

# 1 One-Sample Methods

This chapter discusses nonparametric hypothesis tests and confidence intervals based on the binomial distribution.

## 1.1 A Nonparametric Test of Hypothesis and Confidence Interval for the Median

*Notation:*  $\theta_{0.5}$  is the (unknown) population median of a **continuous** distribution.

Interpret  $\theta_{0.5}$ .

What do heavy-tailed distributions tend to produce?

**Example:**

```
> x = c(20:28, 100)
```

```
> mean(x)
```

```
> mean( x[1:9] )
```

```
> median(x)
```

```
> median( x[1:9] )
```

□

## 1.1.1 Binomial Test

Suppose that under the null hypothesis ( $H_0$ ), the population median is 75.

Sample 40 independent observations, and let  $B$  be the number of observations larger than 75.

What is the distribution of  $B$  under  $H_0$ ?

**Example 1.1.1, p. 11, [table1.1.1.txt](#):** Suppose a certain food product is advertised to contain 75 mg of sodium per serving, but preliminary studies indicate that servings may contain more than that amount. The amount of sodium in the product varies from one serving to another.

Test if the *median* amount of sodium per serving is 75 mg versus the alternative that the *median* is greater than 75 mg, at level  $\alpha = 0.05$ .

$H_0 : \theta_{0.5} = 75$  mg,  $H_a : \theta_{0.5} > 75$  mg

Suppose an observation results in exactly 75 mg.

Should this particular observation support  $H_0$  or  $H_a$ ?

*Note:* Some textbooks recommend discarding *ties* for the *binomial* test, since such (rounded) observations are not informative.

Under  $H_0$ , what proportion of the observations do we expect (on average) to be strictly greater than 75 mg?

# Download the data set.

```
> x = scan(
  "http://educ.jmu.edu/~garrenst/math324.dir/datasets/table1.1.1.txt" )
```

```
> x = scan2( "table1.1.1.txt" )  
> x  
> length( x )
```

Is this a one-sided test or a two-sided test?

Does “26” fall far enough into the tail of the Binomial( $n = 40$ ,  $p = 0.5$ ) distribution to reject  $H_0$  at level  $\alpha = 0.05$ ?

Solve the problem again, this time excluding “ties”; i.e., discard all observations which are 75.

```
> n = sum( x != 75 )
```

Does “26” fall far enough into the tail of the Binomial( $n = 39$ ,  $p = 0.5$ ) distribution to reject  $H_0$ ?

Determine the  $p$ -value.

$R$  has a built-in function called `binom.test`.

The function `binom.test` does not request the original data set, but rather the number of successes and the sample size.

```
> binom.test( 26, 39, alternative="greater" )
```

Textbook uses normal approximation to the binomial.

### **Brief review:**

If the distribution is **symmetric**, how do the population *mean* and population *median* compare?

The *binomial* test typically should be used in place of the *t*-test when the distribution is symmetric and the tails are heavy.

## 1.1.2 Confidence Interval

Explain how confidence intervals and hypothesis tests are related via the following:

$$H_0 : \theta_{0.5} = \theta_H, \quad H_a : \theta_{0.5} \neq \theta_H \quad \text{with } \alpha = 0.05$$

95% confidence interval on  $\theta_{0.5}$

Textbook gives a **normal approximation** for constructing this confidence interval regarding the binomial test.

However, we can compute this confidence interval **exactly** using a function, which effectively performs a hypothesis test (based on the binomial test) on each of the observations and  $\pm\infty$  (using exact binomial probabilities, not normal approximations to binomials).

**Example:** Consider the same data set on sodium content from Table 1.1.1, p. 12, in Example 1.1.2. Construct a 95% confidence interval on the population median (where the only assumption on the population is continuity).

```
> x = scan2( "table1.1.1.txt" )  
> ?quantileCI
```

We are 95% confident that the population median sodium content per serving is between 75.0 mg and 77.1 mg.

**Homework C1.1.1: (OK, already solved below!)** (Understanding nonparametric hypothesis tests and confidence intervals.) A simple random sample of size  $n = 26$  is drawn from a population, whose only assumption is continuity.

(a) Suppose one is interested in testing  $H_0 : \theta_{0.5} = 80$  versus  $H_a : \theta_{0.5} \neq 80$  at level  $\alpha = 0.05$ .

Clearly explain in detail (using more words than mathematics) how to test these hypotheses, based on the binomial test. In other words, explain HOW to use any testing procedures and the NAMES of any random variables used.

- (1) Let  $B$  be the number of observations **greater** than 80.
- (2) If (and only if)  $B$  is significantly small (much less than 13) or significantly large (much greater than 13), then  $H_0$  should be rejected in favor of  $H_a$ .
- (3)  $B$  has a Binomial( $n = 26, p = 0.5$ ) distribution under  $H_0$ .

(b) Suppose one is interested in constructing a 95% confidence interval on the population median, based on the binomial test.

Clearly explain in detail (using more words than mathematics) how to construct this confidence interval, based on the binomial test.

(1) Consider the hypothesis test  $H_0 : \theta_{0.5} = \theta_H$  versus  $H_a : \theta_{0.5} \neq \theta_H$  at level  $\alpha = 0.05$ .

(2) All real values of  $\theta_H$  which fail to reject  $H_0$  at level  $\alpha = 0.05$ , based on the binomial test, comprise the 95% confidence interval.

**End of Homework C1.1.1.** □

**Homework** p. 21: Exercises 1.1\*, 1.2a\*, 1.3ac

**Hints for homework exercise 1.1\***: State the null and alternative hypotheses. **Define** any notation which you need; e.g., population median. Either retype your  $p$ -value as a comment using “#”, or highlight the  $p$ -value in yellow. State the conclusion in statistical terms; hence, state whether you reject  $H_0$  or fail to reject  $H_0$  in favor of  $H_a$ . Also, state the conclusion in regular English; hence, state that you conclude or fail to conclude that the population median is greater than 70. Introduce the question number as a comment using “#” or in red using .html code; e.g., `<span style="color: red"> Exercise 1.1 </span>`.

**Hints for homework exercise 1.2a\***: Find the EXACT (not approximate) confidence interval. Introduce the question number and letter as a comment using “#” or in red using .html code; e.g., `<span style="color: red"> Exercise 1.2(a) </span>`.

## 1.2 Estimating the Population cdf and Percentiles

## 1.2.1 Confidence Interval for the Population cdf

*Recall:* The cumulative distribution function of a random variable  $X$  is

$$F(x) = P(X \leq x) \text{ for all real } x.$$

In most practical situations, the population is unknown, so  $F(x)$  is also unknown.

Take a simple random sample from this unknown population.

For any fixed value of  $x$ , how can we estimate  $F(x)$ ?

The **empirical cdf** is denoted by  $\hat{F}(x)$ .

**Example 1.2.1, p. 15, [table1.2.1.txt](#):** Testing of electrical and mechanical devices often involves an action such as turning a device on and off or opening and closing a device many times. The interest is in the distribution of the number of on-off or open-close cycles that occur before the device fails. The hypothetical data in Table 1.2.1 are the number of cycles (in thousands) that it takes for 20 door latches to fail.

(a) Evaluate the empirical cdf at 25.

```
> x = scan2( "table1.2.1.txt" )
```

```
> length( x )
```

```
> x <= 25
```

```
> sum( x <= 25 )
```

```
> mean( x <= 25 )
```

(b) Using computer software, graph the empirical cdf for this data set.

```
> ?plot.ecdf
```

(c) Suppose we were able to sample more and more observations from this population. Explain how the empirical cdf should appear.

□

*Recall:* The **empirical cdf** is a generalization of a **sample proportion**.

What do we know about sample proportions for large sample sizes?

**Revisit example 1.1.2, p. 12, regarding sodium content, [table1.1.1.txt](#):**

(a) Determine the empirical cdf evaluated at 75.

```
> x = scan2( "table1.1.1.txt" )
```



(b) Graph the empirical cdf.

(c) Is a normal approximation valid for constructing a 95% confidence interval on  $F(75)$ ?

```
> n = length( x )
```

(d) Construct a 95% confidence interval on  $F(75)$ , based on the normal approximation.

```
> c( p.hat - qnorm( 0.975 ) * sqrt( p.hat * (1-p.hat) / n ), p.hat +  
      qnorm( 0.975 ) * sqrt( p.hat * (1-p.hat) / n ) )
```

(e) Construct the *exact* 95% confidence interval on  $F(75)$ , based on binomial probabilities.

In other words, for what values of  $p_0$  would we fail to reject  $H_0$ , when testing

$H_0 : F(75) = p_0$  versus  $H_a : F(75) \neq p_0$  at level 0.05, based on the binomial test?

```
> sum( x <= 75 )
```

```
> length( x )
```

(f) Repeat parts (d) and (e) using a function from the [binom](#) package.

```
> install.packages( "binom" )
```

```
> library( binom )
```

(g) Is the above confidence interval wide or narrow?

## 1.2.2 Inference for Percentiles

*Question:* What is the 25th percentile of a distribution governed by the random variable  $X$ ?

Find the 25th percentile of a  $N(\mu = 500, \sigma = 100)$  distribution.

```
> shadeDist( 432.551, "dnorm", 500, 100 )
```

```
> shadeDist( qnorm( 0.25, 500, 100 ), "dnorm", 500, 100 )
```

What does the 50th percentile represent?

*Notation:*  $\theta_p$  represents the 100 $p$ th percentile, for  $0 \leq p \leq 1$ .

**Example:** Sample 50 observations from an unknown continuous population.

(a) Clearly explain (using more words than mathematics) how we might test

$H_0 : \theta_{0.3} = 200$  versus  $H_a : \theta_{0.3} \neq 200$  at level  $\alpha = 0.05$ .

Determine  $\hat{p}$ , the sample proportion of observations **less** than 200.

If (and only if)  $\hat{p}$  is significantly small (much less than 0.3) or significantly large (much greater than 0.3), then  $H_0$  should be rejected in favor of  $H_a$ .

Another approach:

Determine  $B$ , the number of observations less than 200.

Note that  $B \sim \text{Binomial}(n = 50, p = 0.3)$  under  $H_0$ .

If (and only if)  $B$  is significantly small (much less than 15, since  $15 = np$  under  $H_0$ ) or significantly large (much greater than 15), then  $H_0$  should be rejected in favor of  $H_a$ .

(b) Suppose one is interested in constructing a 95% confidence interval on  $\theta_{0.3}$ , based on the binomial test.

Clearly explain in detail (using more words than mathematics) how to construct this confidence interval, based on the binomial test.

- (1) Consider the hypothesis test  $H_0 : \theta_{0.3} = \theta_H$  versus  $H_a : \theta_{0.3} \neq \theta_H$  at level  $\alpha = 0.05$ .
- (2) All real values of  $\theta_H$  which fail to reject  $H_0$  at level  $\alpha = 0.05$ , based on the binomial test, comprise the 95% confidence interval.

Textbook discusses using the normal approximation to the binomial (or sample proportion) for inference on percentiles.

**Example 1.2.4 (expanded), [table1.2.1.txt](#):** Refer to the *door latch* data in Table 1.2.1.

(a) Determine a point estimate of the 25th percentile; i.e., determine the 25th **sample** percentile.

```
> x = scan2( "table1.2.1.txt" )
```

```
# Sort the observations from smallest to largest.
```

```
> length(x) * 0.25
```

```
> x[ 5 ]
```

```
> ?quantile
```

(b) Construct a 90% confidence interval on the 25th percentile.

```
> ?quantileCI
```

(c) Construct a 95% confidence interval on the 25th percentile.

(d) Construct a 90% confidence interval on the 50th percentile.

(e) Construct a 90% confidence interval on the 75th percentile.

□

**Homework** p. 21: Exercises 1.2b\*, 1.2c\*, 1.3b

**Hints** for homework exercise 1.2b\* and 1.2c\*: Find the EXACT

(not approximate) confidence intervals. Introduce the question **number** and **letter** as a comment using “#” or in red using .html code; e.g., `<span style="color:red"> Exercise 1.2(b) </span>`.

## 1.3 A Comparison of Statistical Tests

Compare the  $t$ -test with the binomial test under specific distributions and sample sizes.

A **type I error** is the event that  $H_0$  is rejected when  $H_0$  is true.

We denote  $\alpha = P(\text{type I error})$ , and often we set  $\alpha = 0.05$ .

A **type II error** is the event that  $H_0$  is NOT rejected when  $H_a$  is true.

**power** =  $1 - P(\text{type II error})$

**Power** is the probability that  $H_0$  is rejected when  $H_a$  is true.

Do we want  $\alpha$  to be large or small?

Do we want **power** to be large or small?

What happens to **power** as  $\alpha$  gets small?

How can we decrease  $\alpha$  and increase **power** simultaneously?

In general, power is good!

## Steps for computing power of a $t$ -test

Compute the power of the  $t$ -test, where the alternative distribution is  $N(\mu_a, \sigma)$ , and we are testing  $H_0 : \mu = \mu_0$  versus  $H_a : \mu > \mu_0$ .

- (1) Find the rejection region in terms of  $T^*$ , such that  $H_0$  is rejected if and only if  $T^*$  is in the rejection region, where  $T^* = (\bar{X} - \mu_0)/(s/\sqrt{n})$ , for a specific value of  $\alpha$ .
- (2) Compute the probability that  $T^*$  is in the rejection region, under the assumption that the population is  $N(\mu_a, \sigma)$ .

*Note:* Step (1) involves using the  $t$ -table, and step (2) involves using the **noncentral**  $t$ -table with **noncentrality parameter**  $= (\mu_a - \mu_0)/(\sigma/\sqrt{n})$ . This table is easily tabulated by  $R$ .

*Note:* Your textbook loses accuracy by approximating the  $t$ -distribution with a normal distribution (twice), and names the test the **CLT test**.

Hence, in homework exercises, replace the term *CLT test* by *t-test*.

### Example 1.3.1, p. 19 (first part; $t$ -test with normal

**alternative):** Refer to example 1.1.2 regarding sodium content of 40 servings of a food product. The hypothesis test is  $H_0 : \mu = 75$  mg versus  $H_a : \mu > 75$  mg, using  $\alpha = 0.05$ . Using the following steps, compute the power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = 75.8$  mg,  $\sigma = 2.5$  mg).

- (a) Define the standardized test statistic.
- (b) Determine the (exact) rejection region for your test statistic in part (a).
- (c) Graph the pdf of your standardized test statistic under  $H_0$ , and shade in the region corresponding to  $\alpha$ .

(d) Determine the noncentrality parameter.

(e) Compute the power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$  (without using `power.t.test`).

Below is the graph of the pdf of  $T^*$  under  $H_a$ .

```
> shadeDist( qt(0.95, 39), "dt", 39, ncp, FALSE )
```

Below is the graph of the pdf of  $T^*$  (approximated by a  $N(\mu = 2.023858, \sigma = 1)$ ) under  $H_a$ .

```
> shadeDist( qnorm(0.95), "dnorm", ncp, 1, FALSE )
```

```
> plotDist( "dnorm", ncp, 1, "dt", 39, ncp )
```

(f) Graph the pdf of  $T^*$  simultaneously under  $H_0$  and under  $H_a$ .

```
> plotDist( "dt", 39, , "dt", 39, ncp )
```

□

### Example 1.3.1, p. 19 (second part; $t$ -test with normal and

**Laplace alternatives):** Refer to example 1.1.2 regarding sodium content of 40 servings of a food product. The hypothesis test is  $H_0 : \mu = 75 \text{ mg}$  versus  $H_a : \mu > 75 \text{ mg}$ , using  $\alpha = 0.05$ .

(a) Compute the power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$ .

```
> ?power.t.test
```

*Note well:* Use `delta > 0` (for values of  $\mu_a$  in  $H_a$ ).

```
> power.t.test( 40, 0.8, 2.5, type="one.sample", alternative="one.sided"
  )
```

(b) Compute the power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = 76.1 \text{ mg}, \sigma = 2.5 \text{ mg})$ .

What should happen to the power?

(c) What power should we obtain by plugging in zero for `delta` in `power.t.test`?

(d) Compute the (exact) power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = \mu_a, \sigma = 2.5 \text{ mg})$ , for values of  $\mu_a$  equal to 74.8, 75, 75.2, 75.4, 75.6, and 75.8.

```
> delta = c(74.8, 75, 75.2, 75.4, 75.6, 75.8) - 75
> power.t.test( 40, delta, 2.5, type="one.sample",
  alternative="one.sided" )
```



- (e) Next, graph the **LAPLACE**( $\mu = 75$ ,  $\sigma = 2.5$ ) probability density function.
- (f) Compute the (approximate) power of the  $t$ -test under the alternative that the sodium content is distributed as **LAPLACE**( $\mu = \mu_a$ ,  $\sigma = 2.5$  mg), for values of  $\mu_a$  equal to 74.8, 75, 75.2, 75.4, 75.6, and 75.8.

□

## Computing power of a binomial test

### Example 1.3.1, p. 19 (third part; binomial test with normal

**alternative):** Refer to example 1.1.2 regarding sodium content of 40 servings of a food product. The hypothesis test is  $H_0 : \theta_{0.5} = 75$  mg versus  $H_a : \theta_{0.5} > 75$  mg, using  $\alpha = 0.05$ . Using the following steps, compute the (exact) power of the binomial test under the alternative that the sodium content is distributed as  $N(\mu = 75.8$  mg,  $\sigma = 2.5$  mg).

- (a) Define the test statistic.
- (b) Graph the pdf of your test statistic under  $H_0$ .

(c) Determine the (exact) rejection region for your test statistic in part (a).

(d) Determine the exact **size** of this test.

```
> shadeDist( 25, "dbinom", 40, 0.5, FALSE, xlab="B" )
```

(e) Determine the probability that a randomly sampled  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$  observation is larger than 75.

(f) Graph the pdf of a  $N(\mu = 75.8, \sigma = 2.5)$  distribution, and shade in the region corresponding to the above probability.

(g) Let  $B$  be the number of observations larger than 75. What is the distribution of  $B$  under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$ ?

(h) Compute the (exact) power of the binomial test under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$  (without using `power.binom.test`).

□

**Example 1.3.1, p. 19 (fourth part; binomial test with Laplace alternative):** Refer to example 1.1.2 regarding sodium content of 40 servings of a food product. The hypothesis test is  $H_0 : \theta_{0.5} = 75$  mg versus  $H_a : \theta_{0.5} > 75$  mg, using  $\alpha = 0.05$ . Using the following steps, compute the (exact) power of the binomial test under the alternative that the sodium content is Laplace-distributed with  $\mu = 75.8$  mg and  $\sigma = 2.5$  mg.

- (a) Define the test statistic.
- (b) Determine the (exact) rejection region for your test statistic in part (a).
- (c) Determine the probability that a randomly sampled Laplace( $\mu = 75.8$  mg,  $\sigma = 2.5$  mg) observation is larger than 75.
- (d) Graph the pdf of a Laplace( $\mu = 75.8$ ,  $\sigma = 2.5$ ) distribution, and shade in the region corresponding to the above probability.

- (e) Let  $B$  be the number of observations larger than 75. What is the distribution of  $B$  under the alternative that the sodium content is distributed as  $\text{Laplace}(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$ ?
- (f) Compute the (exact) power of the binomial test under the alternative that the sodium content is distributed as  $\text{Laplace}(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$  (without using `power.binom.test`).

□

**Example 1.3.1, p. 19 (fifth part; summary; comparing  $t$ -test and binomial test):**

Refer to example 1.1.2 regarding sodium content of 40 servings of a food product. The hypothesis test is  $H_0 : \mu = 75 \text{ mg}$  versus  $H_a : \mu > 75 \text{ mg}$ , using  $\alpha = 0.05$ . (For the binomial test, replace  $\mu$  by  $\theta_{0.5}$  in  $H_0$  and  $H_a$ .)

- (a) Determine the (exact) power of the  $t$ -test under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$ .

```
> power.t.test( 40, 0.8, 2.5, type="one.sample", alternative="one.sided"
 )
```

- (b) Determine the (exact) power of the **binomial** test under the alternative that the sodium content is distributed as  $N(\mu = 75.8 \text{ mg}, \sigma = 2.5 \text{ mg})$ ,

```
> ?power.binom.test # from R-package 'jmuOutlier'
```

```
> power.binom.test( 40, 0.05, "greater", 75, pnorm, 75.8, 2.5 )
```

(c) Which test, the  $t$ -test or the **binomial** test, is better (i.e., more powerful) under this **normal** alternative?

(d) Determine the (approximate) power of the  $t$ -test under the alternative that the sodium content is distributed as Laplace( $\mu = 75.8$  mg,  $\sigma = 2.5$  mg).

(e) Determine the (exact) power of the **binomial** test under the alternative that the sodium content is distributed as Laplace( $\mu = 75.8$  mg,  $\sigma = 2.5$  mg).

(f) Which test, the  $t$ -test or the **binomial** test, is better (i.e., more powerful) under this **Laplace** alternative?

□

### Summary:

- ⊙ Often, the  $t$ -test is *more* powerful than the **binomial** test for *light*-tailed distributions such as the **normal** distribution.
- ⊙ However, the **binomial** test is often *more* powerful than the  $t$ -test for *heavy*-tailed distributions such as the **Laplace** distribution.
- ⊙ If the tails of the distribution are so *heavy* that the population standard deviation  $\sigma$

is not finite, then the  $t$ -test should not be used.

**Homework** p. 22: Exercises 1.4, 1.5a\* (hand-calculations), 1.5(b,c,d)\*

**Hints for homework exercise 1.5\***: Use `rep` or `replicate` to help create this data set. Notice that you are not being asked to compute power, but you will need to calculate  $p$ -values. Either retype your  $p$ -values as a comment using “#”, or highlight the  $p$ -values in yellow. Always replace the textbook’s terminology *CLT test* with *t-test*. Introduce the question number and letter as a comment using “#” or in red using .html code; e.g., `<span style="color: red"> Exercise 1.5(a) </span>`.

**Hints for homework exercise 1.5(a)\***: Solve using *hand calculations*. As a reminder, whenever you are asked for a solution based on *hand calculations*, you may still use *R*, but withOUT any *R*-functions containing the word `test`.

**Hints for homework exercise 1.5(c)\***: This question is somewhat poorly worded, so use the following re-phrasing instead: “Based on the results of parts (a) and (b), for what types of distributions can the binomial test detect departures from the null location parameter much more easily than the  $t$ -test can?”

**Hints for homework exercise 1.5(d)\***: In the spirit of replacing *CLT test* with *t-test*, replace  $Z_\mu$  with  $T = (\bar{X} - \mu_0)/(s/\sqrt{n})$ . When answering the final question, **explain** whether or not your results correspond to intuition.

**Homework** p. 22: Exercises 1.6, 1.7 (using *R*, NOT the binomial Table A1), 1.8

To obtain the power of the binomial test, we must find  $p = P(X > 75)$  when  $\mu = 75.8$ . If  $X$  has a Laplace distribution with mean  $\mu$  and standard deviation  $\sigma$ , then  $(X - \mu)/\sigma$  has the standard form of the Laplace distribution given in Table 0.2.1. For  $x < \mu$ , it can be shown that

$$P(X > x) = .5 + .5\left(1 - e^{-\sqrt{2}|x-\mu|/\sigma}\right)$$

Applying this formula to our example, we find  $p = P(X > 75) = .682$ . From the formula for the power of the binomial test in the power computation for a normal distribution, we find that the power is .76. Thus, the form of the distribution of the population can greatly affect the relative power of two tests.

### Exercises

- 1 The data in the table are simulated exam scores. Suppose the exam was given in the semester after the course content was revised, and the previous median exam score was 70. We would like to know whether or not the median score has increased. Answer the question by applying the binomial test.

Simulated Exam Scores

79	74	88	80	80	66	65	86	84	80	78	72	71	74	86	96	77	81	76	80
76	75	78	87	87	74	85	84	76	77	76	74	85	74	76	77	76	74	81	76

- 2 Refer to the data in Exercise 1.
  - a Make a 90% confidence interval for the median.
  - b Make a 90% confidence interval for the 75th percentile.
  - c Make a 90% confidence interval for  $F(80)$ , the probability that a score is less than or equal to 80.
- 3 The data in the table are the yearly rainfall totals in Scranton, Pa., for the years 1951–1984.

Rainfall Totals (inches) for Scranton, Pa., 1951–1984

21.3	28.8	17.6	23.0	27.2	28.5	32.8	28.2	25.9	22.5	27.2	33.1	28.7	24.8	24.3	27.1	30.6
26.8	18.9	36.3	28.0	17.9	25.0	27.5	27.7	32.1	28.0	30.9	20.0	20.2	33.5	26.4	30.9	33.2

- a Make a 95% confidence interval for the median.
- b Make 90% confidence intervals for the 20th and 80th percentiles.
- c The confidence interval procedure assumes that the observations are independent and identically distributed. Do you think this is a reasonable assumption for the rainfall data? If not, what could cause this assumption to be invalid?

## 22 Chapter 1: One-Sample Methods

- 4 Suppose we test the hypotheses  $H_0: \theta_{.5} = 75$  versus  $H_a: \theta_{.5} > 75$  and, regardless of the data, we reject  $H_0$ . What is the probability of a Type I error? What is the power of the test for values of  $\theta_{.5} > 75$ ?
- 5 Suppose we assume that the population distribution under  $H_0$  is symmetric so that  $\theta_{.5} = \mu$ . Without looking at the data to check the validity of this assumption, we apply the binomial test and the CLT test. Suppose it turns out that 39 data values that are equal to 75.1 and the 40th one is equal to 90.
  - a What decision is reached using the binomial test to test  $H_0: \theta_{.5} = 75$  versus  $H_a: \theta_{.5} > 75$ ?
  - b What decision is reached using the CLT test to test  $H_0: \mu = 75$  versus  $H_a: \mu > 75$ , where the statistic is computed using the sample standard deviation  $S$  in place of the unknown population standard deviation  $\sigma$ ?
  - c Based on the results of parts a and b, what types of distributions that satisfy the alternative hypothesis are particularly easy for the binomial test to detect in comparison to the CLT test?
  - d Replace 90 by other values such as 80, 78, and 76 that are closer to the null hypothesis. Note what happens to the value of  $Z_\mu$ . Does this correspond to intuition?
- 6 Refer to Section 1.3.3. No computations are required to answer the following questions.
  - a What is the value of the power of the binomial test when  $\mu = 75$ ?
  - b What happens to the power as  $\mu$  gets large?
  - c How does increasing the sample size affect the power of the binomial test?
- 7 Suppose we test  $H_0: \theta_{.5} = \theta_H$  versus  $H_a: \theta_{.5} > \theta_H$  using the binomial test with a sample size  $n = 10$ .
  - a If we reject  $H_0$  when  $B \geq 8$ , use the binomial Table A1 to determine the exact probability of a Type I error.
  - b Suppose we observe a value of  $B = b_{\text{obs}}$ . The  $p$ -value is the probability that  $B \geq b_{\text{obs}}$  given that  $H_0$  is true. Find the  $p$ -values for  $b_{\text{obs}} = 5, 6, 7, 8, 9, 10$ .

**Theory and Complements**

- 8 Refer to the derivations of the power functions in Section 1.3. Evaluate and sketch the power functions of the statistics  $Z_\mu$  and  $Z_B$  for values of the mean between 75 and 77 assuming that the populations have normal distributions. Using your sketch, determine the maximum difference between the power functions. Repeat this procedure for the Laplace population distribution.

Below are solutions to a sample exam, to illustrate easy formatting only.



```

1 ---
2 title: "Solutions to sample exam, to illustrate easy formatting only."
3 author: "Name of student"
4 date: "`r Sys.Date()`"
5 output: html_document
6 ---
7
8 ```{r}
9
10     # Always include R-code and R-output.
11     # Introduce your solutions with question number and letter.
12     # All solutions should be in the correct numerical order.
13     # Right-sided comments are not needed in your output.
14
15     library(fastGraph); library(jmuOutlier); library( binom ); course.number <- 324
16
17     # Question #1(a)      # Read in and print the data set.
18     x <- scan("http://educ.jmu.edu/~garrenst/math324.dir/datasets/exercise1.1.txt")
19     x
20
21     # Question #1(b)      # Compute the sample mean. Only one answer is given,
22     # so you need not restate or highlight your output.
23     mean(x)
24
25     # Question #1(c)      # Construct histogram.
26     hist(x)
27
28     # Question #1(d)      # Compute p-value. Restate or highlight in yellow your
29     # p-value, since t.test() produces additional output.
30     t.test( x, mu=80, alternative="less" )
31     # The p-value is 0.06653.
32
33     # Question #1(d)      # Alternative solution. No need to restate or highlight
34     # in yellow your p-value, since only one answer
35     # (i.e., the p-value) is printed.
36     t.test( x, mu=80, alternative="less" )$p.value
37
38     # Question #2          # State the test statistic used for a t-test on an
39     # unknown population mean.
40     #  $T = ( \bar{X} - \mu ) / ( s / \sqrt{n} )$  # This solution is fine,
41     # but do not be sloppy with parentheses.
42     ```

```

```

41
42  $T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$ 
43 \hspace{0.5in}\hbox{Alternatively, using } \LaTeX \hbox{ code.}

```

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \quad \text{Alternatively, using } \LaTeX \text{ code.}$$

```

44
45 Alternative method for introducing solutions:
46
47 <span style="color: red">\color{darkred}Question #3</span>
48 ~~~{r}
49 # Question #3           # Compute 4*(3+7) using R-code.
50                         # Since an erroneous solution of 4*(3+) halts execution
51                         # of all R-code and hence fails to show even your correct
52                         # R-output, then ideally correct the R-code.
53                         # Otherwise, comment out the erroneous R-code,
54                         # but your solution to question #3 would be nonexistent.
55 # 4*(3+)
56 ~~~

```

# Solutions to sample exam, to illustrate easy formatting only.

Name of student

2024-02-08

```
# Always include R-code and R-output.  
# Introduce your solutions with question number and letter.  
# All solutions should be in the correct numerical order.  
# Right-sided comments are not needed in your output.  
library(fastGraph); library(jmuOutlier); library( binom ); course.number <- 324  
  
# Question #1(a)      # Read in and print the data set.  
x <- scan("http://educ.jmu.edu/~garrenst/math324.dir/datasets/exercisel.1.txt")  
x
```

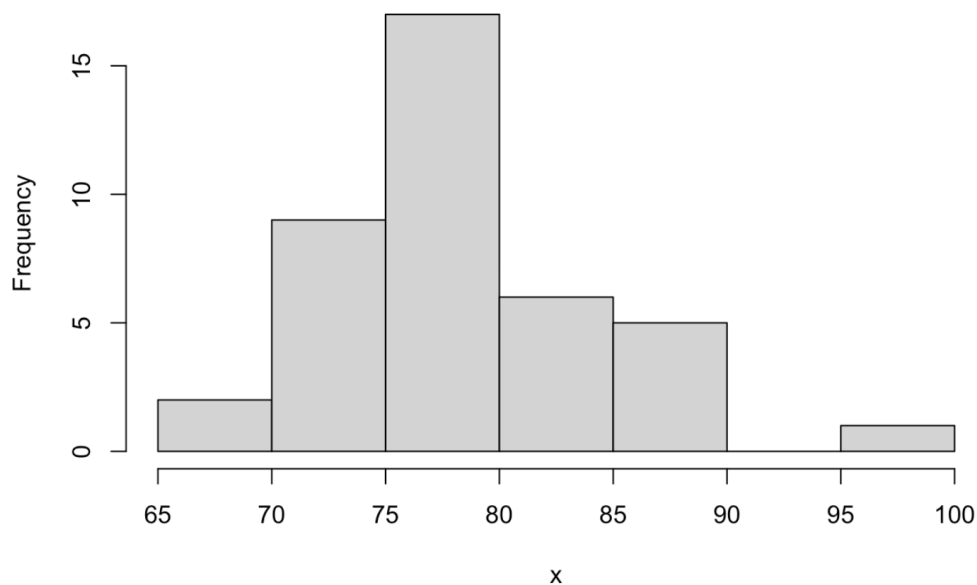
```
## [1] 79 74 88 80 80 66 65 86 84 80 78 72 71 74 86 96 77 81 76 80 76 75 78 87 87  
## [26] 74 85 84 76 77 76 74 85 74 76 77 76 74 81 76
```

```
# Question #1(b)      # Compute the sample mean. Only one answer is given,  
# so you need not restate or highlight your output.  
mean(x)
```

```
## [1] 78.525
```

```
# Question #1(c)      # Construct histogram.  
hist(x)
```

Histogram of x



```
# Question #1(d)      # Compute p-value. Restate or highlight in yellow your
                    # p-value, since t.test() produces additional output.
t.test( x, mu=80, alternative="less" )
```

```
##
## One Sample t-test
##
## data:  x
## t = -1.5342, df = 39, p-value = 0.06653
## alternative hypothesis: true mean is less than 80
## 95 percent confidence interval:
##      -Inf 80.14489
## sample estimates:
## mean of x
##      78.525
```

```
# The p-value is 0.06653.
```

```
# Question #1(d)      # Alternative solution. No need to restate or highlight
                    # in yellow your p-value, since only one answer
                    # (i.e., the p-value) is printed.
t.test( x, mu=80, alternative="less" )$p.value
```

```
## [1] 0.06652973
```

```
# Question #2      # State the test statistic used for a t-test on an
                    # unknown population mean.
# T = ( Xbar - mu ) / ( s / sqrt(n) ) # This solution is fine,
                    # but do not be sloppy with parentheses.
```

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \quad \text{Alternatively, using L<sup>A</sup>T<sub>E</sub>X code.}$$

Alternative method for introducing solutions:

### Question #3

```
# Question #3      # Compute 4*(3+7) using R-code.
                    # Since an erroneous solution of 4*(3+) halts execution
                    # of all R-code and hence fails to show even your correct
                    # R-output, then ideally correct the R-code.
                    # Otherwise, comment out the erroneous R-code,
                    # but your solution to question #3 would be nonexistent.
# 4*(3+)
```