6 Point Estimation

6.1 Some General Concepts of Point Estimation

Definition: An estimator $\hat{\theta}$ is an **unbiased** estimator of θ if $E\hat{\theta} = \theta$.

Definition: The **bias** of an estimator $\hat{\theta}$ is $(E\hat{\theta} - \theta)$.

Example: Let $X_1, \ldots, X_n$ be a **simple random sample** with finite mean μ. Determine a reasonable **estimator** of μ along with its **bias**.

Example: Let $X \sim \text{Binomial}(n,\ p)$. Based on a sample of size **1**, determine a reasonable **estimator** of p along with its **bias**.

Example: Let $X_1, \ldots, X_n$ be a **simple random sample** with mean μ and finite variance σ^2.

(a) Consider the estimator

$$\hat{\sigma}_1^2 = s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Prove that s^2 is **unbiased** for σ^2.

(b) Consider another estimator

$$\hat{\sigma}_2^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Compute the **bias** of $\hat{\sigma}_2^2$ when estimating σ.

(c) Is s unbiased for σ?

Definition: The **standard error** of an estimator is $\sigma_{\hat{\theta}}$.

When the formula for $\sigma_{\hat{\theta}}$ is a function of unknown parameters, we often estimate these unknown parameters and then determine the **estimated standard error**.

Example: Let $X_1, \ldots, X_n$ be a **simple random sample** with mean μ and finite variance σ^2.

(a) Determine the **standard error** of the sample mean, $\bar{X}$.

(b) Determine the **estimated standard error** of the sample mean, $\bar{X}$ (for unknown σ).

Example: Let $X \sim \text{Binomial}(n,\ p)$.

(a) Determine the **standard error** of the sample proportion, $\hat{p}$.

(b) Determine the **estimated standard error** of the sample proportion, $\hat{p}$ (for unknown p).

Remark: The **standard error** measures variability in the estimator and typically **decreases** as n gets large.

Remark: Estimators with small (perhaps zero) bias and small standard error typically are preferable.

6.2 Methods of Point Estimation

The Method of Moments

Definition: For a simple random sample $X_1, \ldots, X_n$, the ***k*th sample moment** is $(1/n)\sum_{i=1}^{n} X^k$, for $k = 1, 2, \ldots$.

Definition: For a randomly sampled observation X from some population, the ***k*th population moment** is EX^k, for $k = 1, 2, \ldots$.

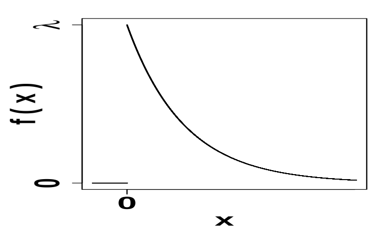
The **method of moments** estimator of some parameter θ is determine by setting the ***k*th sample moment** to the ***k*th population moment**, and then solving for θ.

Remark: When computing **method of moments** estimators, typically we start with $k = 1$, then $k = 2$, and so on.

Example: Let $X_1, X_2, \ldots, X_n$ be a simple random sample from an Exponential(λ) population, where

$$f(x) = \begin{cases} \lambda\, e^{-\lambda\, x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

for a constant $\lambda > 0$. Compute a **method of moments** estimator of λ.



Example: Let $X_1, X_2, \ldots, X_n$ be a simple random sample from a Bernoulli(p) population. Compute a **method of moments** estimator of p.

Example: Let $X_1, X_2, \ldots, X_n$ be a simple random sample from a $N(\mu, \sigma)$ population.

(a) Compute a **method of moments** estimator of μ.

(b) Compute a **method of moments** estimator of σ^2.

(c) Compute a **method of moments** estimator of σ.

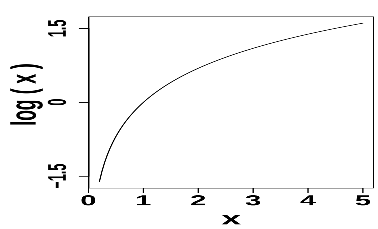
Maximum Likelihood Estimation

Let $x_1, x_2, \ldots, x_n$ be a **simple random sample** from a population with **pdf** $f(x;\theta)$, where θ is an unknown parameter. The **likelihood function** is

$$L(\theta) = f(x_1, x_2, \ldots, x_n;\ \theta) = f(x_1;\theta)\ f(x_2;\theta)\cdots f(x_n;\theta).$$

The value, say $\hat{\theta}$, which maximizes the likelihood function is called the **maximum likelihood estimator** of θ.

Hint: Often, instead of maximizing $L(\theta)$ with respect to θ, maximize $\log\ L(\theta)$, since the logarithm function is monotonic.

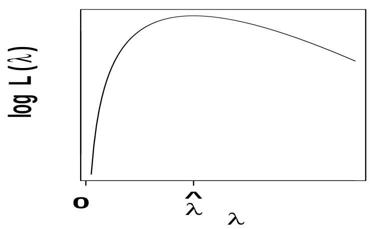


Example: Let $x_1, x_2, \ldots, x_n$ be independent observations from an Exponential(λ) distribution with **pdf**

$$f(x) = \begin{cases} \lambda\ e^{-\lambda\ x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

for a constant $\lambda > 0$.

(a) Determine the **maximum likelihood estimator** of $\boldsymbol{\lambda}$.



(b) Determine the **maximum likelihood estimator** of the population mean of X, where $X \sim$Exponential(λ).

Remark: A function of a **maximum likelihood estimator** is also a **maximum likelihood estimator**.

Remark: The notion of **maximum likelihood estimators** may be extended to more than one variable.

Example 6.17, p. 247: Let $X_1, X_2, \ldots, X_n$ be a simple random sample from a $N(\mu, \sigma)$ distribution, where both μ and σ are unknown. The joint **maximum likelihood estimator** of (μ, σ) is $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$.

What is the **maximum likelihood estimator** of $\boldsymbol{\sigma}$?

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