# 4 Probability

## 4.1 Basic Concepts of Probability

We will understand what the following statements mean:

1. The probability/chance of rain tomorrow is 20%.
2. The probability of inheriting Huntington's disease, if exactly one parent has the disease, is 50%.
3. If two carriers of cystic fibrosis have a child, the child has a 25% chance of having the disease.
4. Only one in every 1,000,000 individuals has the blood type found at the crime scene. If an (innocent) individual is selected at random, the probability that she/he has DNA which match the blood at the crime scene is 1 out of 1,000,000.
5. Suppose we are interested in knowing if the new drug works better than the old drug. In a sample of 200 people, suppose 105 favor the new drug. Do we conclude the new drug is better? This is equivalent to tossing a coin. Now suppose 120? Now suppose 190?

In a **sample survey**, the sample is known and the population is unknown. We use the sample (and the **statistics**) to make inferences about the population.

**Example:**

However, in **probability**, the population is known, and we discuss properties of the sample based on probability.

**Example:**

**Definition:** For a random phenomenon, the **sample space** is the set of all possible outcomes.

**Example:** Suppose items from an assembly line are sampled until a nondefective item is found. Let $N$ denote a nondefective item, and let $D$ denote a defective item. *Assume, hypothetically, an infinite number of defective items and at least one non-defective item.* Determine the sample space of items sampled.

Now, consider probability again.

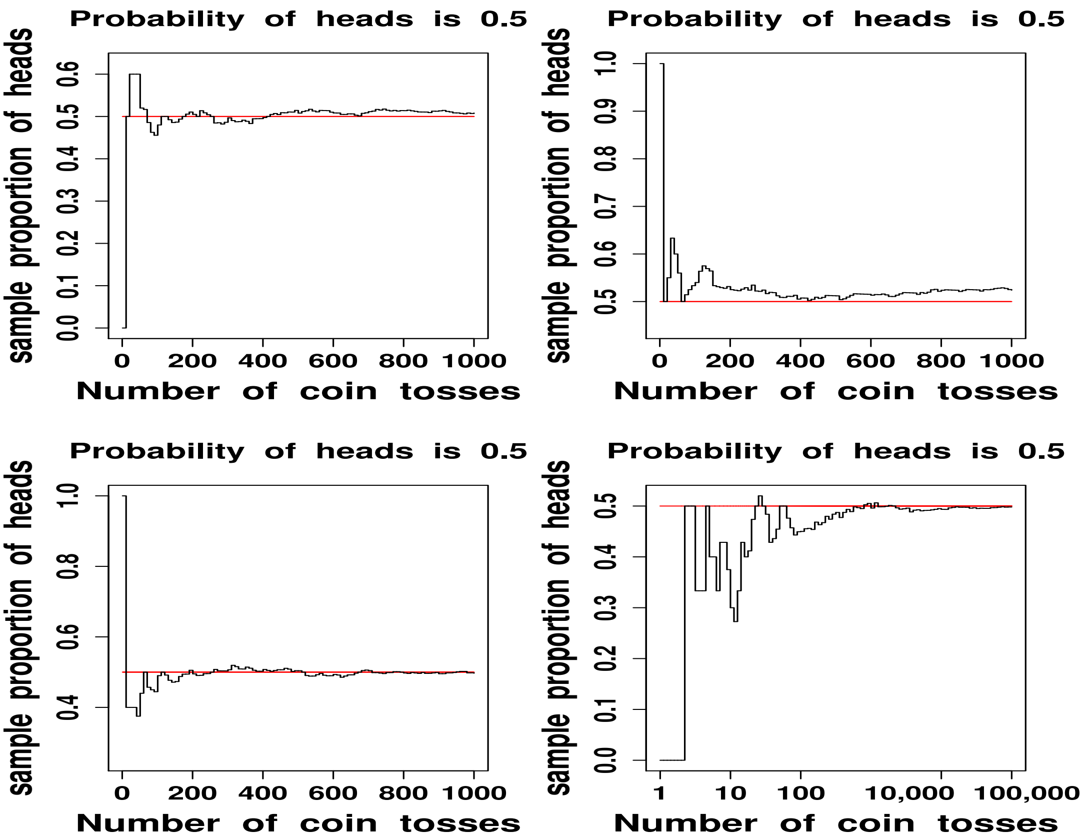
**Definition:** A **probability** is a number between 0 and 1, and reflects the likelihood of occurrence of some outcome.

**Example:** Suppose a coin is **fair**.

**Example:** Suppose a club has 6 females and 4 males.

What does it mean when we say that P(heads)=0.5 for a coin? *Alternative question:* How do we find P(heads) for a coin?

The graphs below represent the sample proportion of heads, in tosses of a fair coin, for a large number of tosses.




**More rigorous definition of *probability*, based on a *long run proportion*:** A **probability** of an outcome is the proportion of times that the outcome is expected to occur when the experiment is repeated many times under identical conditions.

A **law of large numbers** states that a sample proportion, $\hat{p}$, "converges" to the population proportion or probability, over a long run.

**Example:** What happens to the sample proportion of heads after many tosses of the coin?

**Example:** What happens to the sample proportion of females in the club (consisting of 6 females and 4 males) after many selections (WITH replacement)?

# 4.2 The Addition Rule and the Rule of Complements

**Example:** According to the American Red Cross, Greensboro Chapter, 42% of Americans have type $A$ blood.

**Example:** Suppose that P(rain)=0.2. What is P(no rain)?

The above examples use the **complement** rule; i.e., $P(A^c) = 1 - P(A)$.

**Definition:** Two events are **mutually exclusive** if they cannot occur simultaneously.

**Addition rule for mutually exclusive events:** If events $A$ and $B$ are mutually exclusive, then $P(A \textbf{ or } B) = P(A) + P(B)$.

**Example:** Suppose a six-sided die is rolled once. Let $A$ ={roll one}, and let $B$ ={roll two}.

**Example:** Suppose that in a city, 72% of the population got a flu shot, 33% of the population got the flu, and 18% of the population got a flu shot and the flu. Determine P(flu shot $\bigcup$ flu).

**General Addition Rule:** For events $A$ and $B$,

$$P(A \bigcup B) = P(A) + P(B) - P(A \textbf{ and } B).$$

# 4.3 Conditional Probability and the

# Multiplication Rule

**Definition:** Different trials of a random phenomenon are **independent** if the outcome of any one trial is not affected by the outcome of any other trial.

**Example:**

**Definition:** Trials which are not independent are called **dependent**.

**Example:**

**Example:** **Tree diagram** – Use a tree diagram to show the resulting **pairs** when a fair coin is tossed once and a fair **four**-sided die is rolled once.

**Multiplication rule for independent events:** If outcomes $A$ and $B$ are **independent**, then $P(A \text{ and } B) = P(A)P(B)$.

**Example:** *(Know this example.)*

**(a)** What is the likelihood of any particular airplane engine failing during flight?

**(b)** If failure status of an engine is *independent* of all other engines, what is the likelihood that all three engines fail in a three-engine plane?

**(c)** What happened to a three-engine jet from Miami headed to Nassau, Bahamas (Eastern Air Lines, Flight 855, May 5, 1983)?

□

## Conditional Probability: What's the Probability of B, Given A?

**Example:** (fictitious) Suppose that in Rhode Island, there are 100,000 college students. Among these 100,000 Rhode Island college students, 10,000 attend (fictitious) Laplace-Fisher University, and exactly half of the Rhode Island college students are female. Among the 10,000 Laplace-Fisher students, 6,000 are female. Define the events $F = \{\text{Student is female}\}$ and $L = \{\text{Student attends Laplace-Fisher University}\}$.

**(a)** Determine the probability that a randomly selected Rhode Island college student attends Laplace-Fisher University. In other words, what proportion of Rhode Island college students attend Laplace-Fisher University?

**(b)** Determine the probability that a randomly selected Rhode Island college student attends Laplace-Fisher University AND is female.
In other words, what proportion of Rhode Island college students attend Laplace-Fisher University AND are female?

**(c)** Determine the probability that a randomly selected Laplace-Fisher University student is female.
In other words, determine the probability that a randomly selected Rhode Island college student is female, **given** that the student attends Laplace-Fisher University.
In other words, determine the probability that a randomly selected Rhode Island college student is female, **conditional** that the student attends Laplace-Fisher University.
In other words, what proportion of Laplace-Fisher students are female?

□

**Definition:** For events $A$ and $B$, the **conditional probability** of event $B$, given that event $A$ has occurred, is

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}.$$

This above definition is *always true* whenever $P(A) > 0$, regardless of whether $A$ and $B$ are independent or dependent.

**Multiplication rule for conditional probabilities** *(always true)***:**
$P(A \text{ and } B) = P(A)P(B|A)$ if $P(A) > 0$. Similarly, $P(A \text{ and } B) = P(B)P(A|B)$ if $P(B) > 0$.

**Example:** $\spadesuit\heartsuit\clubsuit\diamondsuit$ In a standard deck of 52 shuffled cards, determine the probability that the top card and bottom card are both diamonds. *Notation:* $T = \{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$.

Here, we are sampling **withOUT** replacement.

**Example:** $\spadesuit\heartsuit\clubsuit\diamondsuit$ Suppose that in a standard deck of 52 shuffled cards, the cards are reshuffled after each draw. Determine the probability that the first two cards drawn are diamonds. *Notation:* $D_1 = \{\text{First card drawn is a diamond}\}$ and $D_2 = \{\text{Second card drawn is a diamond}\}$.

Here, we are sampling **WITH** replacement.

**Definition:** (Independence in terms of conditional probabilities) The following four statements are equivalent.

**(a)** Events $A$ and $B$ are **independent**.

**(b)** $P(A \text{ and } B) = P(A)P(B)$

**(c)** $P(A|B) = P(A)$

**(d)** $P(B|A) = P(B)$

**Example:** A fair coin is tossed twice. Let $H_1 = \{\text{First toss is heads.}\}$, and $H_2 = \{\text{Second toss is heads.}\}$. Determine $P(H_2|H_1)$.

□

**Example:** *Revisit earlier example.* According to the American Red Cross, Greensboro Chapter, 42% of Americans have type $A$ blood, 85% of Americans have the Rh factor (positive), and 35.7% of Americans have type $A+$ blood. Are the events {Person has type $A$ blood} and {Person has Rh factor} independent?

□

**Example:** *Revisit earlier example.* ♠♡♣♢ In a standard deck of 52 shuffled cards, let $T = \{\text{Top card is a diamond}\}$ and $B = \{\text{Bottom card is a diamond}\}$. Determine the following probabilities and discuss whether or not $T$ and $B$ are independent.

**(a)** $P(B)$

**(b)** $P(B|T)$

**(c)** $P(T)$

**(d)** $P(T|B)$

**(e)** $P(T \text{ and } B)$

**(f)** $P(T)P(B)$

□

**Example:** *Revisit flu shot example.* Suppose that in a city, 72% of the population got a flu shot, 33% of the population got the flu, and 18% of the population got a flu shot and the flu. *Notation:* Let $S = \{\text{Person got a flu shot}\}$, and let $F = \{\text{Person got the flu}\}$.

**(a)** What proportion of the vaccinated population got the flu?

**(b)** What proportion of the vaccinated population did not get the flu?

**(c)** What proportion of the non-vaccinated population got the flu?

**(d)** What proportion of the non-vaccinated population did not get the flu?

**(e)** Given that a person did not get the flu, what is the probability that the person was vaccinated?

**(f)** Are the outcomes "had flu shot" and "contracted the flu" independent?

□

**Example:** Suppose that 5% of a population is in the armed services, 20% of the armed services in this population is female, and 50% of the population is female. *Notation:* Let $A = \{\text{Person is in the armed services}\}$ and $F = \{\text{Person is female}\}$.

What proportion of the population consists of **females** in the armed services?

□

**Example:** The probability that a randomly selected adult in the United States is wearing a contact lens in the **left** eye and a contact lens in the **right** eye is around __________________. Given that that a randomly selected adult in the United States is wearing a contact lens in the **left** eye, the probability that this adult is wearing a contact lens in the **right** eye should be around __________________.

□

# 4.4 Counting

## The Fundamental Principle of Counting

**Example:** (revisit) A fair coin is tossed once, and a fair four-sided die is rolled once. What is the total number of possible outcomes (i.e., pairs of results)?

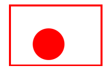
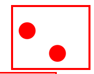
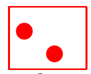
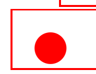
What is the probability of obtaining *heads* on the coin and *two* on the die?

**Example:** Suppose we roll a 3-sided red die once, and a 4-sided blue die once.

**(a)** List all possible outcomes (pairs of results).

**(b)** Determine the total number of outcomes (pairs of results).

**(c)** Assuming that the dice are fair, determine $P(\text{sum} = 4)$.

**(d)** Determine the total number of possible outcomes (where order matters) if we roll the 3-sided red die **five** times and the 4-sided blue die **two** times.
*"Where order matters" implies that rolling a*  *and a*  *on the first and second rolls differs from rolling a*  *and a*  *on the first and second rolls of the red die, respectively.*

**Example:** How many different types of pizzas can be made if **7** toppings are available?

**Example:** Suppose a club consists of **10** members. How many ways can we select one president, one chef, and one custodian, such that individuals **may** hold more than one title?

## Permutations

**Example:** Suppose a club consists of **10** members, and members may **NOT** hold more than one office.

**(a)** How many ways can we select one president, one vice-president, and one secretary?

**(b)** If Larry, Moe, and Curly are in this club, and officers are selected at random, what is the probability that Larry is president, Moe is vice-president, and Curly is secretary?

$\square$

Determine 4!, 5!, 1!, and 0!.

In general,

$$_nP_r = \frac{n!}{(n-r)!}$$

*Note:* With permutations, order **is** relevant.

## Combinations

**Example:** Suppose a club consists of **10** members. How many ways can we select **3** members to serve on the charity committee?

*Note:* A committee consisting of persons (A, B, C) is identical to (A, C, B), (B, A, C), (B, C, A), (C, A, B), and (C, B, A). Thus, order is irrelevant.

$$\text{In general,} \quad {}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

How many ways can we select **7** members **NOT** to serve on the charity committee?

**Example:** **Combinations** can be generated from **Pascal's triangle.** For example, ${}_5C_0 = 1$, ${}_5C_1 = 5$, ${}_5C_2 = 10$, ${}_5C_3 = 10$, ${}_5C_4 = 5$, and ${}_5C_5 = 1$.

$$
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 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\vdots & \vdots & & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

**Example:** ♠♡♣♢ Assume a shuffled standard deck of 52 cards.

**(a)** Determine the number of ways **5** cards can be drawn **WITH** replacement.

**(b)** Determine the number of ways **5** cards can be drawn **withOUT** replacement, where order is **irrelevant**.

**(c)** Determine the probability of drawing **all diamonds** in a 5-card draw.

**(d)** Determine the probability of drawing a **flush** in a 5-card draw.

**(e)** Determine the probability of drawing a **royal flush** in a 5-card draw.

□