5 Joint Probability Distributions and Random Samples

5.1 Jointly Distributed Random Variables

Instead of considering just one random variable X, we are interested in at least two random variables, say, (X, Y).

Discrete case

Definition: The **joint probability mass function** of discrete random variables X and Y is

$$p_{x,y}(x, y) = P(X = x \cap Y = y), \quad \text{and}$$
$$P(\ (X, Y) \in A\ ) = \sum\sum_{(x,y)\in A} p_{x,y}(x, y).$$

Example: Roll a pair of fair **4**-sided dice.
Let $\boldsymbol{X}$ denote the **smaller** of the two numbers on the dice.
Let $\boldsymbol{Y}$ denote the **larger** of the two numbers on the dice.

(a) List the outcomes of the two dice in a table.

(b) Are the outcomes of the two dice **independent**?

(c) Are X and Y **independent**?

(d) Compute $P(X = Y = 1)$.

(e) Compute $P(X = Y = 2)$, $P(X = Y = 3)$, and $P(X = Y = 4)$.

(f) Compute $P(X = 1 \cap Y = 2)$.

(g) List the **joint probability mass function** of $\boldsymbol{(X, Y)}$ in a **table**.

(h) List the **joint probability mass function** of $\boldsymbol{(X, Y)}$ as a **formula**.

(i) Determine $\sum_x \sum_y p(x, y)$.

□

Definition: The **marginal probability mass functions** of X and Y are

$$p_x(x) = P(X = x) = \sum_y p_{x,y}(x, y), \quad \text{and}$$

$$p_y(y) = P(Y = y) = \sum_x p_{x,y}(x, y).$$

Example: *Revisit.* Roll a pair of fair **4**-sided dice.
Let $\boldsymbol{X}$ denote the **smaller** of the two numbers on the dice.
Let $\boldsymbol{Y}$ denote the **larger** of the two numbers on the dice.

(a) Determine the **marginal pmf** of X.

(b) Determine the **marginal pmf** of Y.

(c) List the **marginal pmf** of X and the **marginal pmf** of Y in a **table**.

□

Continuous case

Definition: The **joint probability density function**, $f_{x,y}(x, y)$, of continuous random variables X and Y is defined by

$$P(\ (X,Y) \in A\ ) = \int_A \int f_{x,y}(x,y)\ dx\ dy,$$

for all two-dimensional sets A.

If A is a two-dimensional **rectangle**, then A can be written

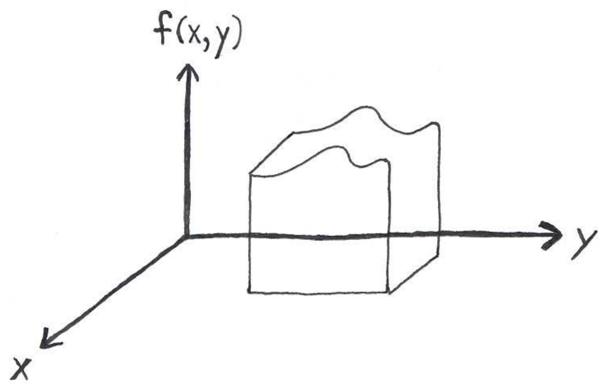
$$A = \{(x,y):\ a \le x \le b,\ c \le y \le d\},$$

for some constants a, b, c, d.

If A is a two-dimensional **rectangle**, then

$$P(\ (X,Y) \in A\ ) = \int_A \int f_{x,y}(x,y)\ dx\ dy$$

$$= \int_c^d \left[\int_a^b f_{x,y}(x,y)\ dx \right] dy = \int_a^b \left[\int_c^d f_{x,y}(x,y)\ dy \right] dx.$$



Example: Consider random variables X and Y with joint probability density function

$$f_{x,y}(x,y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Graph the domain, where the joint **pmf** is positive.

(b) **Prove** that $f_{x,y}(x, y)$ is a valid **joint pmf**.

□

Definition: The **marginal probability density functions** of X and Y are

$$f_x(x) = \int_{-\infty}^{\infty} f_{x,y}(x, y)\; dy, \quad \text{and}$$
$$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x, y)\; dx.$$

Example: *Revisit.* Consider random variables X and Y with joint probability density function

$$f_{x,y}(x, y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the **marginal pdf** of X.

(b) Compute the **marginal pdf** of Y.

□

Example: *Bivariate uniform.* Let

$$f_{x,y}(x, y) = \begin{cases} 1, & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the **marginal pdf** of X.

(b) Compute the **marginal pdf** of Y.

(c) Compute $P(X < 1/3,\ 1/4 < Y < 1/2)$.

□

Independent Random Variables

Definition: Two random variables are **independent** if and only if

$$p_{x,y}(x, y) = p_x(x)p_y(y), \text{ when } X \text{ and } Y \text{ are discrete,}$$

$$f_{x,y}(x, y) = f_x(x)f_y(y), \text{ when } X \text{ and } Y \text{ are continuous,}$$

for all $x, y \in \Re$.

Definition: Two random variables which are NOT independent are called **dependent**.

Example: *Revisit.* Consider random variables X and Y with joint probability density function

$$f_{x,y}(x, y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Mathematically verify whether or not X and Y are independent.

□

Example: *Revisit bivariate uniform.* Let

$$f_{x,y}(x, y) = \begin{cases} 1, & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Mathematically verify whether or not X and Y are independent.

□

Remark: The definition of independence can be generalized to an arbitrary number of random variables.

Conditional distributions

Definition: (Continuous case.) Suppose X and Y have joint **pdf** $f_{x,y}(x,y)$. The **conditional pdf** of Y given $X = x$ is

$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)},$$

which exists if $f_x(x) > 0$.

Definition: (Discrete case.) Suppose X and Y have joint **pmf** $p_{x,y}(x,y)$. The **conditional pmf** of Y given $X = x$ is

$$p_{y|x}(y|x) = \frac{p_{x,y}(x,y)}{p_x(x)},$$

which exists if $p_x(x) > 0$.

Suppose X and Y and **independent** and have joint **pdf** $f_{x,y}(x,y)$, then $f_{y|x}(y|x) = f_y(y)$.

Similarly, suppose X and Y and **independent** and have joint **pmf** $p_{x,y}(x,y)$, then $p_{y|x}(y|x) = p_y(y)$.

Example: *Revisit.* Consider random variables X and Y with joint probability density function

$$f_{x,y}(x,y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the conditional pdf of Y given X.

(b) Compute the conditional pdf of Y given $X = 1/3$.

(c) Compute the conditional pdf of X given Y.

(d) Compute the conditional pdf of X given $Y = 1/3$.

□

Example: *Revisit bivariate uniform.* Let

$$f_{x,y}(x, y) = \begin{cases} 1, & \text{if } 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the conditional pdf of Y given X.

(b) Compute the conditional pdf of X given Y.

Example: *Revisit.* Roll a pair of fair **4**-sided dice.
Let $\boldsymbol{X}$ denote the **smaller** of the two numbers on the dice.
Let $\boldsymbol{Y}$ denote the **larger** of the two numbers on the dice.
Compute the conditional pmf of Y given $X = 2$.

□

5.2 Expected Values, Covariance, and Correlation

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be jointly distributed random variables with **pmf** $\boldsymbol{p(x, y)}$ if **discrete** or **pdf** $\boldsymbol{f(x, y)}$ if **continuous**.

Let $h(x, y)$ be a function of x and y. Then,

$$Eh(X,Y) = \begin{cases} \sum_x \sum_y h(x,y)\ p(x,y), & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y)\ f(x,y)\ dx\ dy, & \text{if } X \text{ and } Y \text{ are continuous} \end{cases}$$

Definition: The **covariance** between random variables X and Y is $\mathrm{Cov}(X, Y) = E(X - \mu_x)(Y - \mu_y)$.

Prove that $E(X + Y) = EX + EY$, if EX and EY are finite.

□

Derive the shortcut formula (where $\sigma_x^2 + \sigma_y^2 < \infty$): $\mathrm{Cov}(X, Y) = EXY - \mu_x\mu_y$.

□

Example: *Revisit.* Consider random variables X and Y with joint probability density function

$$f_{x,y}(x,y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the **mean** of $\boldsymbol{X}$.

(b) Compute the **mean** of $\boldsymbol{Y}$.

(c) Compute the **mean** of $\boldsymbol{XY}$.

(d) Compute the **covariance** between $\boldsymbol{X}$ and $\boldsymbol{Y}$.

□

Definition: The **correlation coefficient** of X and Y is

$$\rho_{x,y} = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x\ \sigma_y},$$

which exists if $0 < \sigma_x < \infty$ and $0 < \sigma_y < \infty$.

Exercise 5.28, p. 221: Suppose that X and Y are **independent continuous** random variables.

(a) Prove that $EXY = (EX)(EY)$.

□

(b) Apply the result from part (a) to exercise 5.25, p. 220.

□

Exercise 5.33, p. 221: Suppose that X and Y are **independent** random variables.

(a) Prove that $\text{Cov}(X, Y) = 0$.

(b) Prove that $\text{Corr}(X, Y) = 0$, if $0 < \sigma_x < \infty$ and $0 < \sigma_y < \infty$.

□

Example: *Revisit.* Consider random variables X and Y with joint probability density function

$$f_{x,y}(x, y) = \begin{cases} 1/y, & \text{if } 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute EX^2.

(b) Compute the **variance** of X.

(c) Compute EY^2.

(d) Compute the **variance** of Y.

(e) Compute the **correlation** of X and Y.

□

Remarks:

(1) If X and Y are **independent** with $0 < \sigma_x < \infty$ and $0 < \sigma_y < \infty$, then

$$\rho_{x,y} = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_x \ \sigma_y} = 0.$$

(2) $-1 \leq \rho_{x,y} \leq 1$ (follows from Schwarz inequality)

(3) If $\rho_{x,y} = \pm 1$, then $Y = aX + b$ for constants $a \neq 0$ and b.

(4) Let a, b, c, and d be constants such that $a \neq 0 \neq c$. *(This is exercise 5.35, p. 221.)*

(a) Then, $\text{Cov}(aX + b,\ cY + d) = ac\ \text{Cov}(X, Y)$ **Prove it!**

(b) Then,

$$\text{Corr}(aX + b,\ cY + d) = \begin{cases} \text{Corr}(X, Y), & \text{if } ac > 0 \\ -\text{Corr}(X, Y), & \text{if } ac < 0 \end{cases}$$

Prove it!

(5) ρ has no units.

(6) ρ is a **population** correlation coefficient and is often estimated by a **sample** correlation coefficient, to be defined in section 12.5.

(7) $\text{Cov}(X, X) =$

(8) For $0 < \sigma_x < \infty$, $\text{Corr}(X, X) =$

(9) Correlation measure **linear** association between X and Y, NOT association in general.

(10) Zero covariance or zero correlation does NOT imply independence, although independence implies zero covariance and zero correlation (for $0 < \sigma_x < \infty$ and $0 < \sigma_y < \infty$).

Example: Compute the **correlation** of X and Y, where where the joint pmf is

$$p_{x,y}(x, y) = \begin{cases} 1/3 & \text{if } (x, y) \in \{(0, 0),\ (-1, 1),\ (1, 1)\} \\ 0, & \text{otherwise} \end{cases}$$

(11) $\rho_{x,y} = \rho_{y,x}$

5.3 Statistics and Their Distributions

Definition: Random variables X_1, X_2, $\ldots$, X_n for a **(simple) random sample** if these observations are **independent** and have the **same** probability distribution.

A **simple random sample** consists of **independent and identically distributed (i.i.d.)** random variables.

If the population size is huge compared to the sample size n, but sampling is performed **withOUT** replacement, then the observations form an **approximate** simple random sample, rather than an **exact** simple random sample, since the observations are **nearly** or **approximately** independent.

Example: Sampling 1000 adults at random **withOUT** replacement from among *all* American adults forms an **approximate** simple random sample.

Definition: A **statistic** is a quantity computed from a sample.

Example:

Recall from section 3.2:

Definition: The **probability distribution** of a **discrete** random variable X consists of the possible values of X along with their associated probabilities.

Definition: The *probability distribution* of a **statistic** is called its **sampling distribution**.

Hence, the **sampling distribution** of a **discrete** *statistic* consists of the possible values of the *statistic* along with their associated probabilities.

The sampling distribution of a sample mean, $\bar{X}$

Example: Consider a population consisting of three marbles in an urn, where the marbles are labeled as $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$. Let x be the value of a marble drawn.

(a) Determine the **probability distribution** of X.

(b) Graph the *probability distribution* of X.

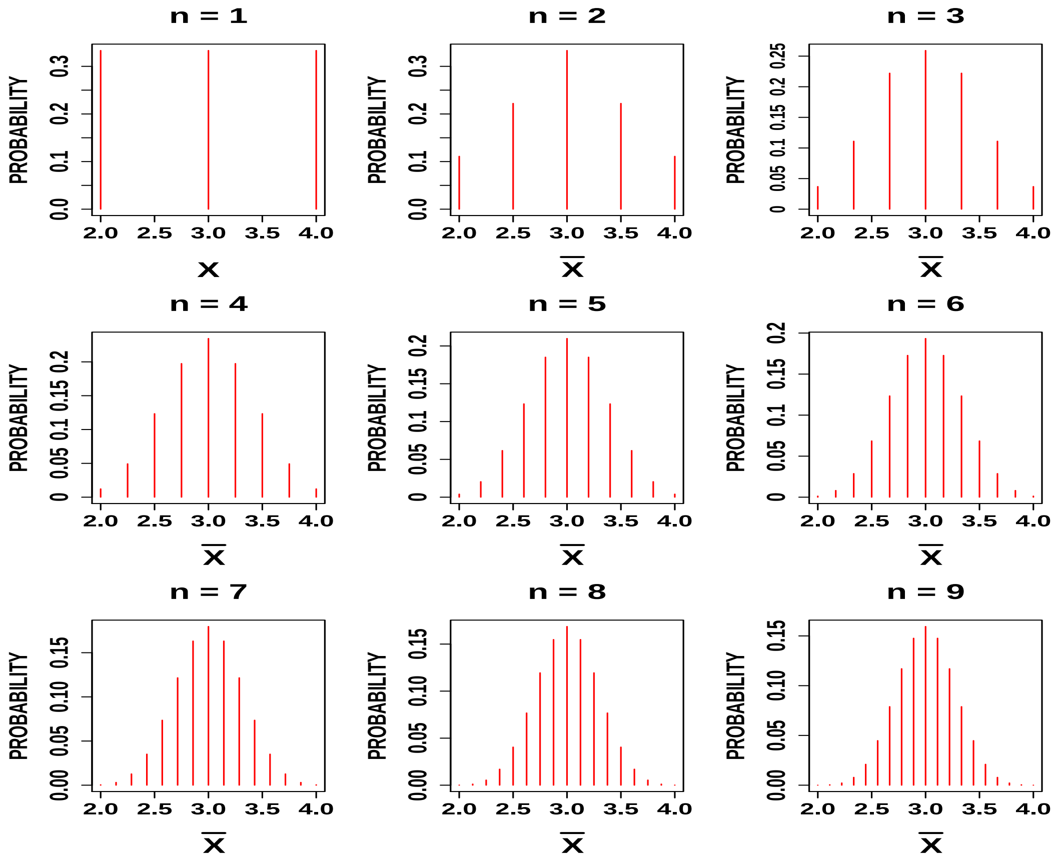
(c) Determine the *mean* of X.

(d) Let $\bar{X}$ be the sample mean, based on **two** observations independently sampled (i.e., **with** replacement) from this population. Determine the **sampling distribution** of $\bar{X}$.

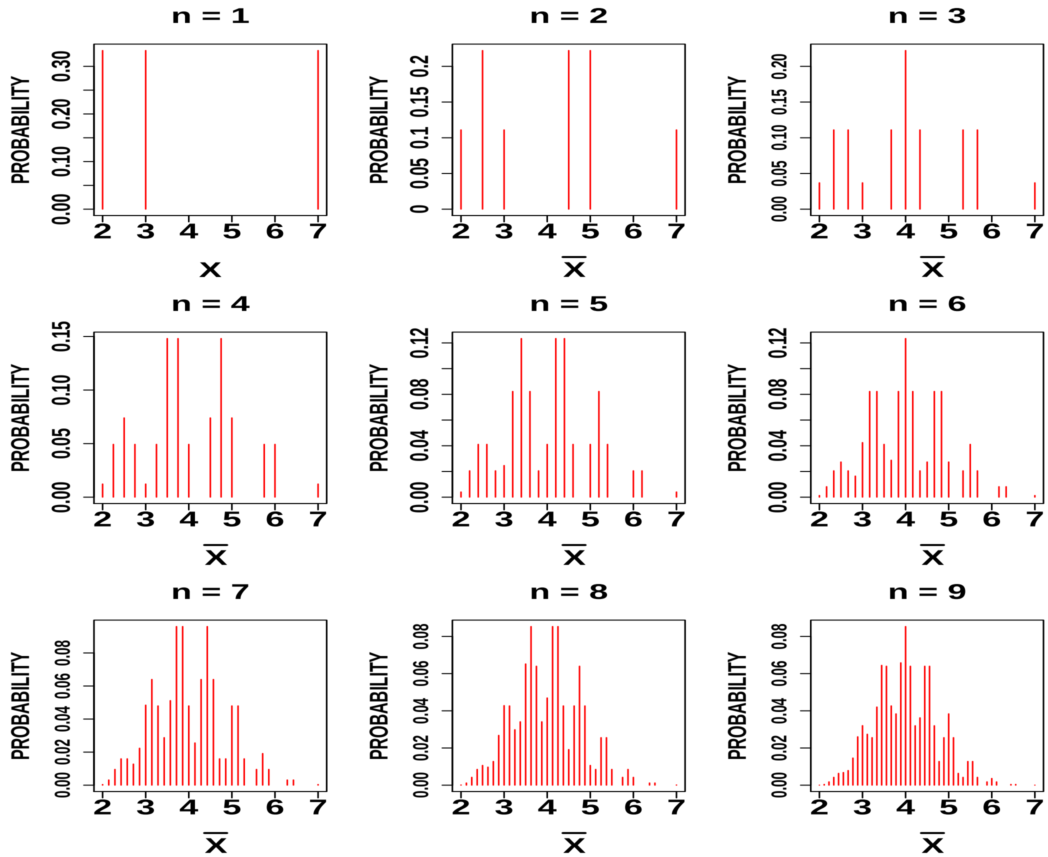
(e) Graph the *sampling distribution* of $\bar{X}$.

(f) Determine the *mean* of $\bar{X}$.

(g) Additional graphs of the *sampling distribution* of $\bar{X}$ are below, based on independent observations and sample size n.



(h) Repeat part (g), using marbles labeled $\boxed{2}$, $\boxed{3}$, and $\boxed{7}$.



□

The sampling distribution of a sample proportion, $\hat{p}$

Recall that a proportion is a special case of a mean.

Example: Sample *independent* observations from a population which is 30% Democrat. Let $\hat{p}$ be the sample proportion of Democrats.

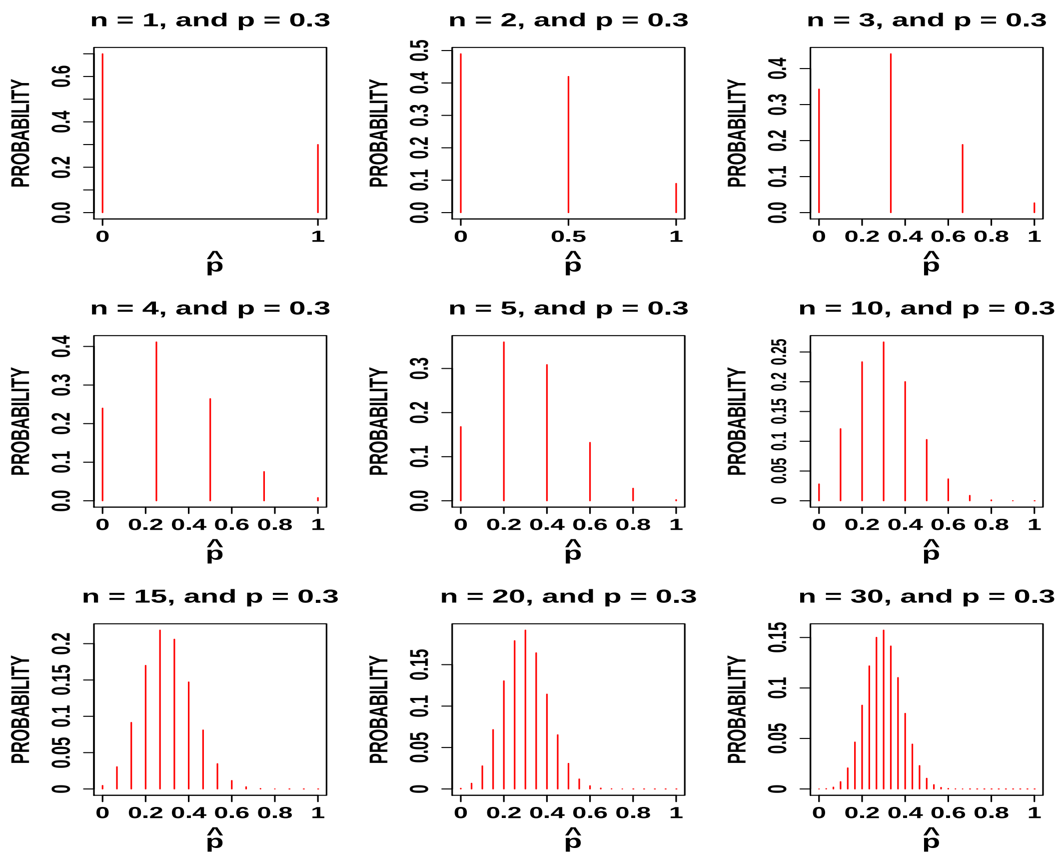
(a) State the **population distribution** in a chart, and construct the *line graph* of the **population distribution**.

Let $X = 0$ if non-Democrat, and $X = 1$ if Democrat.

Note that the *sampling distribution* of $\hat{p}$ for $n = 1$ is the same as the *population distribution* of X.

(b) For $n = 2$, state the **sampling distribution** of $\hat{p}$ in a chart, and construct the *line graph* of the **sampling distribution** of $\hat{p}$.

Consider the graphs below for **sample proportions**, $\hat{p}$, using $p = 0.3$ and $n = 1, 2, 3, 4, 5, 10, 15, 20$, and 30.



(c) What happens to the *sampling distribution* of $\hat{p}$ as the sample size, n, gets larger?

□

5.4 The Distribution of the Sample Mean, $\bar{X}$

Recall: For random variables X and Y, $E(X+Y) = EX + EY$, provided that EX and EY are finite, whether X and Y are independent or dependent.

Example: Sample $X_1, X_2, \ldots, X_n$ either WITH or withOUT replacement from a distribution with finite mean μ.

(a) **Derive** $E\bar{X}$.

$\bar{X}$ is **unbiased** for μ.

(b) Define the **sample total** $T_0 = X_1 + X_2 + \ldots + X_n$. Determine the **mean** of T_0.

□

Let $X_1, X_2, \ldots, X_n$ be a **simple random sample** from a distribution with mean μ and finite variance σ^2.

(a) **Derive** the **standard deviation** of $\bar{X}$.

(b) Define the **sample total** $T_0 = X_1 + X_2 + \ldots + X_n$. Determine the **standard deviation** of T_0.

□

Remark: These formulas for **means** are **exact**, whether sampling is performed WITH or withOUT replacement.

Remark: These formulas for **standard deviations** and **variances** are **exact** for **exact** simple random samples (i.e., WITH replacement) and **approximate** for **approximate** simple random samples.

The Case of a Normal Population Distribution: If $X_1, X_2, \ldots, X_n$ are a **simple random sample** from a $N(\mu, \sigma)$ population, then $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$ and $T_0 \sim N(n\mu, \sigma\sqrt{n})$.

The Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be a **simple random sample** from a distribution with mean μ and positive finite variance σ^2. Then,

$$P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z\right) = P\left(\frac{T_0 - n\mu}{\sigma\sqrt{n}} < z\right) \to P(Z < z),$$

as $n \to \infty$, where Z is a standard normal random variable.

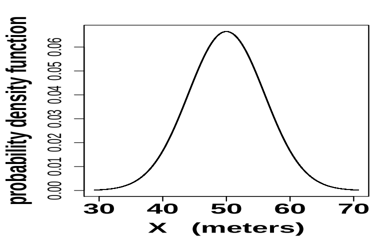
Sample means and sample totals are approximately normal for sufficiently large n, *for most distributions of interest.*

Rule of thumb: Usually $n > 30$ is considered sufficiently large.

The *Central Limit Theorem* holds **approximately** for an **approximate** simple random sample.

However, distributions with at least one **heavy tail** often need n to be very large, in order for the normal approximation to be reasonable.

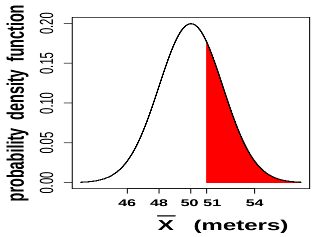
Example: Suppose $X \sim N(\mu = 50 \text{ meters}, \sigma = 6 \text{ meters})$. Sample nine independent observations of X.

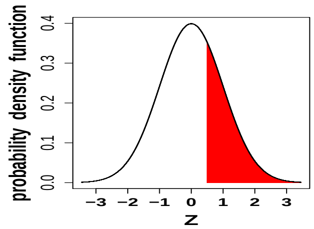


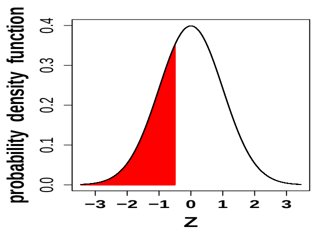
(a) Determine the *mean* of $\bar{X}$.

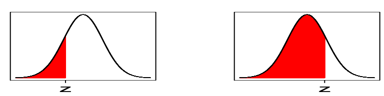
(b) Determine the *standard deviation* of $\bar{X}$; i.e., the *standard error* of $\bar{X}$.

(c) Determine the probability that $\bar{X}$ exceeds 51 meters.







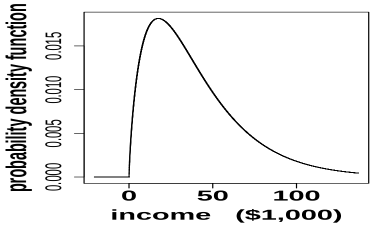


Standard normal table, pp. 722–723

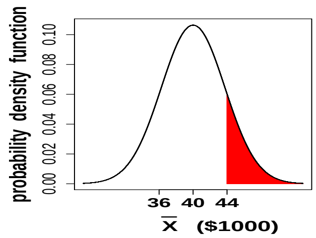
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

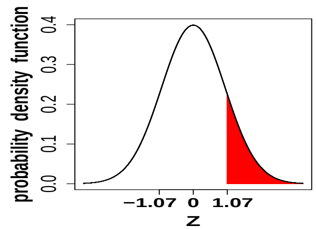
□

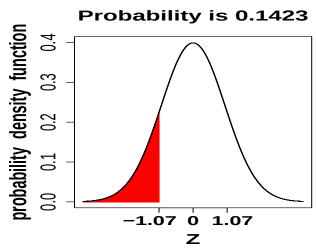
Example: Suppose personal income, X, in the U.S. has mean $\mu = \$40,000$ and standard deviation $\sigma = \$30,000$. Sample **without** replacement.

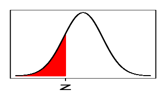


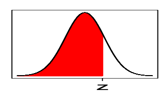
(a) Determine $P(\bar{X} > \$44,000)$, for $n = 64$.







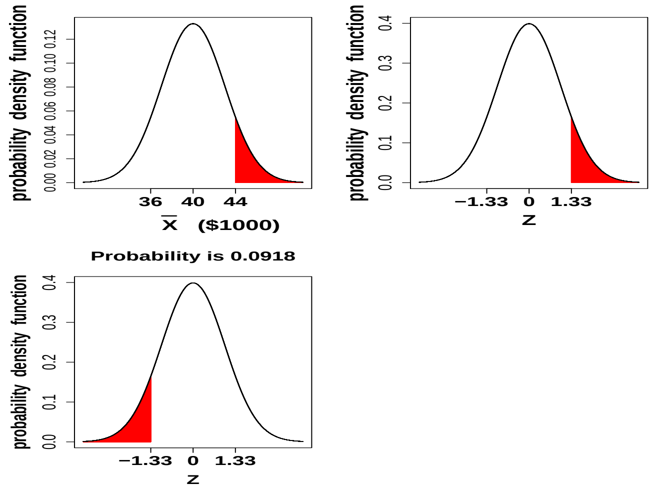


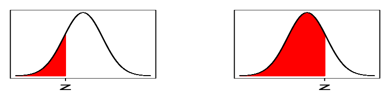


Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(b) Determine $P(\bar{X} > \$44,000)$, for $n = 100$.

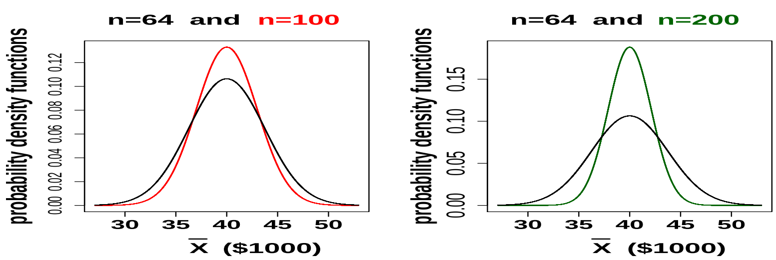




Standard normal table, pp. 722–723

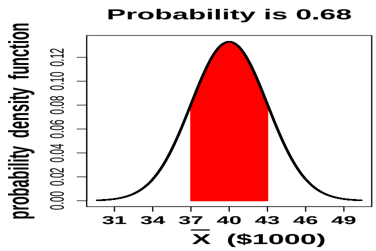
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

(c) What happens to $P(\bar{X} > \$44,000)$ as we increase n to 200?

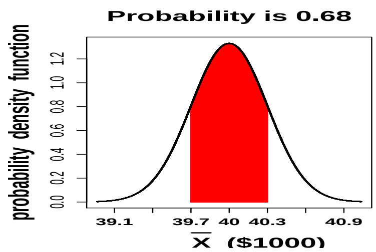


(d) Determine $P(\bar{X} > \$44,000)$, for $n = 10$.

(e) Determine the 68% part of the empirical rule for $n = 100$.



(f) Determine the 68% part of the empirical rule for $n = 10,000$.



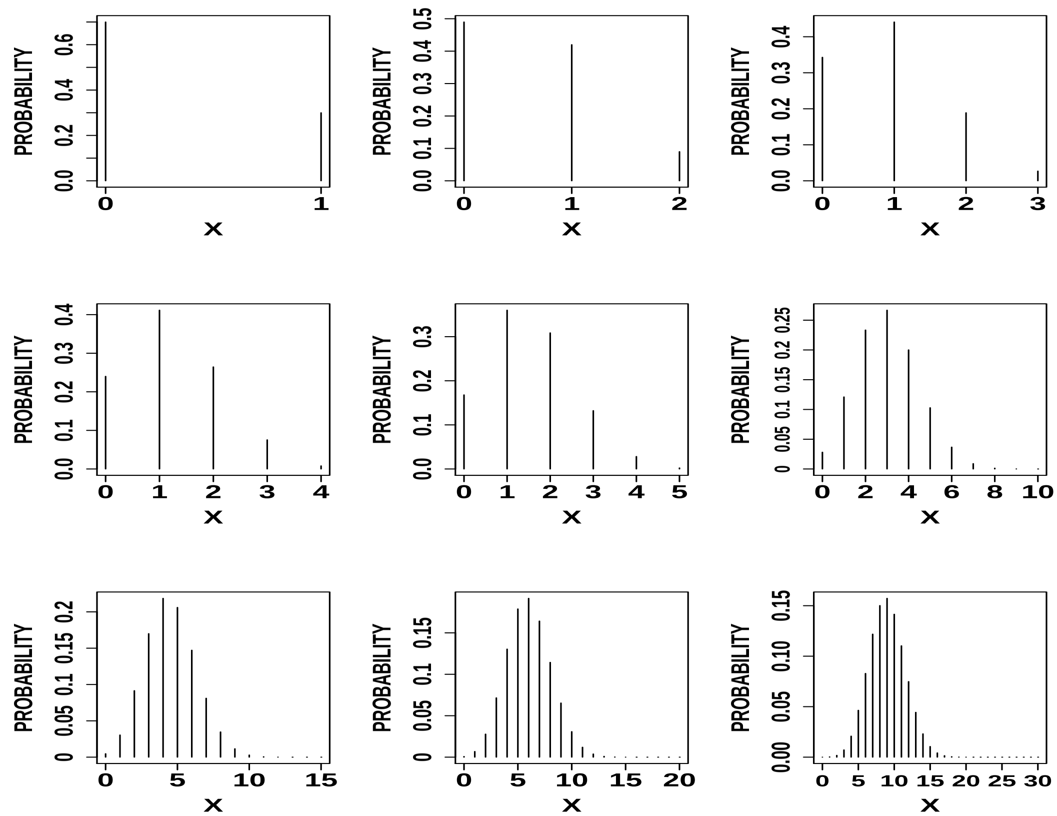
□

For large sample sizes (i.e., $np \geq 10$ and $n(1-p) \geq 10$), a *binomial random variable* and a *sample proportion* are approximately **normally distributed** by the **Central Limit Theorem**.

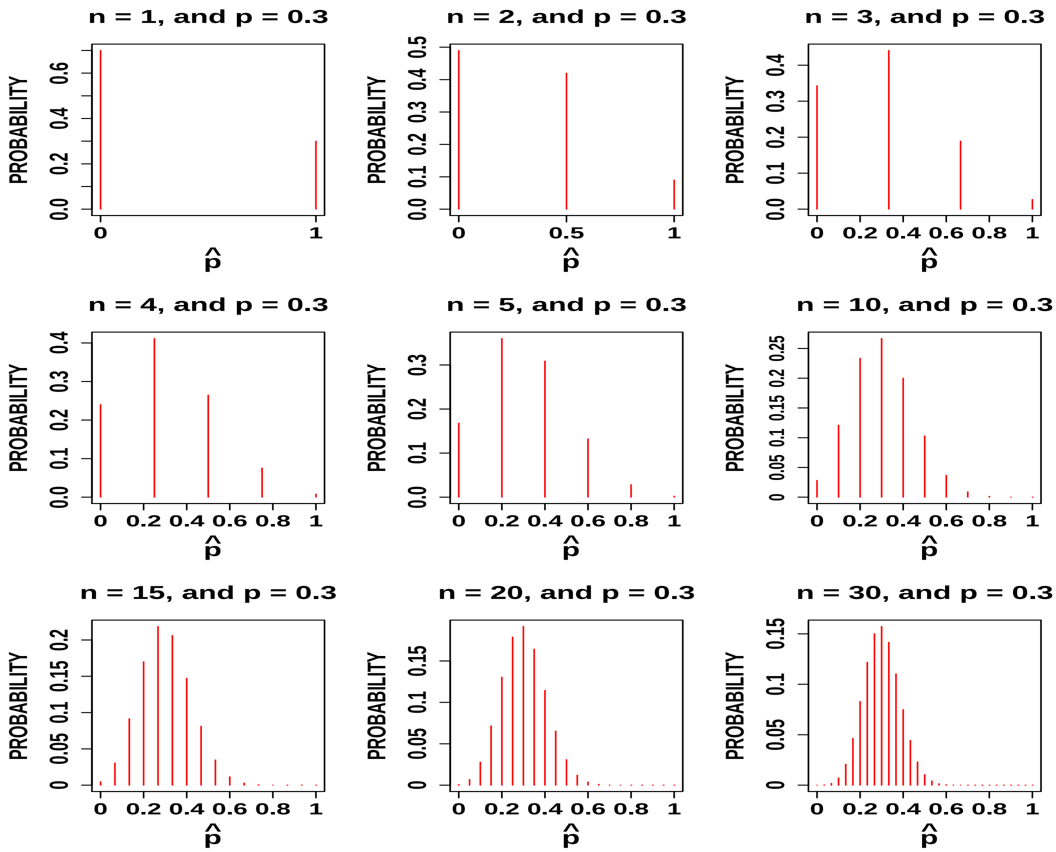
□

Example: Viewing the Central Limit Theorem.

(a) Consider the graphs below for **binomial** random variables, using $p = 0.3$ and $n = 1, 2, 3, 4, 5, 10, 15, 20$, and 30.



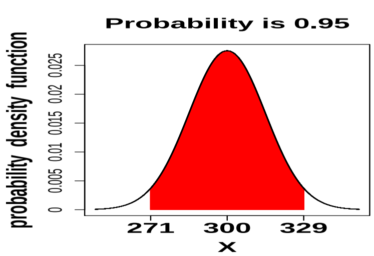
(b) Consider the graphs below for **sampling proportions**, $\hat{p}$, using $p = 0.3$ and $n = 1, 2, 3, 4, 5, 10, 15, 20$, and 30.



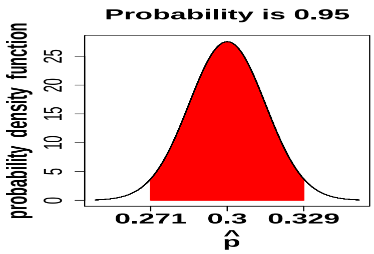
□

Example: *Revisit the Democrats.*

(a) Use the 95% part of the **empirical rule** on the *binomial random variable.*



(b) Use the 95% part of the **empirical rule** on the *sample proportion.*



□

Example: *Virginians who exercise.* According to the Centers for Disease Control and Prevention, about 48% of Virginian adults achieved the recommended level of physical activity.

Recommended physical activity is defined as "reported moderate-intensity activities (i.e., brisk walking, bicycling, vacuuming, gardening, or anything else that causes small increases in breathing or heart rate) for at least 30 minutes per day, at least 5 days per week or vigorous-intensity activities (i.e., running, aerobics, heavy yard work, or anything else that causes large increases in breathing or heart rate) for at least 20 minutes per day, at least 3 days per week or both. This can be accomplished through lifestyle activities (i.e., household, transportation, or leisure-time activities)."

http://apps.nccd.cdc.gov/PASurveillance/StateSumV.asp?Year=2001

www.cdc.gov/nccdphp/dnpa/physical/stats/us_physical_activity/index.htm

Take a sample of size $n = 100$, and let X be the number who achieved the recommended level of physical activity. What is the distribution of X?

□

Case A: Sample **with** replacement from a common population. *Hence, observations are independent.* The observations form an **exact** simple random sample.

Case B: Sample **without** replacement from a common population, but the population size is quite large compared to n. *Hence, observations are nearly independent.* The observations form an **approximate** simple random sample.

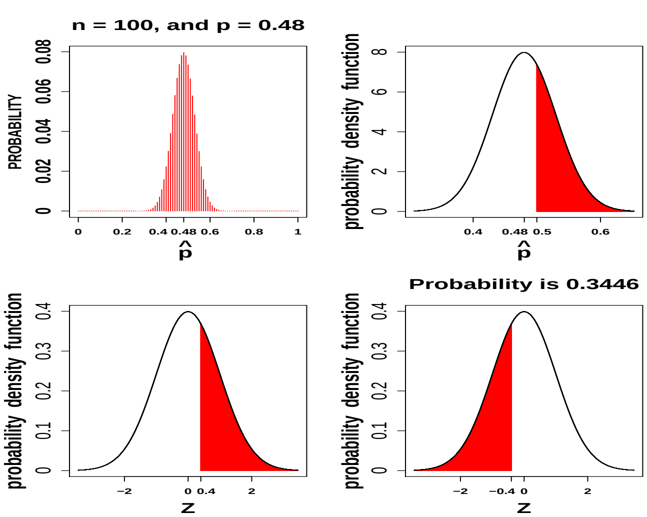
If n is a small percentage of the population size, then sampling **without** *replacement is similar to sampling* **with** *replacement, since sampling the same person more than once would be quite unlikely.*

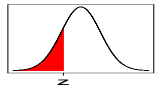
(a) $\mu_{\hat{p}} = p$ *always.*

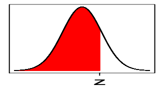
(b) $\sigma_{\hat{p}} = \sqrt{p(1-p)/n}$ (called the **standard error** of $\hat{p}$), exactly for Case A and approximately for Case B.

(c) (A version of the Central Limit Theorem) The sample proportion $\hat{p}$ is approximately normal if {rule of thumb} $np \geq 10$ and $n(1-p) \geq 10$, for Cases A and B.

Example: *Revisit Virginians who exercise.* Determine the probability that a majority of Virginians in a sample of size 100 achieve the recommended level of physical activity.







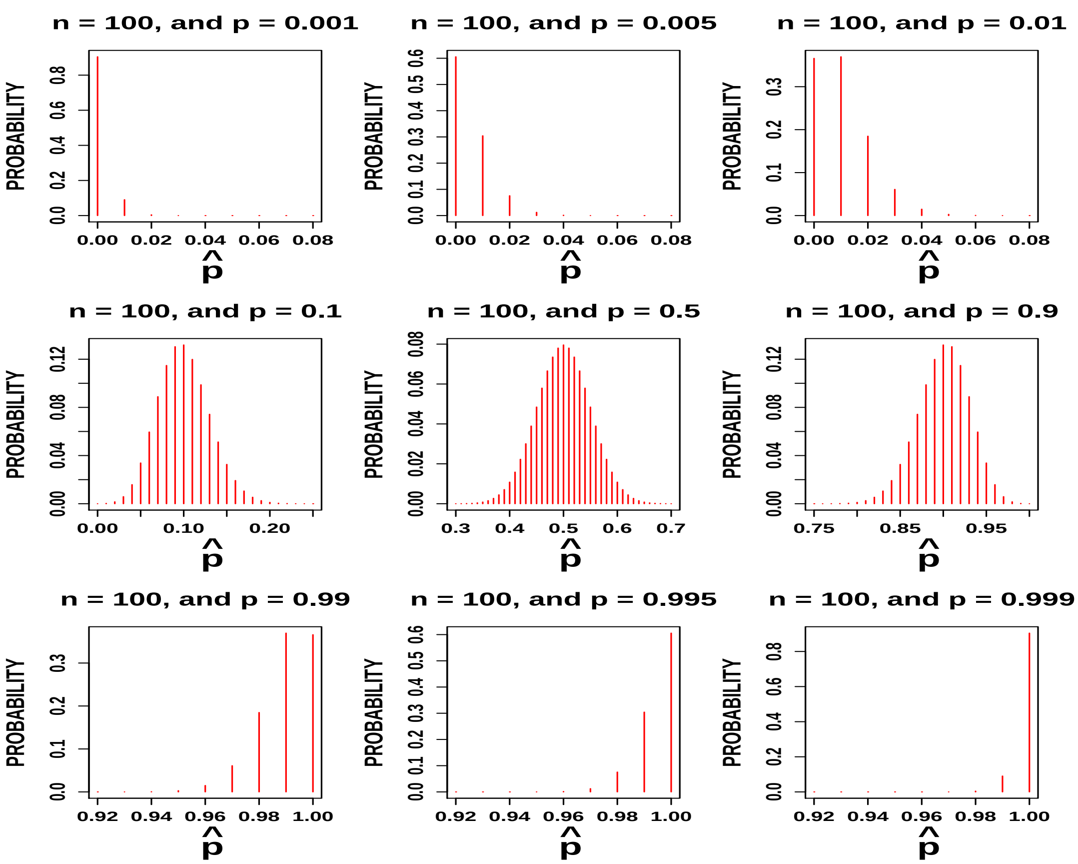
Standard normal table, pp. 722–723

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

□

Why is the rule of thumb needed?

Example: Consider the *sampling distribution* of $\hat{p}$, for $n = 100$ and various p.



□

5.5 The Distribution of a Linear Combination

In this section, let X_1, X_2, …, X_n be random variables, and let a_1, a_2, …, a_n be constants. Then,

$a_1X_1 + a_2X_2 + \ldots + a_nX_n$ is a **linear combination** of the X_is.

What are some examples of linear combinations?

Previously we showed that $E(X+Y) = EX + EY$ for finite EX and EY, regardless of whether X and Y are independent or dependent.

For the **continuous** case, evaluate EaX in terms of $\mu = EX$, for a random variable X with pdf $f_x(\cdot)$ and mean μ, and a constant a.

Evaluate $E(a_1X_1 + \ldots + a_nX_n)$ in terms in the means $\mu_i = EX_i$, for $i = 1, \ldots, n$.

Evaluate $\text{Var}(a_1X_1 + \ldots + a_nX_n)$ in terms in the individual variances (assumed finite) and covariances.

Evaluate $\text{Var}(a_1X_1 + \ldots + a_nX_n)$ for **uncorrelated** X_is, in terms of the individual variances (assumed finite).

Evaluate $E(X_1 - X_2)$ in terms of μ_1 and μ_2 (both assumed finite).

Evaluate $\text{Var}(X_1 - X_2)$ for **independent** X_is, in terms of the individual variances (assumed finite).

The Normal Distribution: A **linear combination** of independent normal random variables is also normally distributed.

Example: Suppose $X \sim N(\mu = 8,\ \sigma = 3)$ and $Y \sim N(\mu = 2,\ \sigma = 4)$ such that X and Y are **independent**. Determine the distribution of $(X - Y)$.

Example: Suppose X and Y are **standard normal** random variables.

(a) Determine the distribution of $(X - Y)$

(b) Suppose X and Y are **independent**. Determine the distribution of $(X - Y)$

(c) Suppose $Y = -X$ with probability 1. Determine the distribution of $(X - Y)$.

(d) Suppose $Y = X$ with probability 1. Determine the distribution of $(X - Y)$.