4 Basic Probability

4.1 Basic Probability Concepts

Definition: For a random phenomenon, the **sample space** is the set of all possible outcomes.

Example: Suppose items from an assembly line are sampled until a nondefective item is found. Let N denote a nondefective item, and let D denote a defective item. *Assume, hypothetically, an infinite number of defective items and at least one non-defective item.* Determine the sample space of items sampled.

Now, consider probability.

Example: According to the American Red Cross, Greensboro Chapter, 42% of Americans have type A blood.

Example: Suppose that P(rain)=0.2. What is P(no rain)?

The above examples use the **complement** rule; i.e., $P(A') = 1 - P(A)$.

Definition: Two events are **mutually exclusive** or **disjoint** if they cannot occur simultaneously.

Addition rule for mutually exclusive events: If events A and B are mutually exclusive, then $P(A \textbf{ or } B) = P(A) + P(B)$.

Example: Suppose a six-sided die is rolled once. Let $A =$\{roll one\}, and let $B =$\{roll two\}.

Example: Suppose that in a city, 72% of the population got the flu shot, 33% of the population got the flu, and 18% of the population got the flu shot and the flu. Determine P(flu shot **or** flu).

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4.2 Conditional Probability

Conditional Probability

Example: (fictitious) Suppose that in Rhode Island, there are 100,000 college students. Among these 100,000 Rhode Island college students, 10,000 attend (fictitious) Laplace-Fisher University (which has no out-of-state students) and exactly half of the Rhode Island students are female. Among the 10,000 Laplace-Fisher students, 6,000 are female. Define the events
$F = \{\text{Student is female}\}$ and $L = \{\text{Student attends Laplace-Fisher University}\}$.

(a) Determine the probability that a randomly selected Rhode Island college student attends Laplace-Fisher University. In other words, what proportion of Rhode Island college students attend Laplace-Fisher University?

(b) Determine the probability that a randomly selected Rhode Island college student attends Laplace-Fisher University AND is female.
In other words, what proportion of Rhode Island college students attend Laplace-Fisher University AND are female?

(c) Determine the probability that a randomly selected Laplace-Fisher University student is female.
In other words, determine the probability that a randomly selected Rhode Island college student is female, **given** that the student attends Laplace-Fisher University.

In other words, determine the probability that a randomly selected Rhode Island college student is female, **conditional** that the student attends Laplace-Fisher University.

In other words, what proportion of Laplace-Fisher students are female?

□

Definition: For events A and B, the **conditional probability** of event A, given that event B has occurred, is

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}.$$

General Multiplication Rule *(always true)***:** For outcomes A and B, then $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$.

This above definition is *always true*, regardless of whether A and B are independent or dependent.

Example: On a hot, sunny day – The probability that a student is wearing an open-toe shoe on the **left** foot and an open-toe shoe on the **right** foot might be around ____________________. Given that that a student is wearing an open-toe shoe on the **left** foot, the probability that the student is wearing an open-toe shoe on the **right** foot should be around ____________________.

□

Independence

Use a tree diagram to show the resulting **pairs** when a fair coin is tossed once and a fair **four**-sided die is rolled once.

Multiplication rule for independent events: If outcomes A and B are **independent**, then $P(A \text{ and } B) = P(A)P(B)$.

Example: *(Know this example.)*

(a) What is the likelihood of any particular airplane engine failing during flight?

(b) If failure status of an engine is *independent* of all other engines, what is the likelihood that all three engines fail in a three-engine plane?

(c) What happened to a three-engine jet from Miami headed to Nassau, Bahamas (Eastern Air Lines, Flight 855, May 5, 1983)?

□

Example: ♠♡♣♢ In a standard deck of 52 shuffled cards, determine the probability that the top card and bottom card are both diamonds. *Notation:* $T = \{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$.

Definition: (Independence in terms of conditional probabilities) The following four statements are equivalent.

(a) Events A and B are **independent**.

(b) $P(A \text{ and } B) = P(A)P(B)$

(c) $P(A|B) = P(A)$

(d) $P(B|A) = P(B)$

Example: *Revisit earlier example.* According to the American Red Cross, Greensboro Chapter, 42% of Americans have type A blood, 85% of Americans have the Rh factor (positive), and 35.7% of Americans have type $A+$ blood. Are the events {Person has type A blood} and {Person has Rh factor} independent?

□

Example: *Revisit earlier example.* ♠♡♣♢ In a standard deck of 52 shuffled cards, let $T = \{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$. Determine the following probabilities and discuss whether or not T and B are independent.

(a) $P(B)$

(b) $P(B|T)$

(c) $P(T)$

(d) $P(T|B)$

(e) $P(T \text{ and } B)$

(f) $P(T)P(B)$

□

Example: A fair coin is tossed twice. Let $H_1 = \{\text{First toss is heads.}\}$, and $H_2 = \{\text{Second toss is heads.}\}$. Determine $P(H_2|H_1)$.

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Example: *Revisit flu shot example.* Suppose that in a city, 72% of the population got the flu shot, 33% of the population got the flu, and 18% of the population got the flu shot and the flu. *Notation:* Let $S = \{\text{Person got the flu shot}\}$, and let $F = \{\text{Person got the flu}\}$.

(a) What proportion of the vaccinated population got the flu?

(b) What proportion of the vaccinated population did not get the flu?

(c) What proportion of the non-vaccinated population got the flu?

(d) What proportion of the non-vaccinated population did not get the flu?

(e) Given that a person did not get the flu, what is the probability that the person was vaccinated?

(f) Are the outcomes "had flu shot" and "contracted the flu" independent?

(g) Determine P(flu shot **or** flu), *a previous question.*

□

Example: Suppose that 5% of a population is in the armed services, 20% of the armed services is female, and 50% of the population is female. *Notation:* Let $A = \{\text{Person is in the armed services}\}$ and $F = \{\text{Person is female}\}$.

What proportion of the population consists of **females** in the armed services?

□

4.3 Bayes' Theorem

Example: Define the events
$D = \{\text{Athlete uses DRUGS}\}$, and
$T = \{\text{Athlete TESTS positive for drugs}\}$.
For the athletes in a particular country, suppose that **10%** of the athletes use drugs, **60%** of the drug-users test positive for drugs, and **1%** of the non-drug-users test positive for drugs.

(a) State the probabilities in terms of the notation D and T.

(b) What proportion of the athletes would test **positive** for drugs? Alternatively, what is the probability that a randomly selected athlete would test **positive** for drugs?

(c) What proportion of the athletes would test **negative** for drugs? Alternatively, what is the probability that a randomly selected athlete would test **negative** for drugs?

(d) What proportion of the **drug-users** would test **negative** for drugs? Alternatively, what is the probability that a randomly selected **drug-user** would test **negative** for drugs?

(e) Given that an athlete tested **positive** for drugs, what is the likelihood that the athlete actually **uses** drugs?

(f) Using a *tree diagram*, repeat part (e).

(g) Given that an athlete tested **positive** for drugs, what is the likelihood that the athlete does **NOT use** drugs?

In general, **Bayes' theorem** is the following:

Let $B_1, B_2, \ldots, B_k$ be **mutually exclusive** such that $\cup_{i=1}^{k} B_i = \mathcal{S}$. Then,

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)},$$

for $i = 1, \ldots, k$.

Read pp. 176–177, Appendix E4: Using Microsoft Excel for Basic Probability.