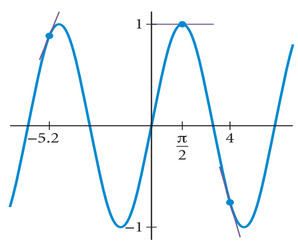
2.6 Derivatives of Trigonometric and Hyperbolic Functions

- ▷ *Derivatives of the six trigonometric functions*
- ▷ *Derivatives of inverse trigonometric functions*
- ▷ *Hyperbolic functions, inverse hyperbolic functions, and their derivatives*

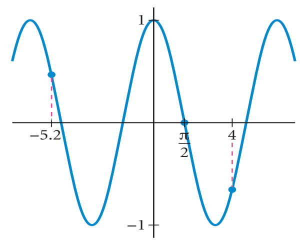
Derivatives of Trigonomteric Functions

Because trigonometric functions have periodic oscillating behavior, and their slopes also have periodic oscillating behavior, it would make sense if the derivatives of trigonometric functions were trigonometric. For example, the two graphs below show the function $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. At each value of x, it turns out that the slope of the graph of $f(x) = \sin x$ is given by the height of the graph of $f'(x) = \cos x$. Check this for the values $x = -5.2$, $x = \frac{\pi}{2}$, and $x = 4$:

Slopes of $f(x) = \sin x$ at three points



Heights of $f'(x) = \cos x$ at three points



The six trigonometric functions have the following derivatives:

Theorem 2.17

Derivatives of the Trigonometric Functions

For all values of x at which the functions below are defined, we have:

(a) $\frac{d}{dx}(\sin x) = \cos x$	**(c)** $\frac{d}{dx}(\tan x) = \sec^2 x$	**(e)** $\frac{d}{dx}(\cot x) = -\csc^2 x$
(b) $\frac{d}{dx}(\cos x) = -\sin x$	**(d)** $\frac{d}{dx}(\sec x) = \sec x \tan x$	**(f)** $\frac{d}{dx}(\csc x) = -\csc x \cot x$

It is important to note that these derivative formulas are only true if angles are measured in radians; see Exercise 5.

Proof. We will prove the formulas for $\sin x$ and $\tan x$ from parts (a) and (c) and leave the proofs of the remaining four formulas to Exercises 81–84.

(a) The proof of the first formula is nothing more than an annotated calculation using the definition of derivative. To simplify the limit we obtain we will rewrite $\sin(x+h)$ with a trigonometric identity. Our goal after that will be to rewrite the limit so that we can apply the two trigonometric limits from Theorem 1.34 in Section 1.6.

$$\begin{aligned}
\tfrac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && \leftarrow \text{definition of derivative}\\
&= \lim_{h\to 0}\frac{(\sin x\cos h+\sin h\cos x)-\sin x}{h} && \leftarrow \text{sum identity for sine}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h} && \leftarrow \text{algebra}\\
&= \lim_{h\to 0}\left(\sin x\,\frac{\cos h-1}{h}+\cos x\,\frac{\sin h}{h}\right) && \leftarrow \text{algebra}\\
&= \sin x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)+\cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) && \leftarrow \text{limit rules}\\
&= (\sin x)(0)+(\cos x)(1)=\cos x. && \leftarrow \text{trigonometric limits}
\end{aligned}$$

(c) We do not have to resort to the definition of derivative in order to prove the formula for differentiating $\tan x$. Instead we can use the quotient rule, the fact that $\tan x = \frac{\sin x}{\cos x}$, and the formulas for differentiating $\sin x$ and $\cos x$:

$$\begin{aligned}
\tfrac{d}{dx}(\tan x) = \tfrac{d}{dx}\left(\tfrac{\sin x}{\cos x}\right) &= \frac{\tfrac{d}{dx}(\sin x)\cdot(\cos x)-(\sin x)\cdot\tfrac{d}{dx}(\cos x)}{(\cos x)^2} && \leftarrow \text{quotient rule}\\
&= \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{\cos^2 x} && \leftarrow \text{derivatives of } \sin x \text{ and } \cos x\\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. && \leftarrow \text{algebra and identities}
\end{aligned}$$

■

Derivatives of Trigonometric Functions

We can use the formulas for the derivatives of the trigonometric functions to prove formulas for the derivatives of the inverse trigonometric functions. Interestingly, although inverse trigonometric functions are transcendental, their derivatives are algebraic:

Theorem 2.18

Derivatives of Inverse Trigonometric Functions

For all values of x at which the functions below are defined, we have:

(a) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ **(b)** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ **(c)** $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

These derivative formulas are particularly useful for finding certain antiderivatives, and in Chapter xxx they will be part of our arsenal of integration techniques. Of course, all of these rules can be used in combination with the sum, product, quotient, and chain rules. For example,

$$\tfrac{d}{dx}(\sin^{-1}(3x^2+1)) = \frac{1}{\sqrt{1-(3x^2+1)^2}}(6x) = \frac{6x}{\sqrt{1-x^2}}.$$

Proof. We will prove the rule for $\sin^{-1} x$ and leave the remaining two rules to Exercises 85 and 86. We could apply Theorem 2.13 here, but it is just as easy to do the implicit differentiation by hand. Since $\sin(\sin^{-1} x) = x$ for all x in the domain of $\sin^{-1} x$, we have:

$$\begin{aligned}
\sin(\sin^{-1} x) &= x && \leftarrow \sin^{-1} x \text{ is the inverse of } \sin x \\
\tfrac{d}{dx}(\sin(\sin^{-1} x)) &= \tfrac{d}{dx}(x) && \leftarrow \text{differentiate both sides} \\
\cos(\sin^{-1} x) \cdot \tfrac{d}{dx}(\sin^{-1} x) &= 1 && \leftarrow \text{chain rule} \\
\tfrac{d}{dx}(\sin^{-1} x) &= \frac{1}{\cos(\sin^{-1} x)} && \leftarrow \text{algebra} \\
\tfrac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \leftarrow \text{since } \sin^2 x + \cos^2 x = 1 \\
\tfrac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1 - x^2}}. && \leftarrow \sin x \text{ is the inverse of } \sin^{-1} x
\end{aligned}$$

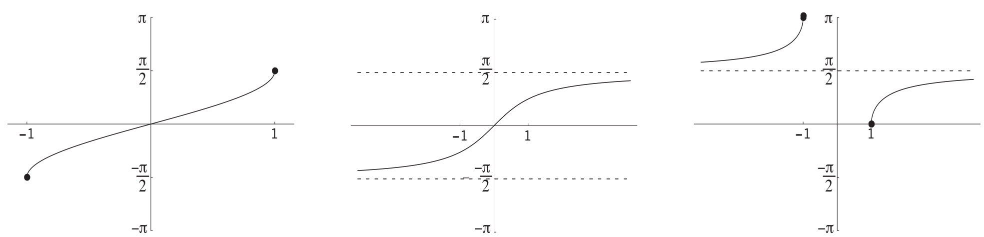
We could also have used triangles and the unit circle to show that the composition $\cos(\sin^{-1} x)$ is equal to the algebraic expression $\sqrt{1 - x^2}$, as we did in Example 4 of Section 0.4. ■

An interesting fact about the derivatives of inverse sine and inverse secant is that their domains are slightly smaller than the domains of the original functions. Below are the graphs of the inverse trigonometric functions and their domains.

$f(x) = \sin^{-1} x$ *has domain* $[-1, 1]$

$g(x) = \tan^{-1} x$ *has domain* $(-\infty, \infty)$

$h(x) = \sec^{-1} x$ *has domain* $(-\infty, -1] \cup [1, \infty)$

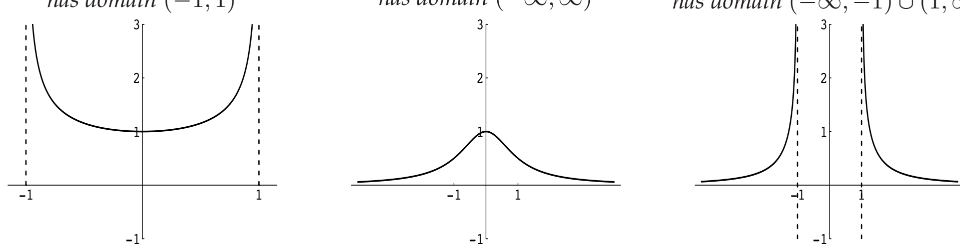


If you look closely at the first and third graphs above you should notice that at the ends of the domains the tangent lines will be vertical. Since a vertical line has undefined slope, the derivative does not exist at these points. This means that the derivatives of $\sin^{-1} x$ and $\sec^{-1} x$ are not defined at $x = 1$ or $x = -1$; see the first and third graphs below.

$f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$ *has domain* $(-1, 1)$

$g'(x) = \dfrac{1}{1 + x^2}$ *has domain* $(-\infty, \infty)$

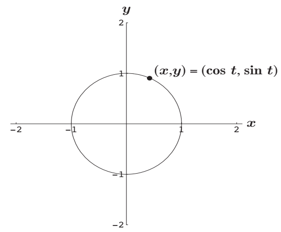
$h'(x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$ *has domain* $(-\infty, -1) \cup (1, \infty)$



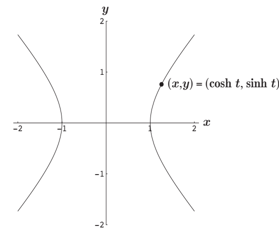
Hyperbolic Functions and Their Derivatives*

The trigonometric functions sine and cosine are ***circular functions*** in the sense that they are defined to be the coordinates of a ***parameterization*** of the unit circle. This means that the circle defined by $x^2 + y^2 = 1$ is the path traced out by the coordinates $(x, y) = (\cos t, \sin t)$ as t varies; see the figure below left.

Points on the circle $x^2 + y^2 = 1$



Points on the hyperbola $x^2 - y^2 = 1$



Now let's consider the path traced out by the hyperbola $x^2 - y^2 = 1$ as shown above right. One parameterization of the right half of this hyperbola is traced out by the ***hyperbolic functions*** $(\cosh t, \sinh t)$ that we will spend the rest of this section investigating.

The hyperbolic functions are nothing more than simple combinations of the exponential functions e^x and e^{-x}:

Definition 2.19 **Hypberbolic Sine and Hyperbolic Cosine**

For any real number x, the ***hyperbolic sine function*** and the ***hyperbolic cosine function*** are defined as the following combinations of exponential functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

The hyperbolic sine function is pronounced "sinch" and the hyperbolic cosine function is pronounced "cosh." The "h" is for "hyperbolic." As we will soon see, the properties and interrelationships of the hyperbolic functions are similar to the properties and interrelationships of the trigonometric functions. These properties will be particularly useful in Chapter **??** when we are attempting to solve certain forms of integrals.

It is a simple matter to use the definition above to verify that for any value of t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$; see Exercise 87. We will usually think of this fact with the variable x, as this identity:

$$\cosh^2 x - \sinh^2 x = 1.$$

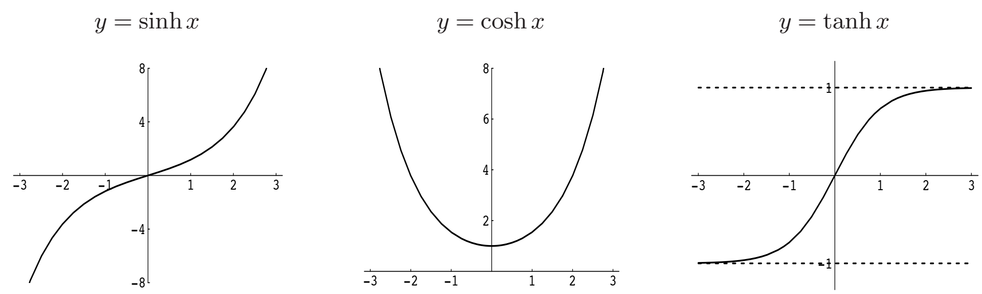
Here we are using the familiar convention that, for example, $\sinh^2 x$ is shorthand for $(\sinh x)^2$. Note the similarity between the hyperbolic identity $\cosh^2 t - \sinh^2 t = 1$ and the Pythagorean identity for sine and cosine. Hyperbolic functions also satisfy many other algebraic identities that are reminiscent of those that hold for trigonometric functions, as you will see in Exercises 88–90.

Just as we can define four additional trigonometric functions from sine and cosine, we can define four additional hyperbolic functions from hyperbolic sine and hyperbolic cosine. We will be primarily interested in the ***hyperbolic tangent function***:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

We can also define csch x, sech x, and coth x as the reciprocals of $\sinh x$, $\cosh x$, and $\tanh x$, respectively.

The graphs of $\sinh x$, $\cosh x$, and $\tanh x$ are shown below. In Exercises 13–16 you will investigate various properties of these graphs.



In Chapter 4 we will see that the middle graph of $y = \cosh x$ is an example of a ***catenary*** curve, which is the shape formed by a hanging chain or cable.

As with any functions that we study, we are interested in finding formulas for the derivatives of $\sinh x$, $\cosh x$, and $\tanh x$. The similarity between hyperbolic functions and trigonometric functions continues here. These derivatives follow a very familiar pattern, differing from the pattern for trigonometric functions only by a sign change.

Theorem 2.20 **Derivatives of Hyperbolic Functions**

For all real numbers x, we have:

(a) $\frac{d}{dx}(\sinh x) = \cosh x$ **(b)** $\frac{d}{dx}(\cosh x) = \sinh x$ **(c)** $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

If you prefer to stay away from the hyperbolic secant function sech x, you can write the third derivative above as $\frac{1}{\cosh^2 x}$.

Proof. The proofs of these differentiation formulas follow immediately from the definitions of the hyperbolic functions as simple combinations of exponential functions. For example,

$$\frac{d}{dx}(\sinh x) \;=\; \frac{d}{dx}\left(\tfrac{1}{2}(e^x - e^{-x})\right) \;=\; \tfrac{1}{2}(e^x + e^{-x}) \;=\; \cosh x.$$

The remaining proofs are left to Exercises 91–92. ■

Although hyperbolic functions may seem somewhat exotic, they work with the other differentiation rules just like any other functions. For example, with the product and chain rules we can calculate:

$$\frac{d}{dx}(5x \sinh^3 x^2) = 5\sinh^3 x^2 + 5x(3\sinh^2 x^2)(\cosh^2 x^2)(2x).$$

The derivatives of the remaining three hyperbolic functions are also very similar to those of their trigonometric cousins, but at the moment we will be focusing only on hyperbolic sine, cosine, and tangent.

Inverse Hyperbolic Functions and Their Derivatives*

For a function to have an inverse, it must be one-to-one. Looking back at the graphs of $\sinh x$, $\cosh x$, and $\tanh x$, we see that only $\cosh x$ fails to be one-to-one. Just as when we defined the trigonometric inverses, we will restrict the domain of $\cosh x$ to a smaller domain on which it is one-to-one. We will choose the restricted domain of $\cosh x$ to be $x \geq 0$. The notation we will use for the inverses of these three functions is what you would expect: $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$.

Since the hyperbolic functions are defined as combinations of exponential functions, it would seem reasonable to expect that their inverses could be expressed in terms of logarithmic functions. This is in fact the case, as you will see in Exercises 95–97. However, our main concern here is to find formulas for the derivatives of the inverse hyperbolic functions, which we can do directly from identities and properties of inverses.

Theorem 2.21 **Derivatives of Inverse Hyperbolic Functions**

For all x at which the following are defined, we have:

(a) $\frac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$ **(b)** $\frac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$ **(c)** $\frac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$

Similar formulas can be developed for the remaining three inverse hyperbolic functions. Notice the strong similarities between these derivatives and the derivatives of the inverse trigonometric functions.

Proof. We will prove the rule for the derivative of $\sinh^{-1} x$ and leave the remaining two rules to Exercises 93 and 94. Starting from the fact that $\sinh(\sinh^{-1} x)$ for all x, we can apply implicit differentiation:

$$\begin{aligned}
\sinh(\sinh^{-1} x) &= x && \leftarrow \text{sinh} -1x \text{ is the inverse of } \sinh x \\
\tfrac{d}{dx}(\sinh(\sinh^{-1} x)) &= \tfrac{d}{dx}(x) && \leftarrow \text{differentiate both sides} \\
\cosh(\sinh^{-1} x)\cdot \tfrac{d}{dx}(\sinh^{-1} x) &= 1 && \leftarrow \text{chain rule, derivative of } \sinh x \\
\tfrac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\cosh(\sinh^{-1} x)}. && \leftarrow \text{algebra} \\
\tfrac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1} x)}} && \leftarrow \text{since } \cosh^2 x - \sinh^2 x = 1 \\
\tfrac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\sqrt{1+x^2}}. && \leftarrow \sinh x \text{ is the inverse of } \sinh^{-1} x
\end{aligned}$$

Compare this proof with our proof earlier in this section for the derivative of $\sin^{-1} x$; the two are very similar. ■

Examples and Explorations

Example 1 ***Differentiating combinations of trigonometric functions***

Find the derivatives of each of the following functions.

(a) $f(x) = \dfrac{\tan x}{x^3-2}$ **(b)** $f(x) = x\,\sin^{-1}(3x+1)$ **(c)** $f(x) = \sec^2 e^x$

Solution.

(a) By the quotient rule and the rule for differentiating tangent, we have:

$$\frac{d}{dx}\left(\frac{\tan x}{x^3-2}\right) = \frac{\frac{d}{dx}(\tan x)\cdot(x^3-2) - (\tan x)\cdot\frac{d}{dx}(x^3-2)}{(x^3-2)^2} = \frac{(\sec^2 x)(x^3-2) - (\tan x)(3x^2)}{(x^3-2)^2}.$$

(b) This is a product of functions, and thus we begin with the product rule. We will also need the chain rule to differentiate the composition $\sin^{-1}(3x+1)$:

$$f'(x) = (1)\cdot\sin^{-1}(3x+1) + x\cdot\frac{1}{\sqrt{1-(3x+1)^2}}(3) = \sin^{-1}(3x+1) + \frac{3x}{\sqrt{1-(3x+1)^2}}.$$

(c) This is a composition of three functions, and thus we need to apply the chain rule twice:

$$\begin{aligned}
\tfrac{d}{dx}(\sec^2 e^x) &= \tfrac{d}{dx}((\sec(e^x))^2) && \leftarrow \text{rewrite so compositions are clear}\\
&= 2(\sec e^x)^1 \cdot \tfrac{d}{dx}(\sec e^x) && \leftarrow \text{first application of chain rule}\\
&= 2(\sec e^x)(\sec e^x)(\tan e^x) \cdot \tfrac{d}{dx}(e^x) && \leftarrow \text{second application of chain rule}\\
&= 2(\sec e^x)(\sec e^x)(\tan e^x)e^x && \leftarrow \text{derivative of } e^x
\end{aligned}$$

Perhaps the trickiest part of this calculation is that the derivative of $\sec x$ has *two* instances of the independent variable: $\frac{d}{dx}(\sec x) = \sec \underline{x} \tan \underline{x}$. This means that in the calculation above, we needed to put the "inside" function e^x into *both* of these variable slots. ■

Example 2 ***Differentiating combinations of hyperbolic functions***

Find the derivatives of each of the following functions.

(a) $f(x) = \ln(\tanh^2(x^3 + 2x + 1))$ **(b)** $f(x) = \sqrt{\cosh^{-1}(e^{3x})}$

Solution.

(a) This is a nested chain rule problem, since $f(x)$ is a composition of multiple functions. We will work from the outside to the inside, one step at a time:

$$\begin{aligned}
f'(x) &= \frac{1}{\tanh^2(x^3+2x+1)} \tfrac{d}{dx}(\tanh^2(x^3+2x+1))\\
&= \frac{1}{\tanh^2(x^3+2x+1)}(2\tanh(x^3+2x+1)) \tfrac{d}{dx}(\tanh(x^3+2x+1))\\
&= \frac{1}{\tanh^2(x^3+2x+1)}(2\tanh(x^3+2x+1))(\text{sech}^2(x^3+2x+1))(3x^2+2).
\end{aligned}$$

(b) Once again we have a nested chain rule situation. Notice in particular how the e^{3x} works with the derivative of inverse hyperbolic cosine:

$$\begin{aligned}
f'(x) &= \tfrac{1}{2}(\cosh^{-1}(e^{3x}))^{-\frac{1}{2}} \tfrac{d}{dx}(\cosh^{-1}(e^{3x}))\\
&= \tfrac{1}{2}(\cosh^{-1}(e^{3x}))^{-\frac{1}{2}} \left(\frac{1}{\sqrt{(e^{3x})^2 - 1}} \right) (3e^{3x}).
\end{aligned}$$

■

Example 3 ***Finding antiderivatives that involve inverse trigonometric functions***

Find a function f whose derivative is $f'(x) = \dfrac{1}{1+4x^2}$.

Solution. Since the derivative of $\tan^{-1} x$ is $\frac{1}{1+x^2}$, we might suspect that the function f we are looking for is related to inverse tangent. We will use an intelligent guess-and-check method to find f. Clearly $f(x) = \tan^{-1} x$ isn't exactly right, since its derivative is missing the "4." A good guess might be $f(x) = \tan^{-1}(4x)$; let's try that:

$$\tfrac{d}{dx}(\tan^{-1}(4x)) \;=\; \frac{1}{1+(4x)^2}\,(4) \;=\; \frac{4}{1+16x^2}.$$

Obviously that wasn't quite right either; but by examining the results we can make a new guess. We might try $\tan^{-1}(2x)$, since the "$2x$" will be squared in the derivative and become the "$4x^2$" we are looking for in the denominator:

$$\tfrac{d}{dx}(\tan^{-1}(2x)) \;=\; \frac{1}{1+(2x)^2}\,(2) \;=\; \frac{2}{1+4x^2}.$$

Now we are getting somewhere; this differs by a multiplicative constant from the derivative $f'(x)$ we are looking for, and that is easy to fix. We need only divide our guess by that constant. Try the function $f(x) = \frac{1}{2}\tan^{-1}(2x)$:

$$\tfrac{d}{dx}(\tfrac{1}{2}\tan^{-1}(2x)) \;=\; (\tfrac{1}{2})\,\frac{1}{1+(2x)^2}\,(2) \;=\; \frac{1}{1+4x^2}.$$

We now know that $f(x) = \frac{1}{2}\tan^{-1}(2x)$ is a function whose derivative is $f'(x) = \frac{1}{1+4x^2}$. Of course, we could also add any constant to $f(x)$ and not change its derivative; for example, $f(x) = \frac{1}{2}\tan^{-1}(2x) + 5$ would also work. In fact, any function of the form $f(x) = \frac{1}{2}\tan^{-1}(2x) + C$ will have $f'(x) = \frac{1}{1+4x^2}$. ■

Example 4 ***Finding antiderivatives that involve hyperbolic functions***

Find a function f whose derivative is $f'(x) = \dfrac{e^x}{\sqrt{e^{2x}-1}}$.

Solution. Until we learn more specific anti-differentiation techniques in Chapter **??**, a problem like this is best done by an intelligent guess-and-check procedure. Given that we have the inverse hyperbolic functions in mind, the best match of the three is the derivative of $\cosh^{-1} x$. Since the expression for $f'(x)$ also involves an e^x, let's revise that guess right away to $\cosh^{-1} e^x$. Now we check by differentiating with the chain rule:

$$\tfrac{d}{dx}(\cosh^{-1} e^x) = \frac{1}{\sqrt{(e^x)^2-1}} \cdot e^x = \frac{e^x}{\sqrt{e^{2x}-1}}.$$

We guessed it on the first try! We have just shown that $f(x) = \cosh^{-1} e^x$ has the desired derivative. ■

? ***Questions.*** Test your understanding of the reading by answering these questions:

- ▷ What trigonometric limits were used to find the derivative of $\sin x$?
- ▷ How can we obtain the derivative of $\sec x$ from the derivative of $\cos x$?
- ▷ What is the graphical reason that the domains of the derivatives of $\sin^{-1} x$ and $\sec^{-1} x$ are slightly smaller than the domains of the functions themselves?
- ▷ How are hyperbolic functions similar to trigonometric functions? How are they different?
- ▷ How can we obtain the derivative of $\sinh^{-1} x$ from the derivative of $\sinh x$?

Exercises 2.6

Thinking Back

Trigonometric and Inverse Trigonometric Values: Find the exact values of each of the quantities below by hand, without using a calculator.

▷ $\sin(-\frac{\pi}{3})$

▷ $\tan(-\frac{\pi}{4})$

▷ $\sec(\frac{5\pi}{6})$

▷ $\sin^{-1} 1$

▷ $\tan^{-1}(\sqrt{3})$

▷ $\sec^{-1}(-2)$

Compositions: For each function k below, find functions f, g, and h so that $k = f \circ g \circ h$. There may be more then one possible answer.

▷ $k(x) = \dfrac{1}{\sin(x^3)}$

▷ $k(x) = \sin^{-1}(\cos^2 x)$

▷ $k(x) = \tan^2(3x + 1)$

▷ $k(x) = \sec(x^3)\tan(x^3)$

Writing trigonometric compositions algebraically: Prove each of the following equalities, which rewrite compositions of trigonometric and inverse trigonometric functions as algebraic functions.

▷ $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

▷ $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

▷ $\sec^2(\tan^{-1} x) = 1+x^2$

▷ $\tan(\sec^{-1} x) = |x|\sqrt{1-\frac{1}{x^2}}$

Concepts

0. *Problem Zero:* Read the section and make your own summary of the material.

1. *True/False:* Determine whether each of the following statements is true or false. If a statement is true, explain why. If a statement is false, provide a counterexample.
   (a) True or False: To find the derivative of $\sin x$ we had to use the definition of derivative.
   (b) True or False: To find the derivative of $\tan x$ we have to use the definition of derivative.
   (c) True or False: The derivative of $\dfrac{x^4}{\sin x}$ is $\dfrac{4x^3}{\cos x}$.
   (d) True or False: If a function is algebraic, then so is its derivative.
   (e) True or False: If a function is transcendental, then so is its derivative.
   (f) True or False: If f is a trigonometric function, then f' is also a trigonometric function.
   (g) True or False: If f is an inverse trigonometric function, then f' is also an inverse trigonometric function.
   (h) True or False: If f is a hyperbolic function, then f' is also a hyperbolic function.

2. *Examples:* Give examples of each of the following. Try to find examples that are different than any in the reading.
   (a) A function that is its own fourth derivative.
   (b) A function whose domain is larger than the domain of its derivative.
   (c) Three non-logarithmic functions that are transcendental, but whose derivatives are algebraic.

3. What limit facts and trigonometric identities are used in the proof that $\frac{d}{dx}(\sin x) = \cos x$?

4. Sketch graphs of $\sin x$ and $\cos x$ on $[-2\pi, 2\pi]$.
   (a) Use the graph of $\sin x$ to determine where $\sin x$ is increasing and decreasing.
   (b) Use the graph of $\cos x$ to determine where $\cos x$ is positive and negative.
   (c) Explain why your answers to parts (a) and (b) suggest that $\cos x$ is the derivative of $\sin x$.

5. The differentiation formula $\frac{d}{dx}(\sin x) = \cos x$ is valid only if x is measured in radians. In this problem you will explore why this derivative relationship does not hold if x is measured in degrees.
   (a) Set your calculator to degree mode and sketch a graph of $\sin x$ that shows at least two periods. If the derivative of sine is cosine, then the slope of your graph at $x = 0$ should be equal to $\cos 0 = 1$. Use your graph to explain why this is not the case when using degrees. *(Hint: Think about your graphing window scale.)*
   (b) Now set your calculator back to radians mode!

6. Suppose you wish to differentiate $g(x) = \sin^2(x) + \cos^2(x)$. What is the fastest way to do this, and why?

7. The following derivatives of the function $f(x) = \cos(3x^2)$ are incorrect. What misconception occurs in each case?
   (a) Incorrect: $f'(x) = (-\sin x)(3x^2) + (\cos x)(6x)$
   (b) Incorrect: $f'(x) = -\sin(6x)$

8. The following derivatives of the function $f(x) = \cos(3x^2)$ are incorrect. What misconception occurs in each case?
   (a) Incorrect: $f'(x) = -\sin(3x^2)$
   (b) Incorrect: $f'(x) = -\sin(3x^2)(6x)(6)$

9. In the proof that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, we used the fact that $\sin(\sin^{-1} x) = x$. It is also true that $\sin^{-1}(\sin x) = x$; could we have started with that inequality instead? Why or why not?

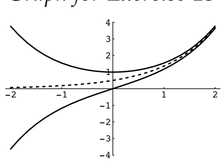
10. Both of the following equations are true: $\tan(\tan^{-1} x) = x$ and $\tan^{-1}(\tan x) = x$. We can find the derivative of $\tan^{-1} x$ by differentiating both sides of one of these equations and solving for $\frac{d}{dx}(\tan^{-1} x)$. Which one, and why?

11. The following derivatives of the function $f(x) = \cos(3x^2)$ are incorrect. What misconception occurs in each case?
    (a) Incorrect: $f'(x) = (-\sin x)(3x^2) + (\cos x)(6x)$
    (b) Incorrect: $f'(x) = -\sin(6x)$

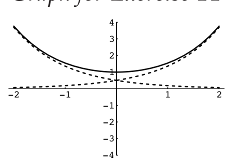
12. Suppose you wish to differentiate $g(x) = \sin^2(x) + \cos^2(x)$. What is the fastest way to do this, and why?

13. The graph below left shows $y = \sinh x$, $y = \cosh x$, and $y = \frac{1}{2}e^x$. For each fact below, **(a)** explain graphically why the fact is true. Then **(b)** prove the fact algebraically using the definitions of the hyperbolic functions.
    (a) $\sinh x \le \frac{1}{2}e^x \le \cosh x$ for all x
    (b) $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{\frac{1}{2}e^x} = 1$ and $\displaystyle\lim_{x\to\infty} \frac{\cosh x}{\frac{1}{2}e^x} = 1$

Graph for Exercise 13



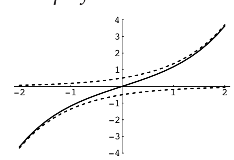
Graph for Exercise 14



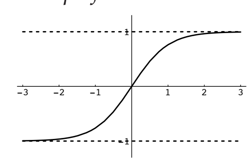
14. The graph above right shows $y = \cosh x$, $y = \frac{1}{2}e^x$, and $y = \frac{1}{2}e^{-x}$. For each fact below, **(a)** explain graphically why the fact is true. Then **(b)** prove the fact algebraically using the definitions of the hyperbolic functions.
    (a) $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$
    (b) $\displaystyle\lim_{x\to-\infty} \frac{\cosh x}{\frac{1}{2}e^{-x}} = 1$

15. The graph below left shows $y = \sinh x$, $y = \frac{1}{2}e^x$, and $y = -\frac{1}{2}e^{-x}$. For each fact below, **(a)** explain graphically why the fact is true. Then **(b)** prove the fact algebraically using the definitions of the hyperbolic functions.
    (a) $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$
    (b) $\displaystyle\lim_{x\to-\infty} \frac{\sinh x}{-\frac{1}{2}e^{-x}} = 1$

Graph for Exercise 15



Graph for Exercise 16



16. The graph above right shows $y = \tanh x$, $y = 1$, and $y = -1$. For each fact below, **(a)** explain graphically why the fact is true. Then **(b)** prove the fact algebraically using the definitions of the hyperbolic functions.
    (a) $-1 \le \tanh x \le 1$
    (b) $\displaystyle\lim_{x\to\infty} \tanh x = 1$ and $\displaystyle\lim_{x\to-\infty} \tanh x = -1$

Skills

Find the derivatives of each of the functions in Exercises 17–50. In some cases it may be convenient to do some preliminary algebra.

17. $f(x) = \dfrac{x^2+1}{\cos x}$
18. $f(x) = 2\cos(x^3)$
19. $f(x) = \cot x - \csc x$
20. $f(x) = \tan^2(3x+1)$
21. $f(x) = 4\sin^2 x + 4\cos^2 x$
22. $f(x) = \sec^2 x^{-1}$
23. $f(x) = 3\sec x \tan x$
24. $f(x) = 3^x \sec x + 17$
25. $f(x) = \sin(\cos(\sec(x)))$
26. $f(x) = \csc^2(e^x)$
27. $f(x) = e^{\csc^2 x}$
28. $f(x) = e^x \csc^2 x$
29. $f(x) = \dfrac{-2^x}{5x\sin x}$
30. $f(x) = \dfrac{\log_3(3^x)}{\sin^2 x + \cos^2 x}$
31. $f(x) = x\sqrt{\sin x \cos x}$
32. $f(x) = \dfrac{\sin x \csc x}{\cot x \cos x}$
33. $f(x) = \dfrac{3x^2 \ln x}{\tan x}$
34. $f(x) = \dfrac{\ln(3x^2)}{\tan x}$
35. $f(x) = \sin(\ln x)$
36. $f(x) = \ln(x \sin x)$
37. $f(x) = \sin^{-1}(3x^2)$
38. $f(x) = 3(\sin^{-1} x)^2$
39. $f(x) = x^2 \arctan x^2$
40. $f(x) = \tan^{-1}(\ln x)$
41. $f(x) = \sec^{-1} x^2$
42. $f(x) = \sin(\sin^{-1} x)$
43. $f(x) = \sin^{-1}(\sec^2 x)$
44. $f(x) = \sin^2(\sec^{-1} x)$
45. $f(x) = \dfrac{\sin^{-1} x}{\tan^{-1} x}$
46. $f(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$
47. $f(x) = \ln(\mathrm{arcsec}(\sin^2 x))$
48. $f(x) = x^{-2} e^{4x} \sin^{-1} x$
49. $f(x) = \sec(1 + \tan^{-1} x)$
50. $f(x) = \dfrac{\sin(\arcsin x)}{\arctan x}$

Find the derivatives of each of the functions in Exercises 51–62. In some cases it may be convenient to do some preliminary algebra. These problems involve hyperbolic functions and their inverses.

51. $f(x) = x \sinh x^3$

52. $f(x) = x \sinh^3 x$

53. $f(x) = \cosh(\ln(x^2+1))$

54. $f(x) = 3\tanh^2 e^x$

55. $f(x) = \sqrt{\cosh^2 x + 1}$

56. $f(x) = \dfrac{\tanh\sqrt{x}}{\sqrt{\sinh x}}$

57. $f(x) = \sinh^{-1}(x^3)$

58. $f(x) = \tanh^{-1}(\tan x^2)$

59. $f(x) = \dfrac{\sinh^{-1} x}{\cosh^{-1} x}$

60. $f(x) = x\sqrt{\tanh^{-1} x}$

61. $f(x) = \sin(e^{\sinh^{-1} x})$

62. $f(x) = \cosh^{-1}(\cosh^{-1} x)$

Use logarithmic differentiation to find the derivatives of each of the functions in Exercises 63–65.

63. $(\sin x)^x$

64. $(\sec x)^x$

65. $(\sin x)^{\cos x}$

In Exercises 66–**??**, find a function f that has the given derivative f'. In each case you can find the answer with an educated "guess-and-check" process.

66. $f'(x) = \dfrac{2x}{\sqrt{1-4x^2}}$

67. $f'(x) = \dfrac{2}{\sqrt{1-4x^2}}$

68. $f'(x) = \dfrac{1}{1+9x^2}$

69. $f'(x) = \dfrac{3x}{1+9x^2}$

70. $f'(x) = \dfrac{1}{9+x^2}$

71. $f'(x) = \dfrac{3}{\sqrt{4-9x^2}}$

In Exercises 72–**??**, find a function f that has the given derivative f'. In each case you can find the answer with an educated "guess-and-check" process. Some of these exercises involve hyperbolic functions.

72. $f'(x) = \dfrac{2x}{\sqrt{1+4x^2}}$

73. $f'(x) = \dfrac{2}{\sqrt{1+4x^2}}$

74. $f'(x) = \dfrac{1}{1-9x^2}$

75. $f'(x) = \dfrac{3x}{1-9x^2}$

76. $f'(x) = \dfrac{1}{9-x^2}$

77. $f'(x) = \dfrac{3}{\sqrt{4+9x^2}}$

Applications

78. In Problem 83 from Section 1.6 we saw that the oscillating position of a mass hanging from the end of a spring, neglecting air resistance, is given by following equation, where A, B, k, and m are constants:

$$s(t) = A\sin\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\cos\left(\sqrt{\tfrac{k}{m}}\,t\right),$$

   (a) Show that this function $s(t)$ has the property that $s''(t) + \frac{k}{m}s(t) = 0$. This is the ***differential equation*** for the spring motion, which means it is an equation involving derivatives that describes the motion of the bob on the end of the spring.

   (b) Suppose the spring is released from an initial position of s_0, and with an initial velocity of v_0. Show that $A = v_0\sqrt{\frac{m}{k}}$ and $B = s_0$.

79. In Problem 84 from Section 1.6 we learned that the oscillating position of a mass hanging from the end of a spring, taking air resistance into account, is given by the following equation, where A, B, k, f, and m are constants:

$$s(t) = e^{\frac{-f}{2m}t}\left(A\sin\left(\tfrac{\sqrt{4km-f^2}}{2m}\,t\right) + B\cos\left(\tfrac{\sqrt{4km-f^2}}{2m}\,t\right)\right),$$

   (a) Show that this function $s(t)$ has the property that $s''(t) + \frac{a}{m}s'(t) + \frac{k}{m}s(t) = 0$ for some constant a. This is the differential equation for spring motion, taking air resistance into account. *(Hint: Find the first and second derivatives of $s(t)$ first and then show that $s(t)$, $s'(t)$, and $s''(t)$ have the given relationship.)*

   (b) Suppose the spring is released from an initial position of s_0, and with an initial velocity of v_0. Show that $A = \frac{2mv_0 + fs_0}{\sqrt{4km-f^2}}$ and $B = s_0$.

80. Suppose your friend Max drops a penny from the top floor of the Empire State Building, 1250 feet from the ground. You are standing about a block away, 250 feet from the base of the building.

   (a) Find a formula for the angle of elevation $\alpha(t)$ from the ground at your feet to the height of the penny t seconds after Max drops it. Multiply by an appropriate constant so that $\alpha(t)$ is measured in degrees.

   (b) Find a formula for the rate at which the angle of elevation $\alpha(t)$ is changing at time t, and use it to determine the rate of change of the angle of elevation at the time the penny hits the ground.

Proofs

81. Use the definition of derivative, a trigonometric identity, and known trigonometric limits to prove that $\frac{d}{dx}(\cos x) = -\sin x$.
82. Use the quotient rule and the derivative of cosine to prove that $\frac{d}{dx}(\sec x) = \sec x \tan x$.
83. Use the quotient rule and the derivative of sine to prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.
84. Use the quotient rule and the derivatives of sine and cosine to prove that $\frac{d}{dx}(\cot x) = -\csc^2 x$.
85. Use implicit differentiation and the fact that $\tan(\tan^{-1} x) = x$ for all x in the domain of $\tan^{-1} x$ to prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.
86. Use implicit differentiation and the fact that $\sec(\sec^{-1} x) = x$ for all x in the domain of $\sec^{-1} x$ to prove that $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$. You will have to consider the cases $x > 1$ and $x < -1$ separately.
87. Prove that for any value of t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$. Bonus question: In fact, these points will always lie on the right hand side of the hyperbola; why?

Use the definitions of the hyperbolic functions to prove that each of the identities in Exercises 88–90 hold for all values of x and y. Note how similar these identities are to those that hold for trigonometric functions.

88. **(a)** $\sinh(-x) = -\sinh x$, and **(b)** $\cosh(-x) = \cosh x$
89. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
90. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Prove each of the differentiation formulas in Exercises 91–94.

91. $\frac{d}{dx}(\cosh x) = \sinh x$
92. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$
93. $\frac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$
94. $\frac{d}{dx}(\tanh^{-1} x)\dfrac{1}{1-x^2}$

Prove that the inverse hyperbolic functions can be written in terms of logarithms as in Exercises 95–97. *(Hint for the first problem: Solve $\sinh y = x$ for y, by using algebra to get an expression that is quadratic in e^y, that is, of the form $ae^{2y} + be^y + c$, and then applying the quadratic formula.)*

95. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, for any x.
96. $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, for $x \geq 1$.
97. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, for $-1 < x < 1$.

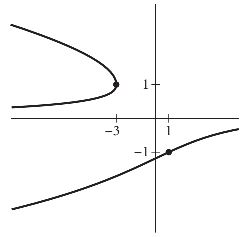
Thinking Forward

Local extrema and inflection points: In the problems below you will investigate how derivatives can help us find the locations of the maxima and minima of a function.

- ▷ Suppose f has a maximum or minimum value at $x = c$. If f is differentiable at $x = c$, what must be true of $f'(c)$, and why?
- ▷ If f is a differentiable function, then the values $x = c$ at which the sign of the derivative $f'(x)$ changes are the locations of the local extrema of f. Use this information to find the local extrema of the function $f(x) = \sin x$. Illustrate your answer on a graph of $y = \sin x$.
- ▷ If f is a differentiable function, then the values $x = c$ at which the sign of the second derivative $f''(x)$ changes are the locations of the inflection points of f. Use this information to find the inflection points of the function $f(x) = \sin x$. Illustrate your answer on a graph of $y = \sin x$.

Appendix A

Answers To Odd Problems



79. If Linda sells magazines at a rate of $\frac{dm}{dt} = 12$ magazine subscriptions per week, and she makes $\frac{dD}{dm} = 4$ dollars per magazine, then by the chain rule, the amount of money she makes each week is $\frac{dD}{dt} = \frac{dD}{dm}\frac{dm}{dt} = 4(12) = 48$ dollars per week.

81. (a) $\frac{dA}{dr} = 2\pi r$. (b) No; Yes. (c) $\frac{dA}{dt} = \frac{d}{dt}(\pi(r(t))^2) = 2\pi r(t)r'(t) = 2\pi r\frac{dr}{dt}$. (d) Yes; Yes. (e) $\frac{dA}{dt}|_{r=24} = 2\pi(24)(2) = 96\pi$.

83. **Proof**: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}(f(x)(g(x))^{-1}) = \frac{d}{dx}(f(x)) \cdot (g(x))^{-1} + f(x) \cdot \frac{d}{dx}((g(x))^{-1}) = f'(x) \cdot (g(x))^{-1} + f(x) \cdot (-g(x))^{-2}g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

85. Mimic the proof in Example 5(a).

87. Let $y = x^{-k}$, where $-k$ is a negative integer. Then $x^k y = 1$, so by implicit differentiation and the product rule, we have $kx^{k-1}y + x^k y' = 0$, and therefore $y' = \frac{-kx^{k-1}x^{-k}}{x^k} = -kx^{-k-1}$.

Section 2.5

1. T, T, F, F, T, T, F, T.

3. No, since the base is a variable. No, since the exponent is a variable.

5. When $k = 1$ we have $\frac{d}{dx}(e^x) = \frac{d}{dx}(e^{1x}) = 1e^{1x} = e^x$. When $b = e$ we have $\frac{d}{dx}(b^x) = \frac{d}{dx}(e^x) = (\ln e)e^x = (1)e^x = e^x$.

7. $\ln x$ is of the form $\log_b x$ with $b = e$.

9. The graph passing through $(0,2)$ is $2(2^x)$; the graph passing through $(0,1)$ and $(1,4)$ is 4^x; the graph passing through $(2,4)$ is 2^x.

11. $f(3) = 2^9 = 512$ and $g(3) = 8^2 = 64$. The solutions to $2^{(x^2)} = (2^x)^2$ are $x = 0$ and $x = 2$.

13. Consider that logarithmic functions are the inverses of exponential functions, and use the definition of one-to-one.

15. Logarithmic differentiation is the process of applying $\ln|x|$ to both sides of an equation $y = f(x)$ and then differentiating both sides in order to solve for $f'(x)$. It is useful for finding derivatives of functions that involve multiple products or quotients, and functions that have variables in both a base and an exponent.

17. $f'(x) = -(2 - e^{5x})^{-2}(-5e^{5x})$

19. $f'(x) = 6xe^{-4x} - 12x^2e^{-4x}$

21. $f'(x) = \frac{(-1)e^x - (1-x)e^x}{e^{2x}}$

23. $e^x(x^2 + 3x - 1) + e^x(2x + 3)$

25. $f'(x) = 3x^2$, with $x > 0$

27. $f'(x) = e^{(e^x)}e^x$

29. $f'(x) = e^{ex+1}$

31. $f'(x) = 0$

33. $f'(x) = -3x^{-4}e^{2x} + x^{-3}(2e^{2x})$.

35. $f'(x) = 2x\log_2 x + \frac{x}{\ln 2} + 3x^2$

37. $f'(x) = \frac{1}{x^2 + e^{\sqrt{x}}}(2x + e^{\sqrt{x}}(\frac{1}{2}x^{-\frac{1}{2}}))$

39. $f'(x) = \frac{1}{2}(\log_2(3x - 5))^{-\frac{1}{2}}(\frac{1}{\ln 2}\frac{1}{3^x - 5}((\ln 3)3^x)$

41. $f'(x) = 2x\ln(\ln x) + x^2\frac{1}{\ln x} \cdot \frac{1}{x}$

43. $f'(x) = (\ln 2)2^{x^2}(2x)$

45. $f'(x) = \begin{cases} (\ln 2)2^x, & \text{if } x < -2 \\ \text{DNE}, & \text{if } x = -2 \\ -\frac{2}{x^3}, & \text{if } x > -2 \end{cases}$

47. $f'(x) = \begin{cases} 2x, & \text{if } x < 1 \\ \text{DNE}, & \text{if } x = 1 \\ \frac{1}{x}, & \text{if } x \geq 1 \end{cases}$

49. $f'(x) = \frac{1}{2}(x\ln|2^x + 1|)^{-\frac{1}{2}}(\ln|2^x + 1| + \frac{(\ln 2)x2^x}{2^x + 1})$

51. $f'(x) = \frac{2^x\sqrt{x^3-1}}{\sqrt{x}(2x-1)}(\ln 2 + \frac{3x^2}{2(x^3-1)} - \frac{1}{2x} - \frac{2}{2x-1})$

53. $f'(x) = x^{\ln x}(2)(\ln x)(\frac{1}{x})$

55. $f'(x) = (\frac{x}{x-1})^x(\ln x - \ln(x-1) + 1 - \frac{x}{x-1})$

57. $f'(x) = (\ln x)^{\ln x}(\frac{\ln(\ln x)}{x} + \frac{1}{x})$

59. $f(x) = \frac{e^{4x}}{3x^5 - 1}$

61. $f(x) = \frac{1}{2}\ln|x^2 + 3| + C$

63. $f(x) = \ln(1 + e^x) + C$

65. (a) $A(t) = 1000e^{0.077t}$; (b) $A(30) \approx \$10{,}074.42$; (c) $t \approx 18$ years.

66. (a) 70 degrees; (b) 350 degrees; (c) 21 minutes. (d) Find $T'(t)$ and show that it is always positive; this means that the temperature of the yam increases over time. Find $T''(t)$ and show that it is always negative; this means that the rate of change of the temperature of the yam decreases over time.

69. $\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = e^{kx} \cdot k = ke^{kx}$.

71. If $f(x) = Ae^{kx}$ is exponential, then $f'(x) = Ake^{kx} = k(Ae^{kx}) = kf(x)$, so $f'(x)$ is proportional to $f(x)$.

73. For $x > 0$ we have $|x| = x$ and therefore $\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}$. For $x < 0$ we have $|x| = -x$ and therefore by the chain rule we have $\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x}$.

Section 2.6

1. T, F, F, T, F, T, F, T.

3. See the proof in the reading.
5. (a) If x is in degrees, then the slope of the graph of $\sin x$ at $x = 0$ is very small, and in particular not equal to $\cos 0 = 1$. To convince yourself of this, graph $\sin x$ (in degrees) together with the line $y = x$ (which has slope 1 at $x = 0$).
7. (a) $\cos(3x^2)$ is a composition, not a product, but the product rule was applied; (b) the chain rule was applied incorrectly, with the derivative of $3x^2$ written on the inside instead of the outside.
9. No, because to differentiate $\sin^{-1}(\sin x) = x$ we would first need to know how to differentiate $\sin^{-1} x$, which is exactly what we would be trying to prove.
11. (a) $\cos(3x^2)$ is a composition, not a product, but the product rule was applied; (b) the chain rule was applied incorrectly, with the derivative of $3x^2$ written on the inside instead of the outside.
13. (a) Use the fact that $\frac{1}{2}e^{-x} \geq 0$ for all x, and split the expressions in the definitions of $\cosh x$ and $\sinh x$ into sums. (b) Calculate the two limits by dividing top and bottom by e^x and show they are both equal to 1.
15. (a) Think about the graph of the sum of $\frac{1}{2}e^x$ and $-\frac{1}{2}e^{-x}$. (b) Calculate $\lim\limits_{x\to-\infty} \dfrac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}e^{-x}}$ by dividing top and bottom by e^{-x}, and Show that this limit is equal to 1.
17. $f'(x) = \frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}$
19. $f'(x) = -\csc^2 x + \csc x \cot x$
21. $f'(x) = 0$
23. $f'(x) = 3\sec x \sec^2 x + 3\sec x \tan^2 x$
25. $f'(x) = \cos(\cos(\sec x))(-\sin(\sec x))(\sec x \tan x)$
27. $f'(x) = e^{\csc^2 x}(2\csc x)(-\csc x \cot x)$
29. $f'(x) = \frac{-(\ln 2)2^x(5x\sin x) + 2^x(5\sin x + 5x\cos x)}{25x^2\sin^2 x}$
31. $f'(x) = \sqrt{\sin x \cos x} + \frac{1}{2}x(\sin x\cos x)^{-\frac{1}{2}}(\cos^2 x - \sin^2 x)$
33. $f'(x) = \frac{(6x\ln x + 3x)\tan x - 3x^2\ln x \sec^2 x}{\tan^2 x}$
35. $f'(x) = \cos(\ln x)(\frac{1}{x})$
37. $f'(x) = \frac{6x}{\sqrt{1-9x^4}}$
39. $f'(x) = 2x\arctan x^2 + x^2(\frac{2x}{1+x^4})$
41. $f'(x) = \frac{2x}{|x^2|\sqrt{x^4-1}}$
43. $f'(x) = \frac{1}{\sqrt{1-\sec^4 x}}(2\sec x)(\sec x \tan x)$
45. $\dfrac{\frac{\tan^{-1} x}{\sqrt{1-x^2}} - \frac{\sin^{-1} x}{1+x^2}}{(\tan^{-1} x)^2}$
47. $f'(x) = \frac{1}{\operatorname{arcsec}(\sin^2 x)} \frac{1}{\sin^2 x\sqrt{\sin^4 x - 1}}(2\sin x \cos x)$
49. $f'(x) = \sec(1+\tan^{-1} x)\tan(1+\tan^{-1} x) \times (\frac{1}{1+x^2})$
51. $f'(x) = \sinh x^3 + 3x^3\cosh x^3$
53. $f'(x) = \sinh(\ln(x^2+1))(\frac{1}{x^2+1})(2x)$
55. $f'(x) = \frac{1}{2}(\cosh^2 x + 1)^{-\frac{1}{2}}(2\cosh x \sinh x)$
57. $f'(x) = \frac{3x^2}{\sqrt{x^6+1}}$
59. $f'(x) = \dfrac{\frac{1}{\sqrt{x^2+1}}(\cosh^{-1} x) - \sinh^{-1} x(\frac{1}{\sqrt{x^2-1}})}{(\cosh -1x)^2}$
61. $f'(x) = \cos(e^{\sinh^{-1} x})e^{\sinh^{-1} x}\frac{1}{x^2+1}$
63. $f'(x) = (\sin x)^x(\ln(\sin x) + \frac{x\cos x}{\sin x})$
65. $f'(x) = (\sin x)^{\cos x}(-\sin x \ln(\sin x) + \frac{\cos^2 x}{\sin x})$
67. $f(x) = \sin^{-1} 2x$
69. $f(x) = \frac{1}{6}\ln(1+9x^2)$
71. $f(x) = \sin^{-1}(\frac{3x}{2})$
73. $f(x) = \sinh^{-1} 2x$
75. $f(x) = -\frac{1}{6}\ln(1-9x^2)$
77. $f(x) = \sinh^{-1}(\frac{3x}{2})$
79. (a) To simplify things, let $C = \frac{\sqrt{4km-f^2}}{2m}$; since k, m, and f are all constants, so is C. Then follow the given hint. (b) Set $s(0) = s_0$ and $s'(0) = v_0$ and solve for A and B.
81. Mimic the proof in the reading for $\frac{d}{dx}(\sin x)$, except using a sum identity for cosine in the second step.
83. $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{(0)(\sin x) - (1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$.
85. Differentiating both sides of $\tan(\tan^{-1} x) = x$ gives us $\sec^2(\tan^{-1} x)\frac{d}{dx}(\tan^{-1} x) = 1$, and therefore $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sec^2(\tan^{-1} x)}$. By one of the Thinking Back problems in this section, we have $\sec^2(\tan^{-1} x) = 1 + x^2$, and therefore $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.
87. $x^2 - y^2 = \cosh^2 t - \sinh^2 t = (\frac{e^t - e^{-t}}{2})^2 - (\frac{e^t + e^{-t}}{2})^2 = \frac{e^{2t} - 2e^t e^{-t} + e^{-2t} - e^{2t} - 2e^t e^{-t} + e^{-2t}}{2} = 1$. For the bonus question, consider whether $\cosh t$ can ever be negative.
89. Expand the right hand side using the definitions and simplify to get the left hand side.
91. Mimic the proof in the reading that was given for the derivative of $\sinh x$.
93. Mimic the proof in the reading that was given for the derivative of $\sinh^{-1} x$.
95. $\frac{e^y - e^{-y}}{2} = x \Longrightarrow e^y - \frac{1}{e^y} = 2x \Longrightarrow \frac{e^{2y}-1}{e^y} = 2x \Longrightarrow e^{2y} - 2xe^y - 1 = 0$. This is quadratic in e^y, and the quadratic formula gives $e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$, and thus $y = \ln(x \pm \sqrt{x^2+1})$. Since logarithms are only defined for positive numbers we have $y = \ln(x + \sqrt{x^2+1})$.

97. Start with $y = \tanh x = \frac{\sinh x}{\cosh x}$ and use the problems above.

Chapter 3

Section 3.1

1. T, T, T, F, F, T, F, F.
3. $f'(1)$ is either 0 or does not exist. If f is differentiable at $x = 1$ then $f'(1)$ must equal 0.
5. $x = -2$, $x = 0$, $x = 4$, $x = 5$.
7. f' has at least two zeros in the interval $[-4, 2]$.
9. There is some $c \in (-2, 4)$ with $f'(c) = -\frac{1}{3}$.
11. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b) = 0$, then there exists at least one value $c \in (a, b)$ for which the tangent line to f at $x = c$ is horizontal.
13. The graph of $f(x) = (x - 2)(x - 6)$ is one example.
15. Your graph should have roots at $x = -2$ and $x = 2$ and horizontal tangent lines at three places between these roots.
17. One example is an "upside-down V" with roots at $x = -2$ and $x = 2$ where the top point of the V occurs at $x = -1$.
19. One example is the function f that is equal to $x + 3$ for $-3 \leq x < -1$ and equal to 0 for $x = -1$.
21. Draw a graph that happens to have a slight "cusp" just at the place where its tangent line would have been equal to the average rate of change.
23. $f'(x) = 0$ at $x = \frac{3}{2}$ and $f'(x)$ does not exist at $x = 3$. f has a local maximum at $x = \frac{3}{2}$ and a local minimum at $x = 3$.
25. $f'(x) = 0$ at $x \approx 0.5$, $x \approx 2$, and $x \approx 3.5$. f has a local minimum at $x \approx 0.5$ and a local minimum at $x = 3.5$. There is neither a maximum nor a minimum at $x \approx 2$.
27. One critical point at $x = -0.65$, a local minimum.
29. Critical points $x = -3$, $x = 0$, $x = 1$, a local minimum, maximum, and minimum, respectively.
31. One critical point at $x = \ln(\frac{3}{2})$, a local maximum.
33. Undefined at $x = 0$. One critical point at $x = \frac{e}{2}$, a local maximum.
35. Critical points at points of the form $x = \pi k$ where k is an integer; local minima at the odd multiples $x = \pi(2k + 1)$, local maxima at the even multiples $x = \pi(2k)$.
37. From the graph, f appears to be continuous on $[-3, 1]$ and differentiable on $(-3, 1)$, and moreover $f(-3) = f(1) = 0$, so Rolle's Theorem applies. Therefore there is some $c \in (-3, 1)$ such that $f'(c) = 0$. In this example there are three such values of c, namely $c \approx -2.3$, $c = -1$, and $c = 0.3$.
39. From the graph, f appears to be continuous on $[0, 4]$ and differentiable on $(0, 4)$, and moreover $f(0) = f(4) = 0$, so Rolle's Theorem applies. Therefore there is some $c \in (0, 4)$ such that $f'(c) = 0$. In this example there are three such values of c, namely $c \approx 0.5$, $c \approx 2$, and $c \approx 3.5$.
41. f is continuous and differentiable everywhere, and $f(0) = f(3) = 0$, so Rolle's Theorem applies; $c = \frac{1}{3}(4 - \sqrt{7})$ and $c = \frac{1}{3}(4 + \sqrt{7})$.
43. f is continuous and differentiable everywhere, and $f(-2) = f(2) = 0$, so Rolle's Theorem applies; $c \approx -1.27279$, $c \approx 0$, and $c \approx 1.27279$.
45. f is continuous and differentiable everywhere, and $\cos(-\frac{\pi}{2}) = \cos(\frac{3\pi}{2}) = 0$, so Rolle's Theorem applies; $c = 0, c = \pi$.
47. f is continuous and differentiable everywhere, and $f(0) = f(2) = 0$, so Rolle's Theorem applies; $c = \sqrt{2}$.
49. f appears continuous on $[0, 2]$ and differentiable on $(0, 2)$; there is one value $x = c$ that satisfies the conclusion of the Mean Value Theorem, rougly at $x \approx 1.2$.
51. f appears continuous on $[-3, 0]$ and differentiable on $(-3, 0)$; there are two values $x = c$ that satisfy the conclusion of the Mean Value Theorem, at $c \approx -2.8$ and $c \approx -0.9$.
53. f is not continuous or differentiable on $[-3, 2]$, so the Mean Value Theorem does not apply.
55. f is continuous and differentiable on $[-2, 3]$; $c \approx -0.5275$ and $c \approx 2.5275$.
57. f is continuous on $[0, 1]$ and differentiable on $(0, 1)$, so the Mean Value Theorem applies; $c \approx 0.4028$.
59. f is continuous and differentiable everywhere, so the Mean Value Theorem applies; $c = cos^{-1}(\frac{2}{\pi}) \approx 0.88$.
61. $C(h)$ is a differentiable function, and $C'(4) = 0.6 \neq 0$, so $C(h)$ cannot have a local minimum at $h = 4$.