRECT ENVELOPE AND SIGN OUT!