RESEARCH SUMMARY AND PROPOSAL

REBECCA FIELD

1. **Overview**

The long term goal of my research is to understand how the structure of a group controls the topological and algebraic geometric invariants of its classifying space. I try to find key examples, the computation of which reveals structure theorems for larger classes of groups. My papers to date are part of a body of work including [F4, Gu, KY, T, Ve, RV, Y] devoted to computing invariants of the classifying space of simple complex algebraic groups. Most of my work concerns reductive algebraic groups but I have also studied p-groups. I am interested in the extent to which these sorts of computations pass to motivic cohomology.

The ultimate goal of my research is theorems relating the structure of the group itself to various invariants on the classifying space of the group using certain basic examples, namely classical groups, exceptional groups and p-groups of a certain length. Algebraic geometric methods also play a part because the algebraic geometric invariant I study (Chow theory) is not a cohomology theory, so slightly different techniques need to be used to attack these.

Before describing my work, let me recall some of the ways these spaces arise in algebraic topology. A vector bundle E on a space X determines a homotopy class of maps from X to the classifying space $BGl(n,\mathbb{C})$ of vector bundles. This allows us to define cohomological invariants of the vector bundle E by pulling back a class in the cohomology of $BGl(n,\mathbb{C})$ via the map that classifies E. All 'natural' cohomological invariants of E arise in this way. This rephrases the problem of finding invariants that distinguish vector bundles—it is equivalent to computing $H^*BGl(n,\mathbb{C})$.

Just as the space $BGl(n,\mathbb{C})$ classifies complex n dimensional vector bundles, $BO(n,\mathbb{C})$ is the classifying space for vector bundles with non-degenerate quadratic form, and $BSO(2n)$ is that of oriented vector bundles with non-degenerate quadratic form.

Aside from their intrinsic interest, these spaces are part of the basic building blocks of algebraic topology. For example, Bott periodicity is a statement about relations between loopings of these spaces as $n \to \infty$.

My work is concerned with a detailed study of algebraic topological and algebraic geometric variants of these spaces. If G is one of the above classical groups, or more generally an algebraic group (for example, a finite group) then its classifying space BG has additional structure. It is an Ind algebraic variety—a direct limit of finite dimensional algebraic varieties. We would like to understand algebraic geometric incarnations of the topology of these varieties.

Put differently, as well as studying topological vector bundles, one wants to study algebraic vector bundles (in the Zariski, etale, ... topologies) and to find cohomological invariants of these. Instead of taking values in singular cohomology, these invariants are now classes in motivic cohomology, Chow groups, algebraic Morava K-theories...and so we must compute these exotic cohomology theories of the classifying spaces.

Finally, the finite dimensional approximations to BG are interesting algebraic varieties themselves. They are of the form $(V-S)/G$ where V is a representation of G and S is the closed subvariety on which G fails to act freely. These spaces are a famous source of examples in algebraic geometry – they were originally introduced by Godeaux, studied by Serre, and have been used by Atiyah and Hirzebruch to provide counterexamples to the integral version of the Hodge conjecture.

2. **Descent for complex oriented cohomology theories and Chow theory**

In this section I will describe work computing E^*BG where E^* is a complex oriented cohomology theory. The work is inspired by the beautiful papers of Kriz and Wilson who describe $E^*BO(n)$ purely in terms of the formal group law attached to the cohomology theory [K][W]. (All groups will be with complex coefficients unless otherwise explicitly stated.) This work is part of a larger joint project with Ian Grojnowski.

The crucial idea which informs our recent work is the use of the descent spectral sequence, which is a version of the bar construction.

Theorem 2.1. *Let $F \to Y \to X$ be any fibration which locally on X admits sections and define*
$Y^n = Y \times_X Y \times_X ... \times_X Y$ the nth fibered product. Then the simplicial space

$$Y \leftleftarrows Y^2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; Y^3 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; Y^4 \qquad \cdots$$

is weakly equivalent to the constant simplicial space X, hence there is a spectral sequence, the descent spectral sequence, converging to E^*X whose E_1 term is the chain complex attached to the cosimplicial complex

$$E^*Y \rightrightarrows E^*Y^2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; E^*Y^3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; E^*Y^4 \cdots$$

where the arrows are induced by the various projection maps.

When one applies this to a spherical fibration, the E_1 term is computable. The content of the differential comes down to computing the swap map $E^*Y^2 \to E^*Y^2$. The computation of E^*BG using descent is even interesting for the case of PSL_2, but the actual answer is well known. To derive a genuinely new result, we turned to the exceptional group G_2, which possesses a spherical fibration $BSL_3 \to BG_2$ with fiber an affine quadric homotopic to S^6. Here ordinary cohomology and K-theory are well known, but the result for a general complex oriented cohomology theory is new (though partial information [an associated graded for MU^*BG_2, the universal complex oriented cohomology theory] had been computed by Kono and Yagita using an Atiyah-Hirtzebruch spectral sequence [KY]). Note that for the descent spectral sequence, the differential $d : E_1 \to E_1$ is defined in terms of the formal group law, so the computation can be carried out for a general complex oriented cohomology theory [FG2].

In fact, the descent spectral sequence for $PSL_2 = SO(3)$ captures most of the difficulty for that of G_2. Namely, both spectral sequences collapse at E_2, and both E_2 terms possess 2-torsion. As E^*BPSL_2 is 2-torsion free, the spectral sequences converge to a non-trivial associate graded for E^*BPSL_2. However, this associated graded is much closer to the actual answer than the one arrived at via the Atiyah-Hirtzebruch spectral sequence. The module structure of the descent spectral sequence reveals that the apparent $2-$torsion is just the residue of the obvious relation $c_3 = \widetilde{c_3}$ due to the standard representation being self dual. Similarly, the $2-$torsion in E_2 for BG_2 is the residue of the relation $c_7 = \widetilde{c_7}$.

Furthermore, while this computation and spectral sequence works for any complex oriented cohomology theory, it does not give the correct answer for algebraic geometric versions of these theories such as higher Chow groups/motivic cohomology. This is because the affine quadric bundle, though trivial étale locally is not trivial in the Nisnevich topology [Vo]. Because of this, the above spectral sequence for Chow theory converges to étale cohomology of BG_2 and not motivic cohomology.

However, the computation, at least for ordinary Chow rings is possible by describing what happens for algebraic cobordism of quadric bundles and then relating this to Chow groups using the "Conner-Floyd theorem" of Morel and Levine[ML], namely that $CH^nX = MGL^{2n,n}X \otimes_{MGL(pt)} \mathbb{Z}$. This determines the Chow groups—but not the higher Chow groups—in terms of algebraic cobordism. The exact extent to which the descent spectral sequence fails to converge to motivic cohomology is of special interest as Chow groups and higher Chow groups are notoriously difficult to compute (for example, this difference could be explained in terms of the existence of higher differentials in the descent spectral sequence). CH^*BG_2 itself is known (by an ingenious set of computations using partial information coming from subgroups of G_2 [Gu]), so our computation is of interest for its relationship to higher Chow groups.

G_2 is certainly not the only group possessing a useful spherical fibration, so this is just the first of a series of papers exploiting the idea of the descent spectral sequence. In a sequel to this paper, I intend to use this technique to compute MU^*Spin and hence CH^*Spin as well as derive my old results on $CH^*BSO(2n)$ from a computation of $MU^*BSO(2n)$ [F] (an associated graded of $MU^*BSO(2n)$ has been determined by Yagita [Y2] but he didn't compute the relations).

3. BSO as an extension of BO by BSp

I will now describe additional joint work with Ian Grojnowski [FG]. We define a new decomposition of the classifying space $BSO(2n)$ (and also the classifying space of its double cover $BSpin(2n)$) which shows the existence a copy of $BSp(2n)$ sitting inside $BSO(2n)$ in a highly non-geometric manner. Remarkably, this gives a surjection of the cohomology of BSO(2n) onto BSp(2n) for *any* cohomology theory.

This is a surprising result—there is no homomorphism of groups from $Sp(2n)$ to $SO(2n)$. However, there are many tantalizing similarities between $SO(2n)$, and $Sp(2n)$. What we have done is find an explanation of these similarities in terms of an exact sequence of their classifying spaces (in the appropriate homotopy category).

Roughly speaking, this copy of $BSp(2n)$ measures the difference between $BO(2n)$ and $BSO(2n)$. Away from the prime 2, this is easy to say precisely:

Theorem 3.1. *Let $E^*(-)$ be a ring valued cohomology theory, such that $1/2$ is an element of $k = E^0(pt)$. Then*

$$E^*(BSO(2n)) = E^*(BO(2n)) \oplus y_n E^*(BSp(2n))$$

where y_n is in $E^{2n}(BSO(2n))$.

When $1/2 \in k$, all three terms in the theorem have been computed for most cohomology theories. However, even when all three terms were known, the relation with $BSp(2n)$ had not been observed.

If $x \in O(2n) - SO(2n)$, then x induces an automorphism θ on $BO(2n)$ by conjugation. As x^2 is homotopic to the identity, this is an involution up to homotopy. So if $1/2 \in k$, we can decompose $E^*(BSO(2n))$ into its eigenspaces. It is immediate from the Leray-Serre spectral sequence that the $+1$ eigenspace is just $E^*(BO(2n), k)$, so the content of the theorem above is the identification of the -1 eigenspace with $E^*(BSp(2n), k)$.

Remarkably, the relation between all three cohomology groups persists at the prime 2. Hence, this relationship gives a completely different method for computing $MU^*BSO(2n)$.

The precise statement can be found in our paper. Rather than explain the complicated patterns caused by 2 torsion, let me explain another feature of the connection between $Sp(2n)$, $SO(2n)$, and $O(2n)$ which is in fact a major ingredient in the proof.

Consider the symmetric space $Gl(2n)/SO(2n)$. The Chow ring of this space was computed in [F3].The main ingredient in [FG] is a conceptually new proof and explanation of this calculation. This symmetric space is a double cover of $Gl(2n)/O(2n)$, and the group of upper triangular matrices - the Borel subgroup B - acts on both of them. An orbit of B on $Gl(2n)/O(2n)$ is homotopy equivalent to a product of circles. The inverse image of such an orbit in $GL(2n)/SO(2n)$ is either a single orbit of the action of B or splits into disjoint copies of a certain canonical orbit.

The orbits whose double covers split are parameterized by symmetric fixed point free permutations on $2n$ letters. This is precisely the set that parametrizes B-orbits on $Gl(2n)/Sp(2n)$. Moreover, the attaching maps between B-orbits are completely determined by this Weyl group data. That is, the attaching maps for this subset of $Gl(2n)/SO(2n)$ and the attaching maps for $Gl(2n)/Sp(2n)$ precisely match up.

However, it *not* the case that there is a copy of $Gl(2n)/Sp(2n)$ sitting inside $Gl(2n)/SO(2n)$ —there are ‘holes’ in the pattern of B-orbits. Nonetheless, we can use this decomposition to explicitly construct a long exact sequence relating these three symmetric spaces. Using this we prove the refinement of theorem 3.1 in which 2-torsion is considered.

When the appropriate version of the theorem of [FG] is applied to the Chow rings of these three symmetric spaces, one recovers a another proof of the main theorem of [F3]:

Theorem 3.2.
$$CH^*(Gl(2n)/SO(2n)) = \mathbb{Z} \oplus y_n\mathbb{Z}$$
where $y_n \in CH^n(Gl(2n)/SO(2n)$.

The original proof was quite different and gave additional information that is not readily accessible from the new proof. It used an inductive argument to decompose $Gl(2n)/SO(2n)$ into $2n$ subvarieties, each of which is trivially fibered over $Gl(2n-2)/SO(2n-2)$ with fiber an open subset of affine space. This gives a completely different interpretation of the class y_n. Namely, it is a transfer via a series of projection maps of the class y_1 in $CH^*(Gl(2)/SO(2))$, which is just $CH^*(\mathbb{CP}^1)$.

In contrast, the new proof is as follows.
$$CH^*(Gl(2n)/SO(2n)) = CH^*(Gl(2n)/O(2n)) + y_nCH^*Gl(2n)/Sp(2n))$$
by the [FG] theorem. But $Gl(2n)/O(2n)$ is the set of non-degenerate symmetric $2n\times 2n$ matrices, while $Gl(2n)/Sp(2n)$ is the set of non-degenerate skew symmetric $2n \times 2n$ matrices. These are both open subsets of affine space, and hence both have Chow rings isomorphic to $\mathbb{Z}$.

4. A more traditional approach to Chow theory: $CH^*BSO(2n)$

Theorems 3.1 and 3.2 are computationally useful. For example, they enable the determination of the Chow ring of $BSO(2n)$:

Theorem 4.1.
$$CH^*(BSO(2n)) \cong \mathbb{Z}[c_2, c_3, ..., c_{2n}, y_n]/(2c_{odd}, y_n.c_{odd}, y_n^2 + (-1)^n 2^{2n-2} c_{2n}),$$
where y_n *maps to* $2^{n-1}\chi$ *in cohomology.*

This is the main theorem of my Ph.D. thesis [F4]. Using a theorem of my advisor Totaro [T] $CH^*Gl(2n)/SO(2n)$ transgresses to give generators of $CH^*BSO(2n)$ as a module over $CH^*BGl(2n) \cong \mathbb{Z}[c_1, ..., c_{2n}]$. The content in the theorem above is the relations satisfied by these generators.

The class $y_n \in CH^*Gl(2n)/SO(2n)$ is identical to the characteristic class defined by Edidin and Graham in [EG], and is the closest one can come to an Euler class in Chow rings.

The representation ring of $SO(2n)$ is generated by exterior powers of the standard representation along with one other representation D_n^+, which is the representation whose highest weight vector is twice that of the half spin representation. It is natural to wonder if the class y_n is generated by $c_n(D_n^+)$ modulo Chern classes of the standard representation. However, I show in [F4] that $c_n(D_n^+)$ only generates $2^{n-2}(n-1)!\, y_n$ and not y_n itself.

Another application of these ideas is to give a completely different method from the one in section 2 for explicit computation of the cobordism ring $MU^*(BSO(2n))$, and the r'th Morava K-theory $K\langle r\rangle^*BSO(2n)$ [F]. These rings were previously known only for $n \leq 3$ [I], stably ("$n = \infty$") [KLW], and modulo relations [Y2].

However, the rings $K\langle r\rangle^*BO(2n)$, and $K\langle r\rangle^*BSp(2n)$ were known—the first was determined by Wilson [W] and $K\langle r\rangle^*BSp(2n)$ is immediately from the fact that $H^*BSp(2n)$ is torsion free.

The theorem of [FG] can be used to piece these results together to form $K\langle r\rangle^*BSO(2n)$.

5. $RO(G)$ graded theory

I am also working on a project with Laura Scull at the University of British Columbia to understand the relationship between equivariant concepts in algebraic geometry and algebraic topology via a detailed examination of equivariant resolution of singularities for singular subvarieties with group action.

In algebraic geometry, the most widely known definition of equivariant intersection theory is that of Edidin and Graham [EG2]. They define $CH^*_G X$ by considering the Borel construction $X \times_G EG$ on X as an *Ind* algebraic variety, and letting $CH^*_G X$ be $CH^* X \times_G EG$, the inverse limit of the ring of algebraic cycles on finite dimensional approximations of the Borel construction mod rational equivalence (this is *all* algebraic cycles, not just G-invariant cycles).

In algebraic topology, there is also the notion of an $RO(G)$-graded equivariant theory [M] [GLM]. In such a theory, one can also suspend by spheres S^V which are the one point compactifications of representations V of G. The cohomology of the Borel construction is the $\mathbb{Z}$-graded part of an $RO(G)$-graded cohomology theory.

We hope to understand the compatibility between these two types of constructions in algebraic geometry using equivariant resolution of singularities [K]. The literature contains many versions of equivariant theories that may be relevant. For example Mark Levine has a new equivariant intersection theory which behaves well under base change, and hence may well be closer to algebraic topological notions of equivariant theories than the currently available notion of Edidin and Graham. This project as a whole is related to the absence of descent in algebraic K-theory, and attempts to repair this by G. Carlsson, Morel and Voevodsky.

References

[B] F. Bogomolov, The Brauer group of quotient spaces of linear representations, *Math. USSR Izvestia*, Vol 30 (1988) 455-485.

[EG] D. Edidin and W. Graham, Characteristic classes and quadric bundles, *Duke Mathematical Journal*, Vol 78 no 2 (1995) 277–299.

[EG2] D. Edidin and W. Graham, Equivariant intersection theory, *Invent. Math.* 131 no 3 (1998) 595–634.

[F] R. Field, MU^*BSpin, CH^*BSpin and related phenomena, in preparation.

[F3] R. Field, The Chow ring of the symmetric space $Gl(2n,\mathbb{C})/SO(2n,\mathbb{C})$, math.AG/0103139, submitted to *Transformation Groups*.

[F4] R. Field, The Chow ring of the classifying space $BSO(2n,\mathbb{C})$, math.AG/0411424, submitted to *Duke Mathematical Journal*

[FG] R. Field and I.Grojnowski, $BSO(2n)$ as an extension of $BO(2n)$ by $BSp(2n)$, 2002, preprint.

[FG2] R. Field and I.Grojnowski, MU^*BG_2, CH^*BG_2 and descent, 2007, preprint.

[GLM] L. Gaunce, Lewis, Jr., and M. Mandell, Equivariant universal coefficient and Kunneth spectral sequences, preprint.

[Gu] P.Guillot, The Chow rings of G_2 and $Spin(7)$, preprint, math.AG/0508122.

[I] K.Inoue, The Brown-Peterson cohomology of BSO(6), *J. Math. Kyoto Univ.*, 32 no 4 (1992) 655–666.

[K] F. Kirwan, Partial desingularisations of quotients of nonsingular varieties and their Betti numbers, *Annals of Mathematics*, 2nd ser vol 122 no 1 (1985) 41–85.

[KLW] N. Kitchloo, G. Laures, and W.S. Wilson, The Morava K-theory of spaces related to BO, *Advances in Mathematics*, 189(1) (2004) 192–236.

[KY] A.Kono and N.Yagita, Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups, *Trans. Amer. Math. Soc.*, 339 no 2 (1993), 781–798.

[K] I. Kriz, Morava K-theory of classifying spaces: some calculations, *Topology* 36 no 6 (1997), 1247–1273.

[M] J.P. May, An introduction to equivariant stable homotopy, *Equivariant Homotopy and Cohomology Theory*, CBMS no 91 (1996) 79–96.

[ML] M.Levine and F.Morel, *Algebraic Cobordism*, manuscript.

[MV] F.Morel and V.Voevodsky, A_1-homotopy theory of schemes, *Inst. Hautes Études Sci. Publ. Math.*, no 90:45–143, 1999.

[RV] L.Rojas and A.Vistoli, On the Chow rings of classifying spaces for classical groups, preprint.

[T] B. Totaro, The Chow ring of a classifying space, *Proc.Symp.Pure Math., K-Theory* Seattle, 1997.

[W] W.S. Wilson, The complex cobordism of $BO(n)$, *J.London Math.Soc.* 29 (1984), 352–366.

[Ve] G.Vezzosi, On the Chow ring of the classifying stack of $PGL_{3,\mathbb{C}}$, preprint, 1999.

[Vo] V.Voevodsky, The Milnor conjecture, preprint.

[Y] N.Yagita, Examples for the mod p motivic cohomology of classifying spaces, *Trans. Amer. Math. Soc.*355:4427-4450, 2003.

[Y2] N. Yagita, Applications of Atiyah-Hirzebruch spectral sequences for motivic cobordisms, preprint.

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