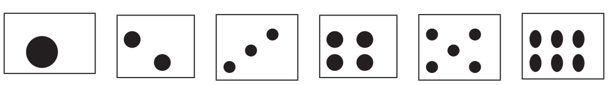
5 Some Important Discrete Probability Distributions

5.1 The Probability Distribution for a Discrete Random Variable

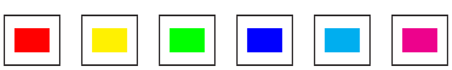
What are some **continuous** random variables?

Discrete distributions

Example: *(Discrete case)* Roll a (not necessarily fair) six-sided die once. The possible outcomes are the **faces** (i.e., dots) on the die.



A different die might have six colors for the six sides (like a Rubik's cube).



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Example: *(Discrete case)* Toss a (not necessarily fair) coin once.

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Definition: The **probability distribution** of a **discrete** random variable X consists of the possible values of X along with their associated probabilities.

We sometimes use the terminology **population distribution**.

Example: Fair (six-sided) die. Let X be the numerical outcome. Determine the probability distribution of X, graph this probability distribution, and compute the mean of the probability distribution.

Example: Toss a fair coin 3 times. Let X = number of heads.

Let Y = number of matches among the three tosses.

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Terminology: The **expected value** of X is the **mean** of a random variable X.

Example: (hypothetical) Suppose an airline often overbooks flights, because past experience shows that some passengers fail to show.

Let the random variable X be the number of passengers who cannot be boarded because there are more passengers than seats.

x	$P(x)$
0	0.6
1	0.2
2	0.1
3	0.1
sum	1

(a) Compute the expected value of X.

(b) Suppose each unboarded passenger costs the airline $100 in a ticket voucher. Compute the average cost to the airline per flight in ticket vouchers.

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Brief review of means and standard deviations

The **mean**, μ, of a random variable is the **average** of all outcomes in the population, and is the limiting value of $\bar{X}$ as n gets large.

The **standard deviation**, σ, of a random variable measures the **spread** of all outcomes in the population, and is the limiting value of s as n gets large.

The **variance**, σ^2, of a random variable also measures the spread in the population, and is the limiting value of s^2 as n gets large.

σ is more intuitive than σ^2, partly because σ has the same units as the original data.

5.3 Binomial Distribution

Example: Toss an unfair coin 5 times, where $\pi = P(\text{heads}) = 0.4$.
Let X be the number of heads.
Suppose we want to determine $P(X = 2)$.

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Factorials: $m! = m(m-1)(m-2)\cdots 1$ for positive integers m.

Example: Compute 4!, 5!, 1!, and 0!.

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A **Bernoulli** trial can have two possible outcomes, *success* or *failure*.

Definition of a **binomial** random variable X.

1. Let n, the number of Bernoulli trials, be **fixed** in advance.
2. The Bernoulli trials are **independent**.
3. The probability of success of a Bernoulli trial is π, which is the same for all observations.

Let X be the number of *successes*. Then, X is a binomial(n, π) random variable.

$$P(X = x) = \frac{n!}{x!\ (n-x)!}\ \pi^x\ (1-\pi)^{n-x}, \quad \text{for } x = 0, 1, 2, \dots, n$$

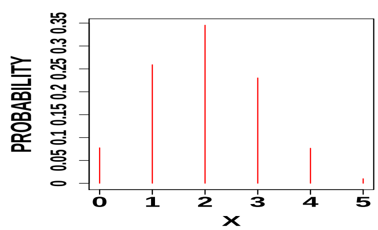
Example: $n = 5$ tosses of an unfair coin.

Assume $\pi = P(heads) = 0.4$ (OR 40% Democrats from a huge population).

Let X be the number of heads.

Determine the probability distribution of X and construct the line graph.

x	$P(x)$
0	0.0776
1	0.2592
2	0.3456
3	0.2304
4	0.0768
5	0.01024



Determine the probability of obtaining *at least one* heads among the five coin tosses.

In 100 tosses of this coin, on average, how many heads do you expect?

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Mean and standard deviation of a binomial random

variable

$\mu = n\pi, \qquad \sigma^2 = n\pi(1-\pi), \qquad \sigma = \sqrt{n\pi(1-\pi)}$

Example: *Revisit.* Let $X \sim \text{Binomial}(n = 5, \pi = 0.4)$. Compute the *mean* and *standard deviation* of X.

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Example: Consider a *huge* population where 30% of the people are Democrats. Let X be the number of Democrats in a sample of size 1000. Compute the mean and standard deviation of X.

For large sample sizes (i.e., $n\pi \geq 5$ and $n(1-\pi) \geq 5$), a *binomial random variable* and a *sample proportion* are approximately **normally distributed** by the **Central Limit Theorem**.

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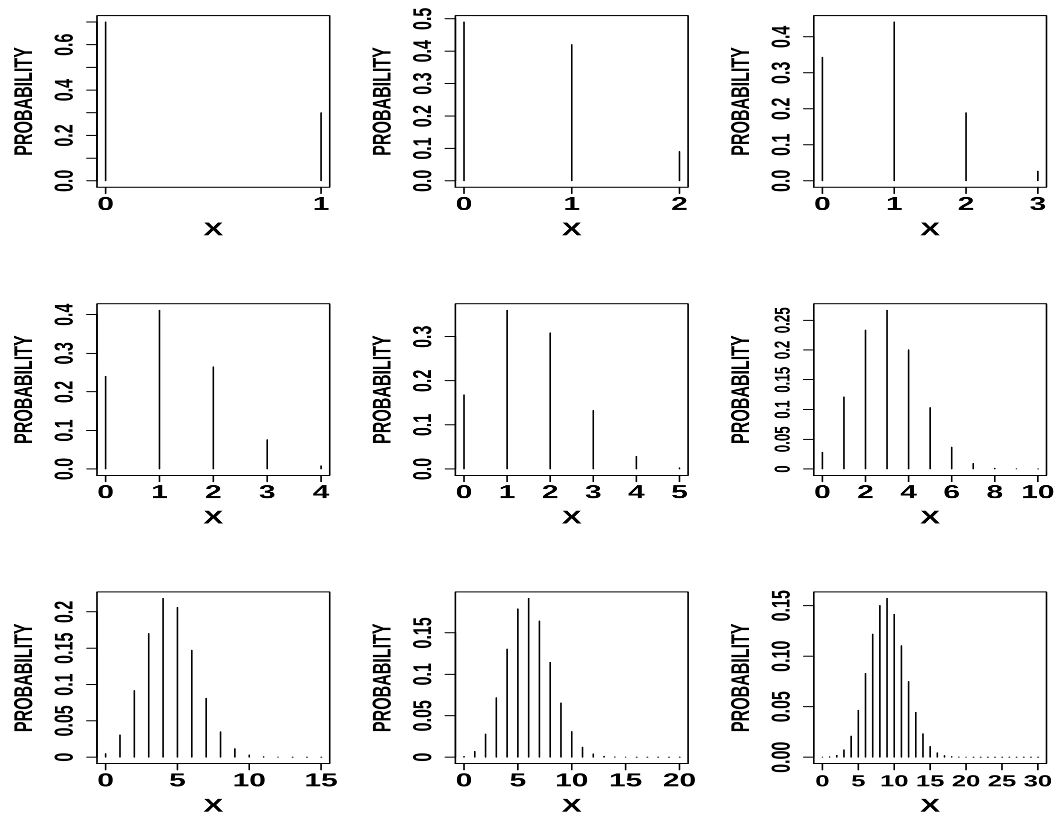
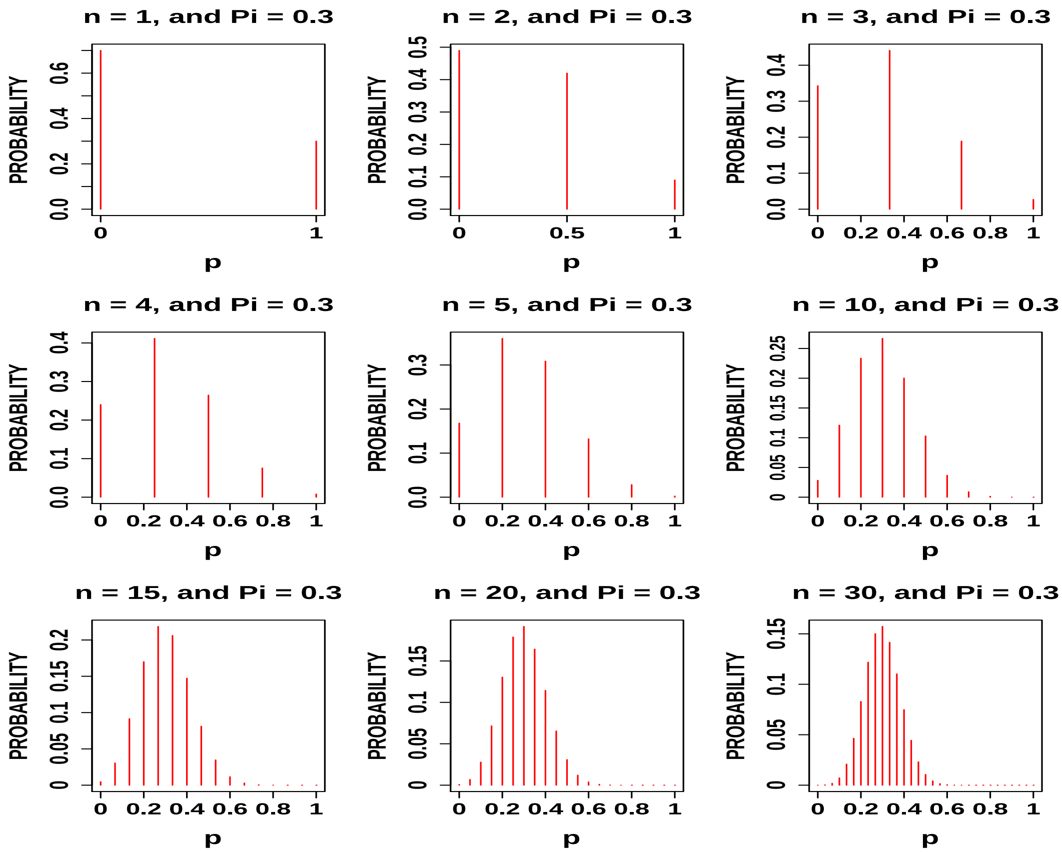
Example: Viewing the Central Limit Theorem.

(a) Consider the graphs below for **binomial** random variables, using $\pi = 0.3$ and $n = 1, 2, 3, 4, 5, 10, 15, 20$, and 30.



(b) Consider the graphs below for **sample proportions**, p, using $\pi = 0.3$ and $n = 1, 2, 3, 4, 5, 10, 15, 20,$ and 30.



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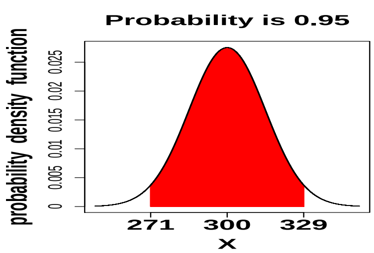
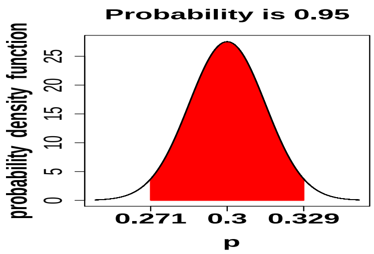
Example: *The Democrats.*

(a) Use the 95% part of the **empirical rule** on the *binomial random variable.*



(b) Use the 95% part of the **empirical rule** on the *sample proportion.*



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Read p. 194, Microsoft Excel.

5.4 Poisson Distribution

Consider the **Binomial($\boldsymbol{n, \pi}$)** distribution, such that $\boldsymbol{n}$ is **huge**, $\boldsymbol{\pi}$ is small, but $\boldsymbol{n\pi}$ is **moderate** (i.e., neither huge nor small).

Example: *Radioactive decay.* Consider a radioactive substance containing 3,000,000 atoms, such that decaying atoms are **independent** of each other, and
$\pi = P(\text{A particular atom decays in the next day}) = 1/1,000,000.$
Compute the **mean** number of atomic decays in the next day.

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Consider letting $n \to \infty$ and $\pi \to 0$ such that $n\pi \to \lambda$, a positive constant, where $X \sim \text{Binomial}(n,\ \pi)$.
In this limit, $X \sim \text{Poisson}(\lambda)$.

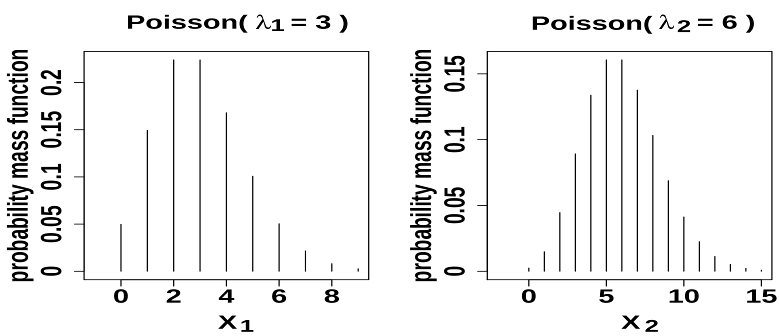
If $X \sim \text{Poisson}(\lambda)$ for $\lambda > 0$, then

$$P(X = x) = \frac{1}{x!}\,\lambda^x\, e^{-\lambda}, \text{ for } x = 0, 1, 2, \ldots$$

Example: *Revisit radioactive decay.* Consider a radioactive substance containing 3,000,000 atoms, such decaying atoms are **independent** of each other, and $\pi = P(\text{A particular atom decays in the next day}) = 1/1,000,000$.

(a) Let X_1 be the number of decays in **one** day. Determine the probability that **at least one** atom decays in the **next day**.

(b) Let X_2 be the number of decays in **two** days. Determine the probability that **at least one** atom decays in the next **two** days.



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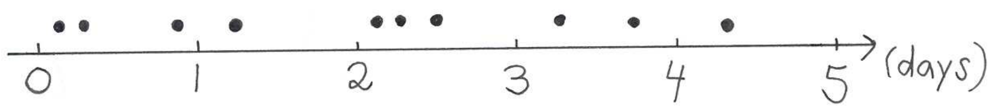
Remark: Typically, when a **Binomial**(n, π) distribution is reasonably approximated by a **Poisson**(λ) distribution, n and π are difficult to determine (or estimate), but λ can be estimated from the data. **How?**

The Poisson Process is explained by the following:

(a) The probability of a **success** (such as a radioactive decay) in the next day is **independent** of its past.

(b) The **mean** of a Poisson process based on **two** days is **twice** as large as the **mean** of the same Poisson process based on **one** day.



Example: Consider the number of recombinations (breaks) in DNA (chromosome pairs) when DNA strands are passed to offspring.

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Read p. 200, Microsoft Excel.

Read pp. 212–214, Appendix E5: Using Microsoft Excel for Discrete Probability Distributions.