4 Continuous Random Variables and Probability Distributions

4.1 Continuous Random Variables and Probability Density Functions

Continuous distributions

Definition: A random variable X is **continuous** if its set of possible values consists of interval(s) of numbers.

For a **continuous** distribution, we construct the **probability density function (pdf)** of X.

The **pdf** often is denoted $\boldsymbol{f(x)}$.

Rules for a continuous histogram.

(1) The total area under the pdf is 1; i.e., $P(\mathcal{S}) = \int_{-\infty}^{+\infty} f(x)\,dx = 1$.

(2) The **probability** of the random variable taking a value in the interval from "a" to "b" is the **area** under the pdf within this interval; i.e., $P(a < X < b) = \int_a^b f(x)\,dx$.

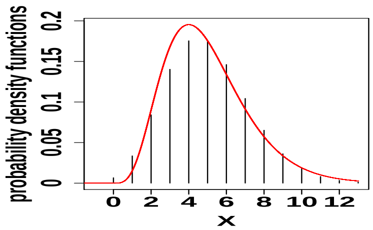
(3) The pdf is nonnegative; i.e., $f(x) \geq 0$, for all $x \in \Re$.

Suppose X is a (absolutely) continuous random variable, and a is a constant.

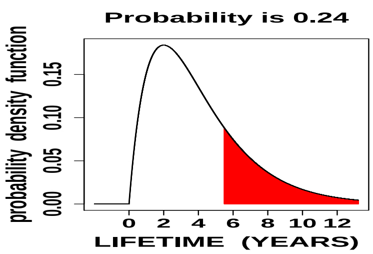
(a) What is $P(X = a)$?

(b) How does $P(X \leq a)$ compare with $P(X < a)$?

□



Example: Let X be the lifetime of a computer CPU in years, as shown in the graph below. Determine the probability that a new CPU lasts at least 5.5 years.



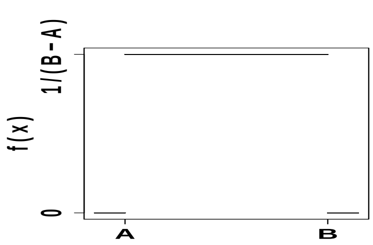
□

Uniform distribution

A **uniform** distribution has pdf

$$f(x) = \begin{cases} \frac{1}{B-A}, & \text{if } A \le x \le B \\ 0, & \text{otherwise} \end{cases}$$

for *real* constants A and B such that $A < B$.



Example: Show that the above **uniform pdf** is valid.

□

Example: Suppose $X \sim \text{Uniform}(A = 30,\ B = 40)$. Determine:

(a) $P(X < 32)$

(b) $P(37 < X < 39)$

(c) $P(31.27 < X < 33.27)$

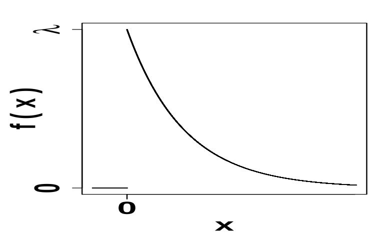
□

Exponential distribution (to be discussed in more detail in section 4.4)

An **exponential** distribution has pdf

$$f(x) = \begin{cases} \lambda\ e^{-\lambda\ x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

for a constant $\lambda > 0$. *You need NOT memorize this formula.*



Example: Show that the above **exponential pdf** is valid.

□

4.2 Cumulative Distribution Functions and Expected Values

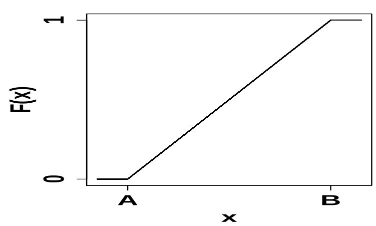
Definition: The **cumulative distribution function**, denoted **cdf**, $\boldsymbol{F(x)}$ for a continuous random variable X is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy, \ \forall x \in \Re,$$

where $f(\cdot)$ is the pdf of X.

Example: Let $X \sim \text{Uniform}(A,\ B)$.

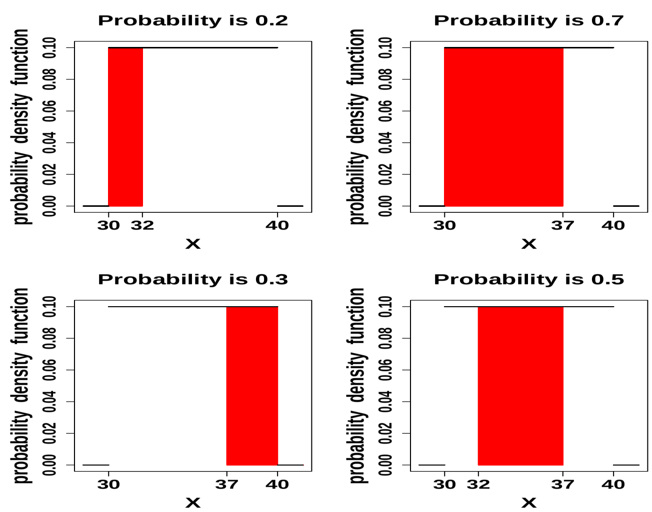
(a) Determine the **cdf** of X.



(b) Determine $P(X \leq 32)$, for $A = 30$ and $B = 40$, using $F(\cdot)$ from part (a).

(c) Determine $P(X > 37)$, for $A = 30$ and $B = 40$, using $F(\cdot)$ from part (a).

(d) Determine $P(32 < X < 37)$, for $A = 30$ and $B = 40$, using $F(\cdot)$ from part (a).



□

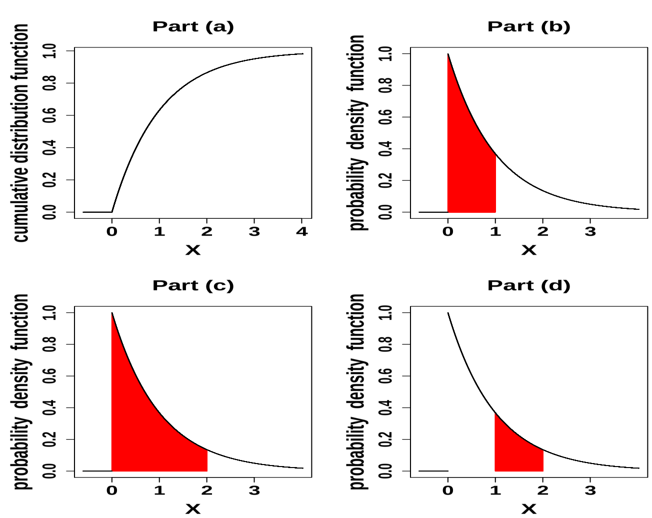
Example: Let $X \sim \text{Exponential}(\lambda)$.

(a) Determine the **cdf** of X.

(b) Determine $P(X \leq 1)$, for $\lambda = 1$, using $F(\cdot)$ from part (a).

(c) Determine $P(X \leq 2)$, for $\lambda = 1$, using $F(\cdot)$ from part (a).

(d) Determine $P(1 \leq X \leq 2)$, for $\lambda = 1$, using $F(\cdot)$ from part (a).



□

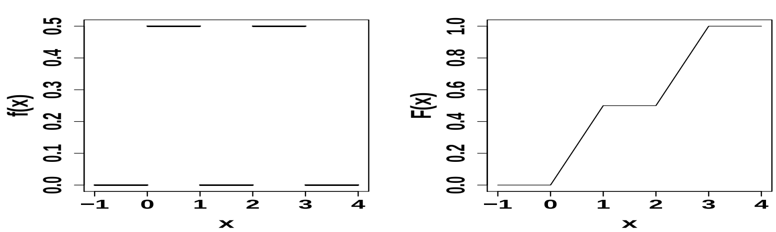
Suppose X is a continuous random variable with pdf $f(x)$ and cdf $F(x)$, then for all $x \in \Re$,

$$F(x) = \int_{-\infty}^{x} f(y)\, dy, \quad \text{and}$$

$$f(x) = \frac{dF(x)}{dx}, \text{ due to } \ldots$$

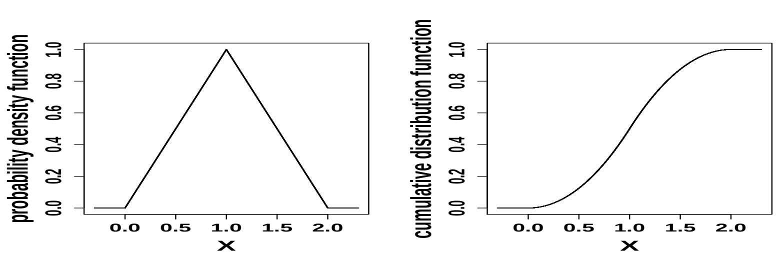
Remark: The **cdf** of a (absolutely) continuous random variable may have a countable number of nondifferential points.

Example: Consider the following **pdf** and corresponding **cdf**.



□

Example: *Triangular distribution.* The sum of **two** independent **Uniform(0, 1)** random variables produces a **triangular** random variable, whose **pdf** is shown below (left graph).



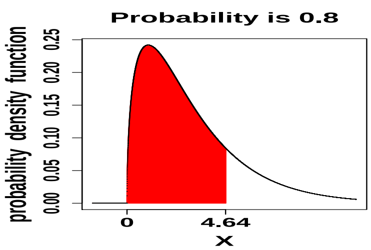
(a) Determine the formula for the **pdf** $\boldsymbol{f(x)}$.

(b) Determine the formula for the **cdf** $\boldsymbol{F(x)}$.

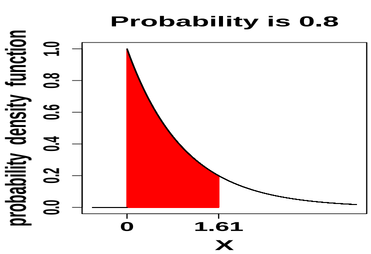
□

Percentiles of a Continuous Distribution

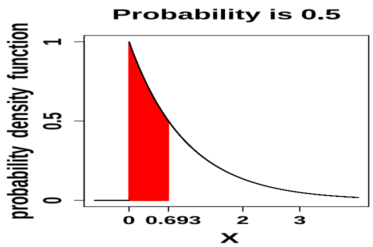
Example: For the following pdf, determine the 80th percentile.



Example: Find the **80th** percentile of an Exponential($\lambda = 1$) distribution.



Example: Find the **median** an Exponential($\lambda = 1$) distribution.



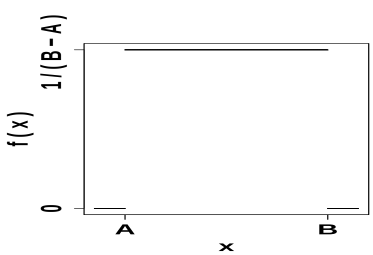
□

Expected Values

Definition: The **expected value** or **mean** of a continuous random variable X with **probability density function** $f(x)$ is

$$\mu_x = EX = \int_{-\infty}^{+\infty} x \; f(x) \; dx.$$

Example: Derive the **mean** of a Uniform(A, B) distribution.



□

For a **symmetric** random variable (i.e., $f(x)$ is symmetric) with a finite **mean**, compare the **mean** and **median**.

Example: Derive the **mean** of an Exponential(λ) distribution.

□

Definition: The **expected value** of a **function**, $h(x)$, of a continuous random variable X with **probability density function** $f(x)$ is
$Eh(X) = \int_{-\infty}^{+\infty} h(x)\ f(x)\ dx$.

Definition: If X is a **continuous** random variable with pdf $f(x)$, then the **variance** of X is

$$\sigma_x^2 = E(X - \mu_x)^2 = \int_{-\infty}^{+\infty} (x - \mu_x)^2\ f(x)\ dx.$$

Example: Derive the " shortcut formula for σ^2 " on p. 142;
i.e., prove that $\sigma^2 = EX^2 - \mu^2$ for the **continuous** case, where $\sigma^2 < \infty$.

□

The **standard deviation** of X is

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{EX^2 - \mu_x^2}.$$

Example: Derive the **variance** and **standard deviation** of a Uniform(A, B) distribution.

□

Example: Derive the **variance** of an Exponential(λ) distribution.

□

Example: Suppose X is a continuous (or discrete) random variable and a and b are constants. In terms of μ_x and σ_x, rewrite the following expressions:

(a) $E(aX + b)$

(b) σ^2_{aX+b}

(c) σ_{aX+b}

□

4.3 The Normal Distribution

The **normal** or **Gaussian** distribution is bell-shaped and symmetric.

Often, sample sums and sample averages are approximately **normal**, as expressed by the Central Limit Theorem (to be defined in section 5.4).

Examples of real-life applications of the **normal** distribution are listed in your textbook (beginning of section 4.3, p. 159).

The **probability density function** of a **normal** random variable is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

You need NOT memorize this formula.

It can be shown that:

(a) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$ (using Polar coordinates)

(b) $EX = \mu$

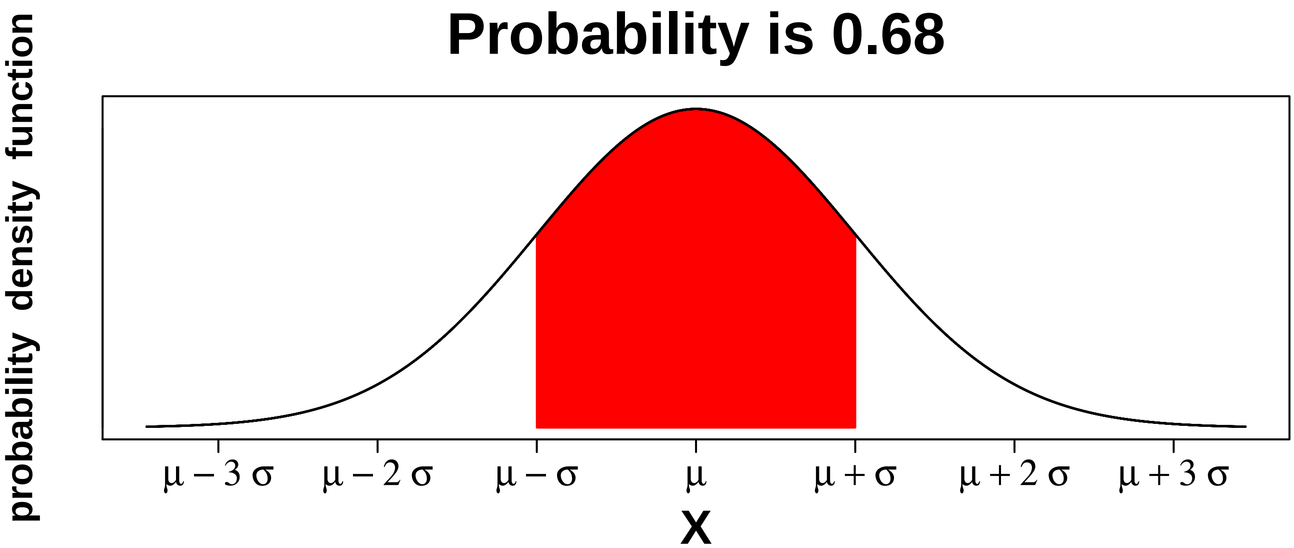
(c) $\mathrm{Var}(X) = \sigma^2$

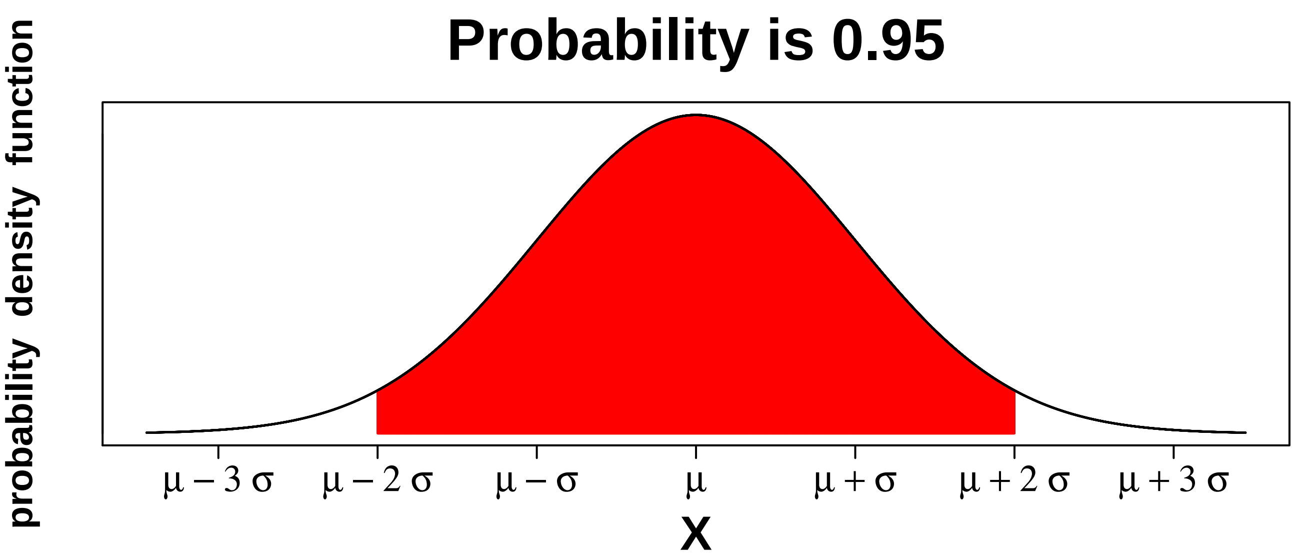
Empirical Rule

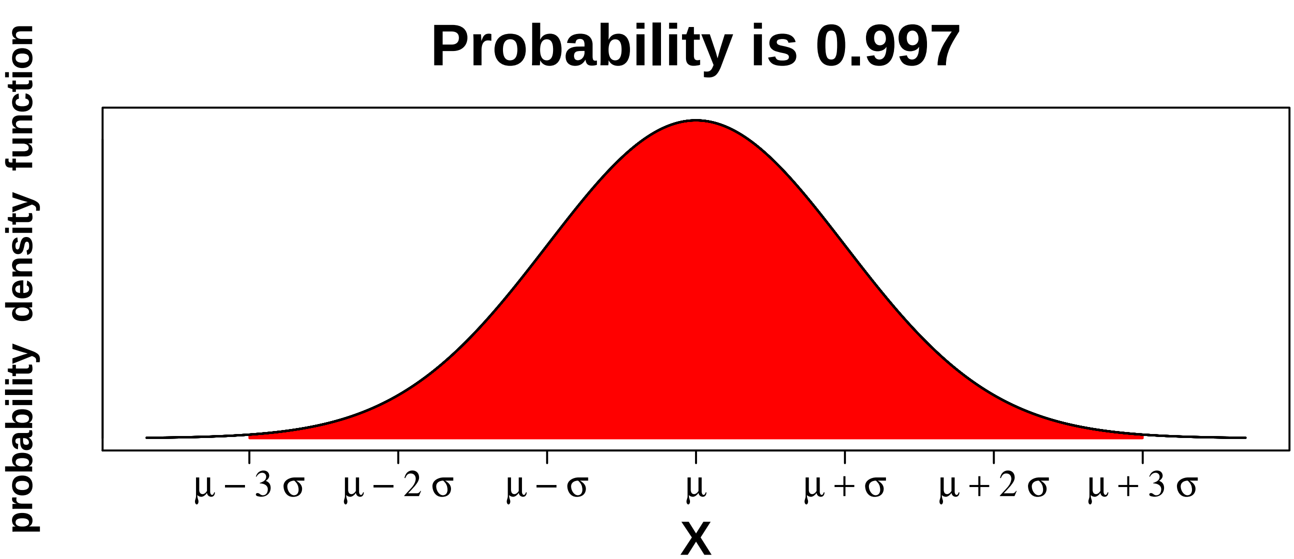
If a large number of observations are sampled from an approximately normal distribution, then (usually)

1. Approximately 68% of the observations fall within **one** standard deviation, σ, of the mean, μ.

2. Approximately 95% of the observations fall within **two** standard deviations, σ, of the mean, μ.
3. Approximately 99.7% of the observations fall within **three** standard deviations, σ, of the mean, μ.


Probability is 0.68
probability density function
μ − 3 σ
μ − 2 σ
μ − σ
μ
μ + σ
μ + 2 σ
μ + 3 σ
X


Probability is 0.95
probability density function
μ − 3 σ
μ − 2 σ
μ − σ
μ
μ + σ
μ + 2 σ
μ + 3 σ
X


Probability is 0.997
probability density function
μ − 3 σ
μ − 2 σ
μ − σ
μ
μ + σ
μ + 2 σ
μ + 3 σ
X

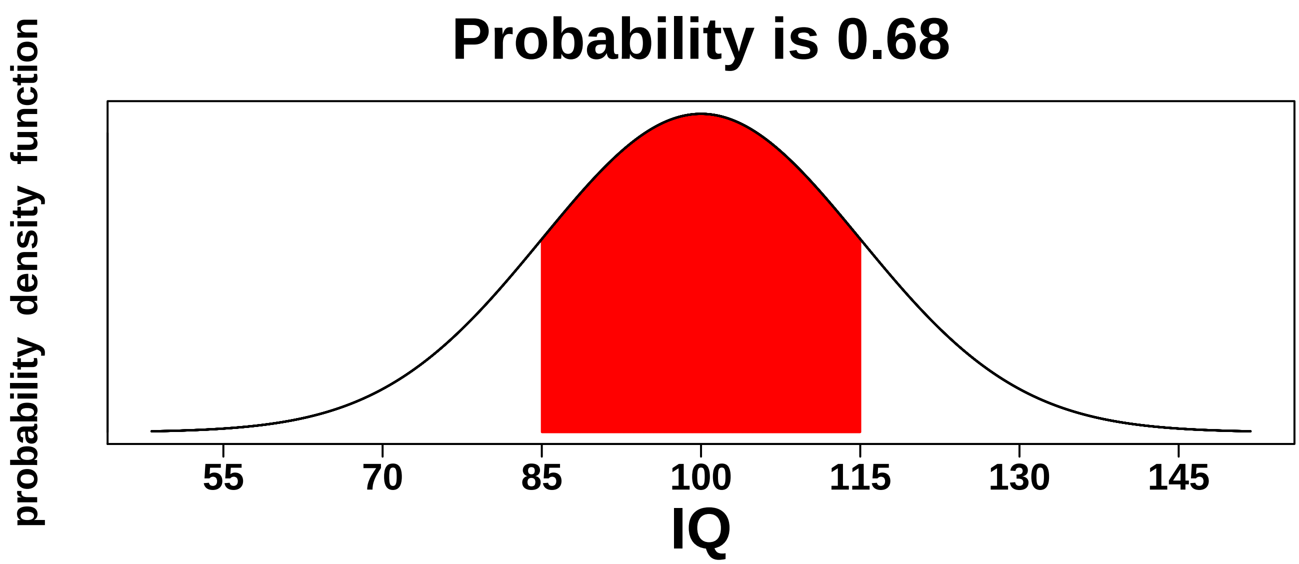
Example: IQ scores of normal adults on the Weschler test have a symmetric bell-shaped distribution with a mean of 100 and standard deviation of 15.

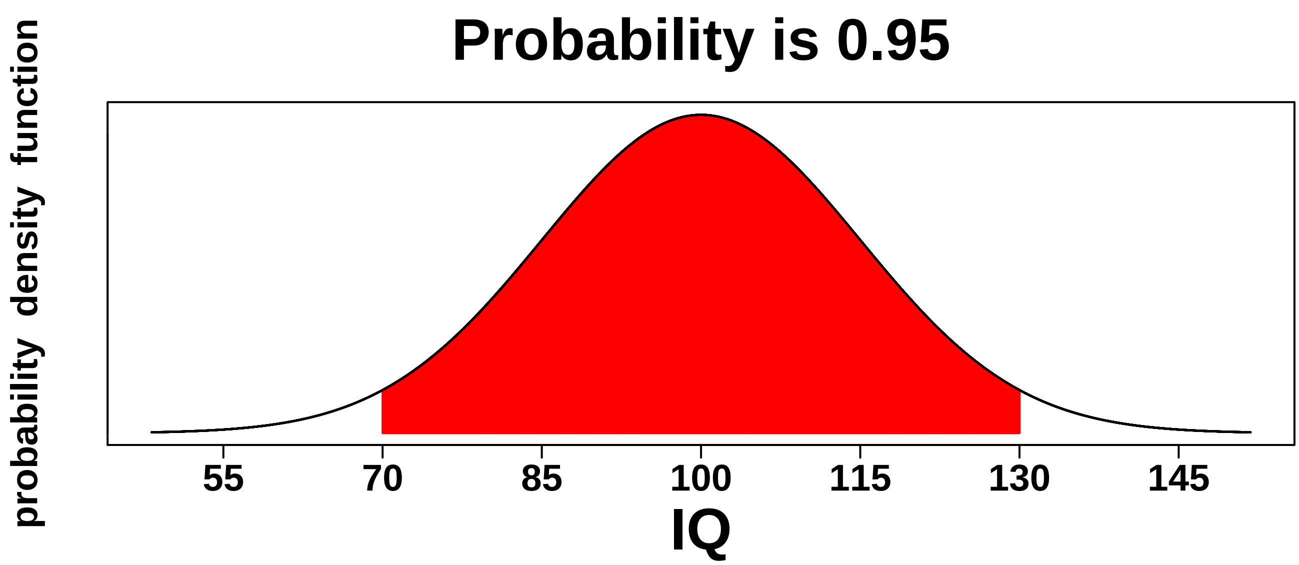
(a) If 1000 adults are sampled, approximately how many have IQs between 85 and 115?

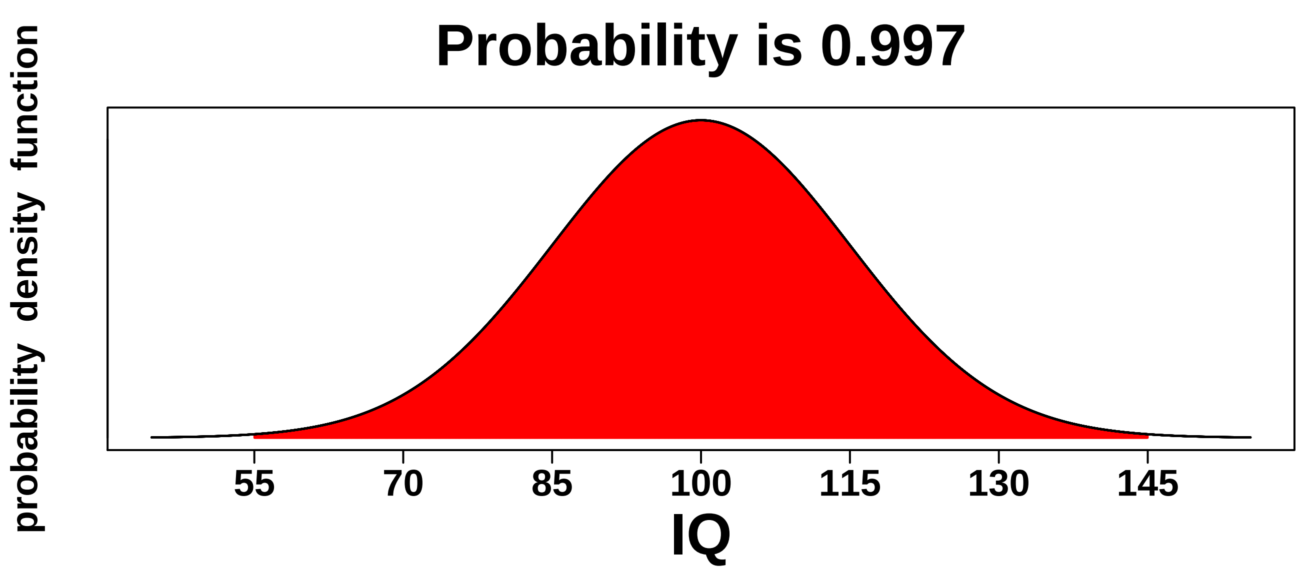
(b) If 1000 adults are sampled, approximately how many have IQs between 70 and 130?

(c) If 1000 adults are sampled, approximately how many have IQs between 55 and 145?

(d) If 1000 adults are sampled, approximately how many have IQs greater than 130?


Probability is 0.68
probability density function
55
70
85
100
115
130
145
IQ


Probability is 0.95
probability density function
55
70
85
100
115
130
145
IQ


Probability is 0.997
probability density function
55
70
85
100
115
130
145
IQ

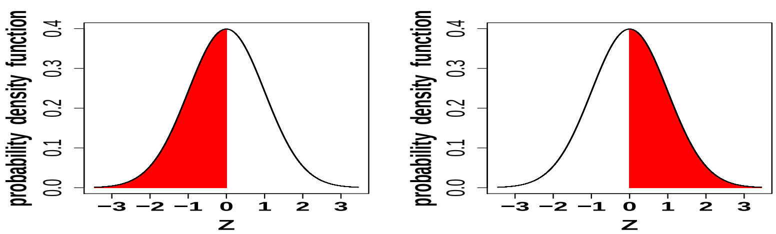
The standard normal distribution

Notation: $Z \sim N(0, 1)$.

Z represents the number of standard deviations, σ, away from the mean, μ.

Z is the "standardized" variable, known as the Z-score, and has **no units**.

Example: Compute $P(Z < 0)$, $P(Z \leq 0)$, $P(Z > 0)$, and $P(Z \geq 0)$.



□

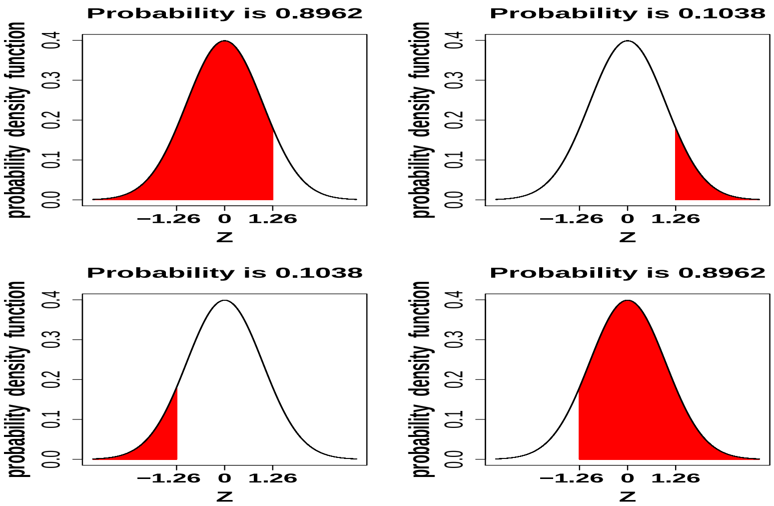
Example: *Using the standard normal table.* Let Z be a standard normal random variable.

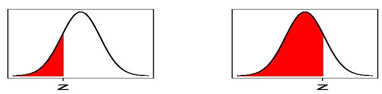
(a) Determine $P(Z < 1.26)$.

(b) Determine $P(Z > 1.26)$.

(c) Determine $P(Z < -1.26)$.

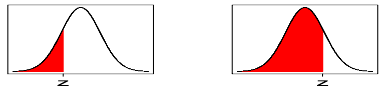
(d) Determine $P(Z > -1.26)$.





Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

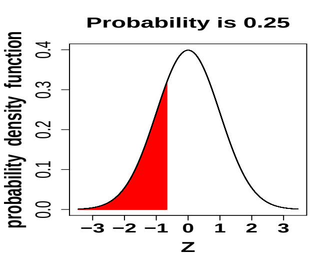
□

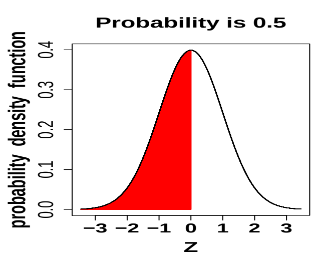
Example: *Using the standard normal table in reverse.* Let Z be a standard normal random variable.

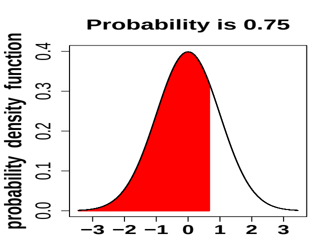
(a) Determine the 25th percentile of Z.

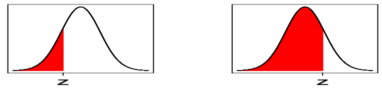
(b) Determine the 50th percentile of Z.

(c) Determine the 75th percentile of Z.



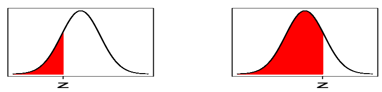






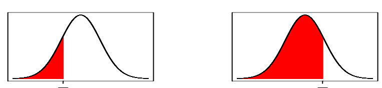
Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

□

Again consider $X \sim N(\mu, \sigma)$.

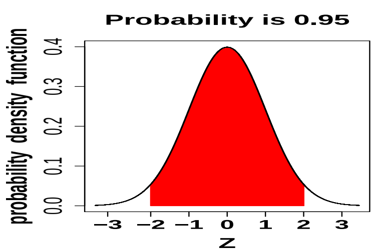
$$Z = \frac{X - \mu}{\sigma}$$

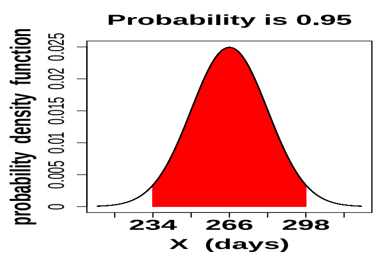
Reverse table look-up uses $X = \mu + \sigma Z$

$$X \leftrightarrow Z \leftrightarrow \text{probability}$$

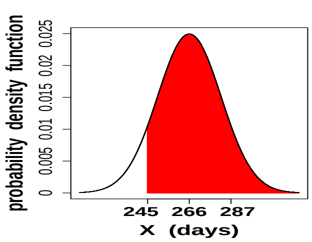
Example: The length of human pregnancies from conception to birth varies according to a distribution which is approximately normal with mean 266 days and standard deviation 16 days.

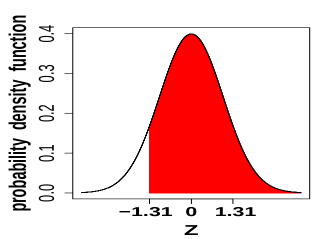
(a) Show the empirical rule regarding 95%.

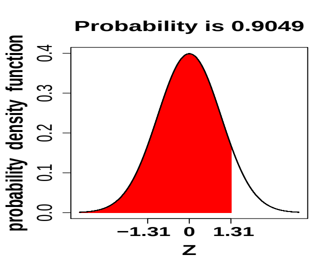


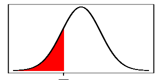


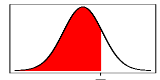
(b) What proportion of pregnancies last more than 245 days?





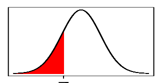


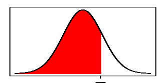




Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

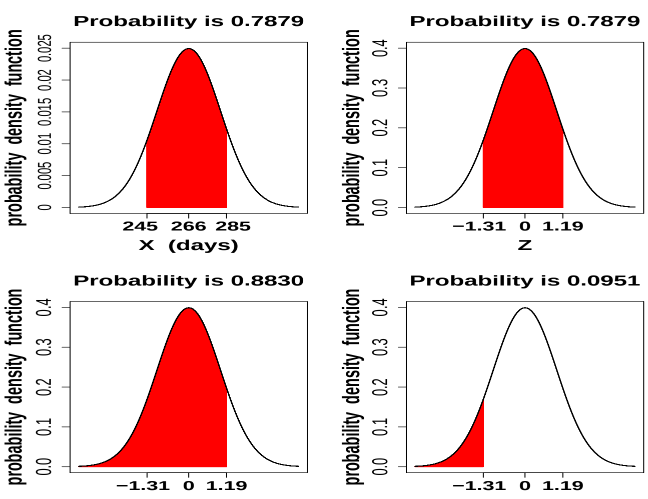


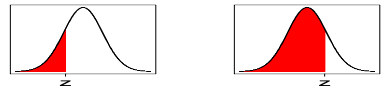


Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(c) What proportion of pregnancies last between 245 and 285 days?

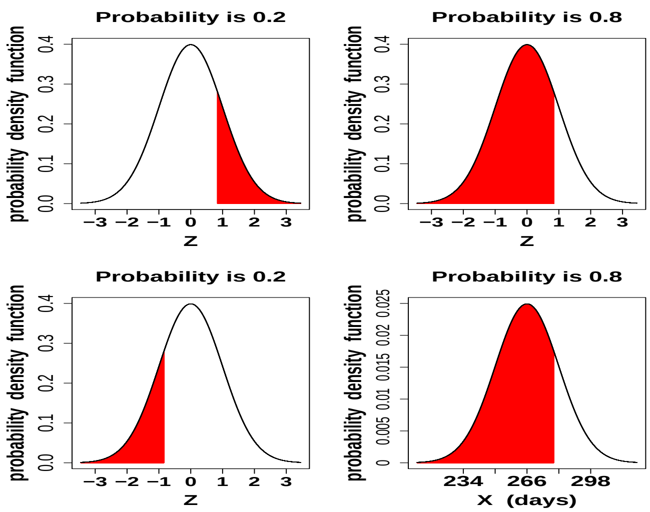


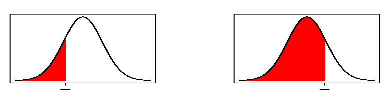


Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

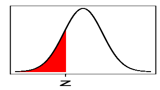
(d) How long do the longest 20% of pregnancies last?

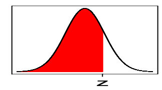




Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

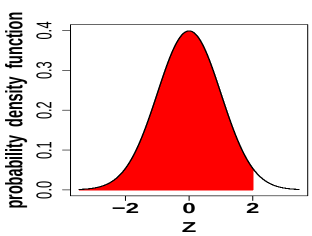


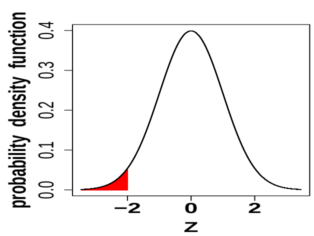


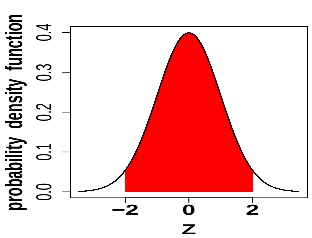
Standard normal table, pp. 722–723

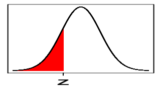
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

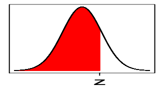
Example: Let $X \sim N(\mu, \sigma)$. Using the standard normal table, verify the empirical rule regarding 95%. In other words, compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$ to four significant digits.





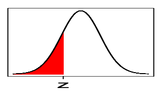


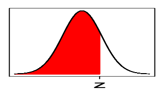




Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮





Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

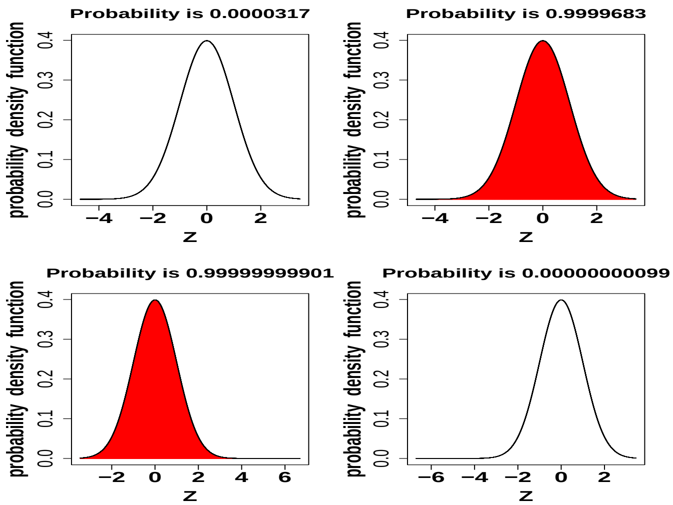
Example: (off the charts)

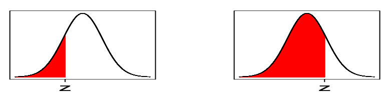
(a) Determine $P(Z < -4)$

(b) Determine $P(Z > -4)$

(c) Determine $P(Z < 6)$

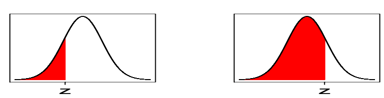
(d) Determine $P(Z < -6)$





Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

□

The Normal Approximation to the Binomial Distribution

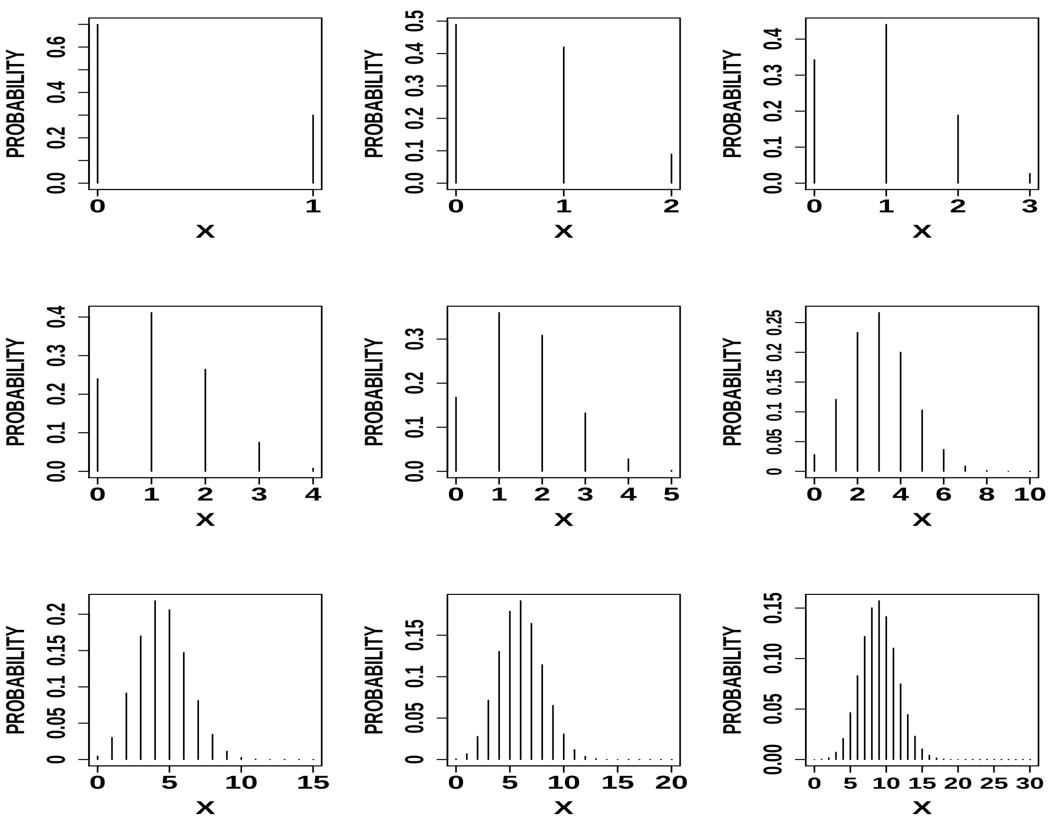
Suppose $X \sim \text{Binomial}(n,\ p)$.

Then, $\mu_x = EX = np$ and $\sigma_x = \sqrt{np(1-p)}$.

Rule of thumb: If $\min\{np,\ n(1-p)\}$ is sufficiently large, say, at least **10**, then X is approximately $N(\mu_x,\ \sigma_x)$.

This result follows from the Central Limit Theorem (to be defined in section 5.4), since a Binomial random variable is a sample sum of Bernoulli random variables.

Example: *Viewing the normal approximation to the binomial distribution.*
Consider the graphs below for **binomial** random variables, using $p = 0.3$ and $n = 1,\ 2,\ 3,\ 4,\ 5,\ 10,\ 15,\ 20$, and 30.



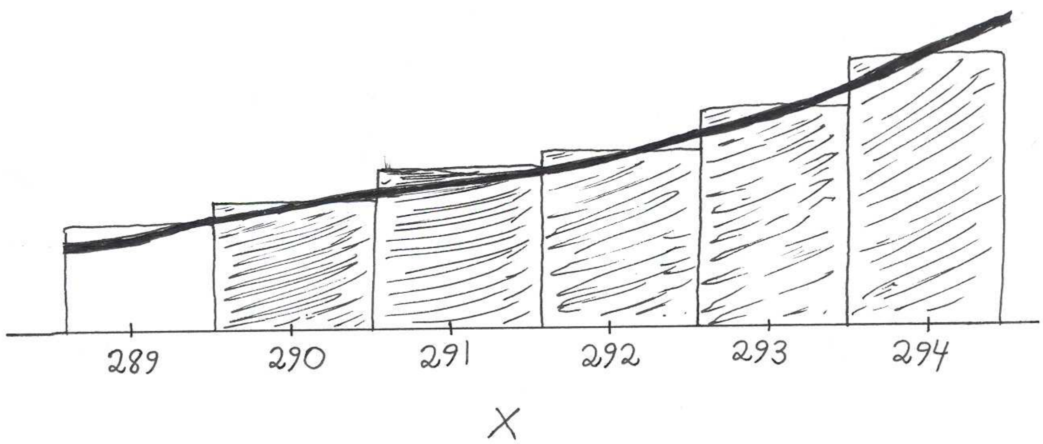
Example: Toss a coin 1000 times where $P(\text{heads}) = 0.3$, and let X be the number of heads.

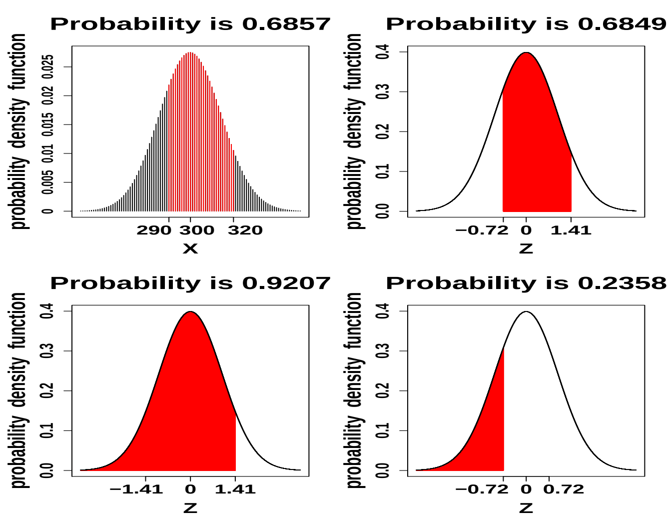
(a) State the **exact** distribution of X.

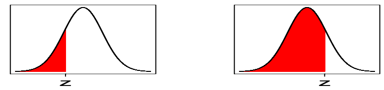
(b) Compute the **mean** and **standard deviation** of X.

(c) Check the **rule of thumb**.

(d) Calculate $P(290 \le X \le 320)$ using the normal approximation **with continuity correction**.

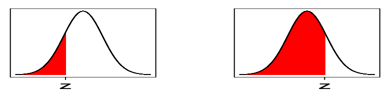






Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

□

4.4 The Gamma Distribution and Its Relatives

Exponential Distribution

Remark: The **exponential** distribution is a special case of a **gamma** distribution.

Remark: The **exponential** distribution is **memoryless**.

If $X \sim$ Exponential, then it can be shown that

$P(X \geq t + t_0 | X \geq t_0) = P(X \geq t)$, for any $t, t_0 \geq 0$.

Example: Radioactive decay.

Example: Failure time of computer chip.

Remark: The **exponential** distribution is related to the **Poisson** distribution.

The amounts of time separating responses in a **Poisson** process are *independent and identically distributed* **exponential** random variables.

Example: Radioactive decay.

Let Y be the number of **responses** in one day.

Let X be the number of **days** in between responses.

Remark: The **chi-squared** distribution (to be examined in chapter 7) is another special case of a **gamma** distribution.

4.5 Other Continuous Distributions

Examples of other continuous distributions include the **Weibull** distribution, the **Beta** distribution (of which a special case is the **uniform** distribution), and the **Lognormal** distribution.

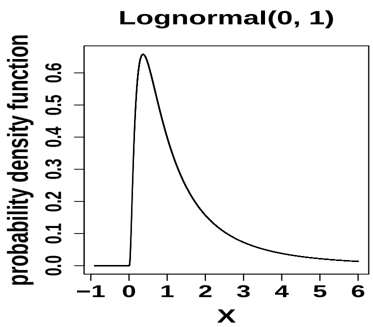
Lognormal distribution

If $Y \sim N(\mu,\ \sigma)$, then $X = e^Y$ is **lognormal** with parameters μ and σ.

What are the possible values of X?

$$EX = e^{\mu+\sigma^2/2}, \quad \text{and}$$

$$\text{Var}(X) = \left(e^{2\mu+\sigma^2}\right)\,\left(e^{\sigma^2} - 1\right)$$

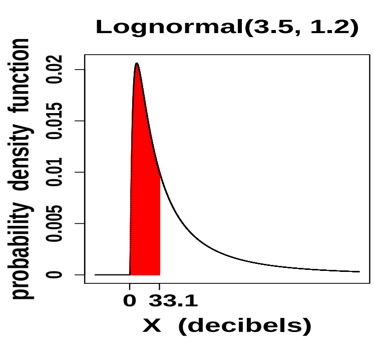


Derive the **lognormal cdf** in terms of the **normal cdf**, $\Phi(\cdot)$.

Exercise 4.72, p. 185: Let $X \sim \text{lognormal}(\mu, \sigma)$.

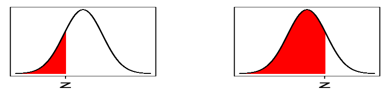
(a) Compute the median $\tilde{\mu}$ of X.

Determine $\tilde{\mu}$ for the **lognormal** distribution of Exercise 4.71, where $\mu = 3.5$ and $\sigma = 1.2$, regarding *received power of radio signals between two cities.*



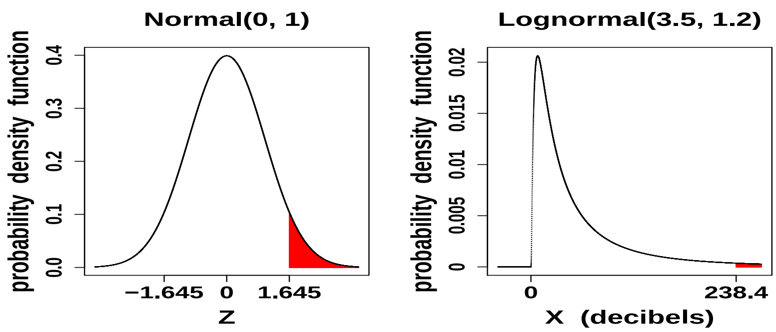
(b) The notation z_α represents the $100(1-\alpha)$ percentile for a **standard normal** distribution. Determine the $100(1-\alpha)$ percentile of X.

Regarding the **lognormal** distribution of Exercise 4.71, where $\mu = 3.5$ and $\sigma = 1.2$, what value will *received power* exceed only 5% of the time?



Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



4.6 Probability Plots

Probability plots are used for determining whether or not data came from a particular distribution, often a **normal** distribution.

Algorithm:

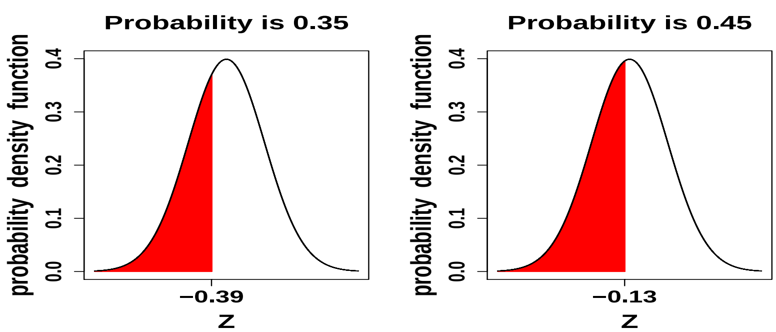
(a) Order the n observations from smallest to largest.

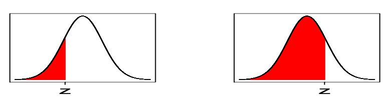
(b) Graph the ordered observations against the **standard normal percentiles**.

(c) Look for deviations from **linearity**.

The **standard normal percentiles** are based on the **probabilities** $(i - 0.5)/n$ for $i = 1, 2, \ldots, n$.

Example: Determine the **standard normal percentiles** for **10** observations.





Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Example: Consider the following 10 observations from some population: {9.7, 2.9, 7.6, 4.3, 5.4, 13.6, 6.5, 0.4, 8.5, 11.2}.

The **sorted** observations are {0.4, 2.9, 4.3, 5.4, 6.5, 7.6, 8.5, 9.7, 11.2, 13.6}.

(a) Construct the **normal probability plot** of these 10 observations.

(b) Construct the **normal probability plot** of these 10 observations, but replace "13.6" by "20".

(c) Construct the **normal probability plot** of these 10 observations from part (a), but replace "0.4" by "−5".

(d) Construct the **normal probability plot** of these 10 observations from part (a), but replace "13.6" by "20" and "0.4" by "−5".