Chapter 4

Induction

There is one additional method of proof, known as mathematical induction, that is especially important in discrete mathematics. Since it is rather more complicated than the proof techniques considered previously, we will devote an entire chapter to it.

4.1 Chains of Implications

One of the endearing properties of conditional statements is the possibility of fashioning them into long chains of implications. That was precisely what we did in constructing direct proofs in chapter two.

For example, suppose I tell you, “If it rains I will go to the movies,” and then later I tell you, “If I go to the movies I will eat popcorn.” You would be entirely justified in concluding that the conditional statement “If it rains I will eat popcorn” is true as well. We can express this symbolically as follows: Let P be the statement “It rained,” let Q be the statement “I will go to the movies,” and let R be the statement “I will eat popcorn.” Then we are led to the following valid argument:

- P is true.
- $P \to Q$.
- $Q \to R$. Therefore,
- R is true.

Of course, this argument is valid regardless of the meanings of P, Q and R.

To put it another way, if P implies Q and Q implies R, then it is valid to conclude that P implies R.

This particular chain of implications involved three propositions, but there is no reason to stop there. Suppose I have k propositions, labeled $P_1, P_2, \ldots, P_k$. Further suppose that I know $P_1 \rightarrow P_2$, $P_2 \rightarrow P_3$ and so on until I reach $P_{k-1} \rightarrow P_k$. Then, once I have shown that P_1 is true I am immediately justified in concluding that P_k is true as well. Given P_1, the remaining propositions fall like dominoes.

It is upon this foundation that mathematical induction is based.

4.2 How Induction Works

Let us be more specific. Suppose that $P(x)$ is some proposition about positive integers. Here we are using x as a stand-in for some unspecified integer. Consider the following argument:

- $P(1)$ is true.
- If n is a positive integer such that $P(n)$ is true, then $P(n+1)$ is true. Therefore,
- $P(2)$ is true.

This is a valid argument, and that has some curious consequences.

The action revolves around the if-then statement in the second line. It tells us that as soon as we know our proposition P is true for any given integer n, we are justified in concluding that P is true for the next integer as well. If I know $P(17)$ is true, then I conclude that $P(18)$ is true as well. Given $P(72)$, I get $P(73)$ for free. In the argument above we used this implication to go from the assumption that $P(1)$ was true to the conclusion that $P(2)$ was also true.

So now we have $P(2)$. What if we run the argument again, starting with "$P(2)$ is true" instead of "$P(1)$ is true"? In this case the second line would permit us to conclude that $P(3)$ is true. And if we run the argument yet again, now starting with "$P(3)$ is true?" We would have concluded that $P(4)$ is true.

Perhaps you see where this is leading. The argument could be run repeatedly, with the conclusion of each run forming the initial assumption for the next one. In so doing, we would inevitably conclude that the proposition P is true for all positive integers. To see that, suppose we know that $P(1)$ is true and that for all integers n, we have $P(n) \to P(n+1)$. Finally, suppose we wanted to prove that the proposition P is true for a particular integer k. We could prove this via the following chain of implications:

$$P(1) \to P(2),\ P(2) \to P(3), \ldots, P(k-1) \to P(k).$$

Once we know $P(1)$ to be true, the remaining propositions fall into place.

The logic discussed above is sound, but is it really practical to prove anything this way? The surprising answer is yes. Clumsy as it seems, mathematical induction is quite useful in proving statements about the positive integers. We will see several examples of this in the remainder of the chapter.

Before moving along, let us clear up one potential point of confusion. We have already used the word "induction" in chapter one. There we were describing a style of reasoning used by scientists to conclude, from the fact that a theory successfully predicted the results of one hundred experiments, that it would successfully predict the result of the one hundred first. Though we are using the word differently here, to refer to a method of proving propositions about the positive integers, the two uses of the word are related. In both cases the intent is to learn something new by extrapolating from known, concrete examples. The main difference is that in mathematical induction, the extrapolation is justified by rigorous logic. In science, the extrapolation is justified only by the fact that such extrapolations have worked so well in the past.

4.3 Two Examples

Something amusing happens when we begin adding up consecutive odd numbers. Observe:

$$1 = 1 \qquad 1+3=4 \qquad 1+3+5=9 \qquad 1+3+5+7=16$$
$$1+3+5+7+9=25 \qquad 1+3+5+7+9+11=36.$$

Based on this evidence we might be led to think that the sum of the first n odd numbers is n^2. By this we mean that, for example, the first four odd numbers are 1, 3, 5 and 7, and their sum is 16 which is four squared. I think you will find that the seventh odd number is 13 and that the sum of all the odd numbers between one and thirteen inclusive is 49 which is seven squared.

We will prove this conjecture by induction. Before doing so, however, let us pause to discuss a few points concerning mathematical style.

If we are going to perform calculations concerning sums of arbitrary odd numbers, we need some convenient way of writing down such sums. Since a number is odd precisely when it is one more than some multiple of two, we can say that any odd number can be written as $2k + 1$ for some integer k.

In the present circumstance, more precision is required. It is not just some arbitrary collection of odd integers we are summing, but actually the first n such numbers. It would be nice to have a way of expressing the n-th odd number solely in terms of n. After some experimentation we might stumble on to the fact that the n-th odd number is given by $2n - 1$. By this we mean that the second odd number is three, which is also $2(2) - 1$ and the tenth odd number is nineteen, which is $2(10) - 1$, and so on. Thus, our goal will be to prove the following:

Theorem 1. *For any integer n,*

$$1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) = n^2.$$

Incidentally, since we are discussing questions of how mathematics is written down, please notice the proper use of the the three dots in the sum above. After listing enough of the summands to make the pattern clear, we put a plus sign, then the three dots, then another plus sign.

Also notice that if the n-th odd number is $2n - 1$, then the odd number coming just before it is $2(n - 1) - 1 = 2n - 3$.

All this is fine if you have hours to spend writing out enough terms of your sum to make the pattern clear to your reader, but mathematicians are busy people and therefore seek greater concision than we have obtained so far. Toward that end, they have devised a simple abbreviation to unburden themselves of the awful task of writing out so many terms of their sums. Instead of the equation given above, they would restate the theorem as follows:

Theorem 2. *For any positive integer n,*

$$\sum_{k=1}^{n}(2k - 1) = n^2.$$

This is read as follows: "The sum as k goes from 1 to n of $2k - 1$ is equal to n^2." By this we mean that the individual summands are recovered by first setting $k = 1$ (thereby obtaining 2(1)-1=1 as the first term in the sum), then setting $k = 2$ (thereby obtaining 2(2)-1=3 as the second term in the sum) and continuing until we reach $k = n$ (which gives us $2n - 1$ as the last term in the sum).

The variable k is known as the ***index of summation***. There is nothing special about the letter k. Any other letter would have worked just as well.

What is required to prove this theorem inductively? First we must show the theorem is true for the special case when $n = 1$. Having done that, we must then show that whenever the theorem is true for an integer n, it must, as a matter of logic, be true for $n+1$ as well. Specifically, we must prove the following if-then statement: If

$$\sum_{k=1}^{n}(2k-1) = n^2$$

then

$$\sum_{k=1}^{n+1}(2k-1) = (n+1)^2.$$

Look carefully at the sums given above. The only difference between them is the extra term, namely $(2n + 1)$, in the second sum. The first n terms of the sums are identical. Keep that in mind as you read the following proof:

Proof. When $n = 1$ our theorem asserts that $1 = 1$ which is certainly true. Though it is not necessary for the proof, we observe that our previous calculations show the theorem is true when $n = 2, 3, 4, 5, 6$ as well.

Now suppose that for some value of n we have

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

If we add $2n + 1$ to both sides we get

$$\left[\sum_{k=1}^{n}(2k-1)\right] + (2n+1) = n^2 + (2n+1).$$

Since $2n + 1$ is the next odd number after $2n - 1$, we can rewrite this as

$$\sum_{k=1}^{n+1}(2k-1) = n^2 + 2n + 1 = (n+1)^2,$$

which is exactly what we wanted. Thus, our theorem is true by induction. $\square$

Our second example is slightly more complicated, but pay close attention to the logic we use at each step. Instead of thinking about sums of consecutive odd numbers, this time we will consider sums of consecutive squares.

Theorem 3. *For any positive integer n we have*

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Notice that when $n = 1$ we obtain the equation $1 = \frac{1(2)(3)}{6}$ which is true. When $n = 2$ we obtain

$$1 + 4 = 5 = \frac{2(2+1)(2(2)+1)}{6} = \frac{30}{6},$$

which is also true.

Proof. We have already shown the theorem is true when $n = 1$. Now assume that for some particular integer n we know that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We need to show that from this assumption it follows that

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

This expression was obtained from the preceding one by replacing n with $n + 1$ everywhere n appeared. As in the previous example, the second sum differs from the first only in its final term. We could obtain the second sum from the first by adding $(n + 1)^2$ to both sides.

We now carry out the following computation:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \left[\sum_{k=1}^{n} k^2\right] + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1)+6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

Since this is exactly what we needed to show, the theorem is true by induction. □

4.4 Strong vs. Weak Induction

As a method of proof induction is both useful and fascinating. Nonetheless, just as sometimes in life you must forsake the Oreos and eat the spinach instead, we must pause here to ponder certain logical subtleties.

The sort of induction we have used thus far is known as "weak" induction. It is so named because the assumption that is made to kick off the induction, namely that the proposition P is true for one particular integer x, is the bare minimum we must assume to get the induction going.

Sometimes in life a stiletto is adequate, and for those times weak induction is sufficient. Other times, alas, there is just no substitute for a well-oiled chainsaw. It is with those times in mind that mathematicians have developed the principle of strong induction.

Not for us the minimal assumption that our proposition P is true for one particular integer x. In strong induction we assume the proposition is true not just for the integer x, but for every positive integer smaller than x as well.

Let us express this difference using the notation we developed in chapter two. Weak induction has the following form:

$$[P(1) \wedge \forall k(P(k) \to P(k+1))] \to \forall n\ P(n).$$

In plain English we would say, "If $P(1)$ is true, and if for all integers k we have that $P(k)$ implies $P(k+1)$, then $P(n)$ is true for all n."

Strong induction, by contrast, looks like this:

$$[P(1) \wedge (\forall k(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1))] \rightarrow \forall n\ P(n).$$

This time we are saying, "If $P(1)$ is true, and if for all integers k we know that when P is true for all integers between one and k inclusive this implies that P is true for $k+1$, then we know that $P(n)$ is true for all n.

An example should help to clarify things:

Theorem 4. *Every positive integer greater than one is either prime or can be written as the product of prime numbers.*

Proof. The number two is itself prime, so the theorem is true in this case.

Now suppose that the theorem is true for each of the integers $2, 3, 4, \ldots, k$, for some integer k. Then all of these integers are either prime or are the product of primes. We need to show this implies the same for $k+1$. The number $k+1$ is either prime or it is not prime. If it is prime then we are done. If $k+1$ is not prime then there must be two integers d_1 and d_2, neither of them equal to one, such that

$$k+1 = d_1 d_2.$$

Furthermore, since $d_1, d_2 \neq 1$, we know that both d_1 and d_2 are smaller than k. It then follows from our assumption that both of them are either prime or can be written as the product of primes. If both are prime then we have written $k+1$ as the product of primes and we are done.

If either of them is not prime, then write out their prime factorizations. Having done that, we can combine the prime factorizations of d_1 and d_2 to obtain a factorization of $k+1$. $\square$

In this case, weak induction would not have been up to the task of proving the desired result. The only things we knew about d_1 and d_2 were that they were smaller than $k+1$ and divided $k+1$. With no further information we had no basis for concluding that they could be written as the product of primes. Therefore, it was necessary for us to assume that any number smaller than $k+1$ could be written as the product of primes. Strong induction allows us to make that assumption; weak induction does not.

Incidentally, the theorem above is one part of the fundamental theorem of arithmetic mentioned in chapter three.

Strong and weak induction are logically equivalent. Both are sound methods for proving theorems about positive integers. This may prompt you to ask why we are making such a fuss about distinguishing between them. Well, it is important when carrying out a proof that you be explicit regarding your assumptions. Though logically equivalent, the fact remains that different assumptions go into each one. Therefore, you must tell your reader the particular form of induction you are using.

So why not just always use strong induction? After all, strong induction gives you more information to work with, thereby simplifying the work involved in proving the result.

In writing a proof you should always make the weakest assumptions that allow you to prove what you want. The weaker your assumptions, the more powerful your theorem will be. For many tasks weak induction is perfectly adequate. Why make added assumptions when you do not need them?

4.5 Two More Examples

Mathematical induction is especially useful in proving formulas concerning sums of positive integers. But it arises in numerous other contexts as well. Two such examples are considered here.

Theorem 5. *If S is a set containing n elements, then there are 2^n subsets of S.*

Recall that if A and B are sets, then we say that A is a subset of B if every element in A is also in B. Thus, if S is the set $\{1, 2\}$, then the subsets of S are

$$\emptyset, \{1, 2\}, \{1\} \text{ and } \{2\}.$$

Also recall that the empty set is a subset of every set. We will have more to say about sets in the next chapter.

Notice that our two-element set possessed four subsets. According to our theorem, a three-element set will possess eight subsets; a four-element set will possess sixteen subsets, and so on. We can prove this by induction.

Proof. When $n = 1$, our theorem says that if S is a one-element set, then S will possess $2^1 = 2$ subsets. Those sets are $\emptyset$ and S itself. So the statement is true for $n = 1$.

Now we assume the statement is true for some integer n. In other words, if a set S has n elements, then it possesses 2^n subsets. We must show that this assumption implies that if some other set T has $n+1$ elements, then it possesses 2^{n+1} subsets.

To do that, assume that T has $n+1$ elements. Let x be an arbitrary element in T. To determine the number of subsets of T, notice that all such subsets either contain x or do not contain x. We will consider each of these cases separately.

Let S be the set $T - \{x\}$. In other words, S is the set T with the element x removed.

Any subset of T that does not contain x can be viewed as a subset of S. We know that S has n elements. It then follows from our assumption that S possesses 2^n subsets. From this we conclude that there are 2^n subsets of T that do not contain x.

What about subsets of T that do contain x? Well, notice that any subset of T that contains x can be viewed as a subset of S with the element x inserted into it. We further observe that if X and Y are two different subsets of S, then they will still be different sets after x is added to each of them.

It follows that there are as many subsets of T containing x as there are subsets of S; namely 2^n. Since the total number of subsets of T is obtained by adding the number of subsets containing x to the number not containing x, we have that the total number of subsets of T is

$$2^n + 2^n = 2(2^n) = 2^{n+1}$$

as desired. □

Our second example involves two formulas from trigonometry that you knew at one time but have probably forgotten. Specifically:

- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\sin(A+B) = \sin A \cos B + \sin B \cos A$

Let me also remind you that since we define $i = \sqrt{-1}$, we have that $i^2 = -1$.

Theorem 6. *For any positive integer n, we have*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Proof. We will use induction, of course. When $n = 1$ our theorem says that

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta,$$

which is plainly true.

Now suppose the result is true for some integer n. In other words, for some particular value of n we can assume that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

We now carry out the following computation:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta) \\ &= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta) \\ &= (\cos n\theta)(\cos\theta) - (\sin n\theta)(\sin\theta) \\ &\qquad + i((\sin n\theta)(\cos\theta) + (\cos n\theta)(\sin\theta)) \\ &= \cos(n\theta + \theta) = i\sin(n\theta + \theta) \\ &= \cos[(n+1)\theta] + i\sin[(n+1)\theta]\end{aligned}$$

as desired. □

Since this amusing result was first established by the French mathematician Abraham DeMoivre, we now refer to it as ***DeMoivre's Formula***.

4.6 The Well-Ordering Principle

Mathematical induction is a useful device for proving theorems about positive integers. It is not useful at all in proving theorems about rational numbers or real numbers. Why is that?

It is a curious fact that any set of positive integers must have a smallest element. A consequence of this is the observation that the positive integers are ordered in such a way that for any particular integer larger than one we can identify an immediate predecessor and an immediate successor. It is this property that sets the positive integers apart from the rational numbers and the real numbers. These latter sets are certainly ordered, meaning that given any two rational or real numbers a and b we know that either they are equal, or one of them is larger than the other. But there is no next rational (or real) number after a and no rational (or real) number that immediately precedes a.

An ordered set S with the property that any subset of S must have a smallest element is said to be ***well-ordered***. Thus, the positive integers are well-ordered, whereas the rational numbers and the real numbers are not. Well-ordering is precisely the property we need to make induction work.

To see this, recall that induction is premised on first showing that a result is true for the smallest element in an ordered set. When working with the positive integers that smallest element is one. The next step involved showing that assuming the truth of the claim for any particular element implies its truth for the next element in the ordering. If our set possesses elements that lack either an immediate predecessor or an immediate successor, our chain of implications will break down (with the obvious exceptions that the smallest element has no predecessor and the largest element, if there is one, has no successor).

On the other hand, it is possible to construct proofs by induction on any set that is well-ordered. Suppose S is a well-ordered set in which we have identified some smallest element. Call that element x. Since S is well-ordered we know that the subset $S - \{x\}$ has a smallest element, call it y. Then we can say that y must be the immediate successor of x. If we then apply the well-ordering property to the subset $S - \{x, y\}$, we would produce the immediate successor of y. This process could be continued indefinitely. Since we can be certain that no elements of S will be left out of the chain, we can use induction to prove assertions about S.

Let me conclude by mentioning that the well-ordering principle also leads to an amusing paradox. It goes like this: Every positive integer can be uniquely described by some string of letters. We can certainly describe a number by writing out its name, but we can also give other descriptions. For example, the number "Four" could also be described as "The square of the smallest prime."

Now, there have to be certain numbers for which the shortest string of letters that uniquely describes them is more than two hundred letters long. After all, there are infinitely many positive integers but only finitely many strings containing fewer than two hundred letters. (Incidentally, in the next chapter we will see how to determine precisely how many strings with no more than two hundred letters there are). Therefore, define the set S to be the set of all positive integers that can not be described by a string possessing fewer than two hundred letters. Since S is a subset of the integers, and since the integers are well-ordered, we know that S must have a smallest element. Call this number x.

Then the situation is this: The number x is "the smallest positive integer that can not be described by a string possessing fewer than two hundred letters." But the phrase in quotation marks uniquely identifies x with a string possessing fewer than two hundred letters.

Think about that until it goes away.

4.7 Problem Solving Skills

4.7.1 What Sorts of Problems are Solved by Induction?

It is only through practice that you develop some facility for choosing the best proof technique for a given problem. But there are a few tell-tale signs that suggest when to use induction.

Two things are essential to any successful induction proof:

1. The statement to be proved must assert that something is true for every positive integer.

2. If P is our proposition and n is a positive integer, there must be some way of relating $P(n)$ to $P(n+1)$.

The necessity of item one above follows from the way induction is defined. Our definition of an induction proof, given in the second section of this chapter, assumes that our goal was to say something about the positive integers.

Item two becomes especially clear in the context of the summation formulas we proved in section three. Suppose $P(n)$ is an assertion about the sum of the first n integers in some sequence (for example, the first n odd numbers in our first example, and the first n perfect squares in the second). Then $P(n+1)$ differs from $P(n)$ only by containing one additional term. Since our inductive hypothesis inevitably allows us to make a statement about the sum $P(n)$, we can learn something about $P(n+1)$ simply by adding one more term to both sides of a given equation.

In proving DeMoivre's formula the situation was different. In that case n was appearing in the exponent of an algebraic expression. To go from

n to $n+1$ in that case we used the fact that $x^{n+1} = x^n x$. In this case $x = \cos\theta + i\sin\theta$. The proof about the number of subsets of a set with n elements was a bit more complex in this regard. It is not immediately clear that the subsets of an $(n+1)$-element set really are related in some way to the subsets of an n element set. The trick we used there (of dividing the subsets of a set S into those that do, and do not, contain a given element) is worth remembering.

There is another class of problems to which induction can frequently be applied: those involving sequences. As an example, consider the Fibonacci numbers. You may recall that this sequence is defined by:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad 55 \quad 89 \ \ldots,$$

with each number after the first two being equal to the sum of the preceding two numbers. Let us denote the n-th number in this sequence as f_n. We would then have $f_1 = 1$, $f_3 = 2$, $f_7 = 13$ and $f_{11} = 89$, for example. Then we know that

$$f_n = f_{n-1} + f_{n-2}.$$

Proposition 1. *Let f_n be the n-th Fibonacci number. Then*

$$f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1.$$

When $n = 1$ our proposition asserts that

$$f_2 = f_3 - 1.$$

Since $f_2 = 1$ and $f_3 = 2$, this statement is true.

Though it is not necessary for our inductive proof, let us work out one more concrete example. If $n = 5$ then the proposition says

$$f_2 + f_4 + f_6 + f_8 + f_{10} = f_{11} - 1,$$

which is equivalent to

$$1 + 3 + 8 + 21 + 55 = 89 - 1.$$

This last equation is easily seen to be true.

Now we can proceed with our proof.

Proof. We have already shown the proposition is true when $n = 1$. Now suppose that for some particular value of n we have

$$f_2 + f_4 + \ldots + f_{2n-2} + f_{2n} = f_{2n+1} - 1.$$

We need to show this implies that

$$f_2 + f_4 + \ldots + f_{2n} + f_{2n+2} = f_{2n+3} - 1.$$

We notice that:

$$(f_2 + f_4 + \ldots + f_{2n-2} + f_{2n}) + f_{2n+2} = (f_{2n+1} - 1) + f_{2n+2}.$$

But we also notice that

$$f_{2n+1} + f_{2n+2} = f_{2n+3}.$$

It follows that the right hand side of the equation above is equal to $f_{2n+3} - 1$ and the proof is complete. □

4.7.2 The Logic of Induction Arguments

Many people find the logic behind induction arguments a bit confusing. The difficulty revolves around that if-then statement. It sure seems as if we are assuming what we are trying to prove. If we are allowed to assume that the proposition is true for some particular integer n in order to establish that it is also true for $n + 1$, why not simply assume it to be true for $n + 1$ and be done with it?

The answer is that the if-then statement is only one step in an inductive proof. It is not the whole thing. There is another step in which we verify that the proposition really is true for some particular positive integer. It is the combination of both these steps that makes a complete proof.

When I make the statement "If it rains then I will go to the movies," I am not making any assumptions about the weather. I am simply telling you what will happen if it rains. By the same token, if I say "If the sum of the first three odd integers is three squared, then the sum of the first four odd integers is four squared," I am not making any assumption about what the sum of the first three odd integers actually is.

To put it another way, suppose you are trying to show that the proposition $P(n)$ is true for any positive integer n. You are welcome to show that $P(n)$

implies $P(n+1)$, but by itself that does not prove your proposition. It is only when you add the extra ingredient that $P(1)$ is actually true that you obtain a correct proof.

Perhaps you are thinking that if $P(n)$ really does imply $P(n+1)$, then surely $P(n)$ must be true for *some* value of n. If so, consider the following example:

Proposition 2. *Let n be a positive integer satisfying $n \geq 2$. Suppose that in every set of n horses all the horses have the same color. Then in any set of $n+1$ horses, all the horses have the same color as well.*

Proof. Let S be a set containing $n+1$ horses. Label those horses $H_1, \ldots, H_{n+1}$. By our assumption, we must have that the horses $H_1, H_2, \ldots, H_n$ all have the same color. Similarly, the n horses $H_2, H_3, \ldots, H_{n+1}$ must all have the same color. The only way this is possible is for all $n+1$ horses to have the same color, and the proof is complete. □

There is nothing wrong with that proof. It really is true that if you assume that in every set of n horses all the horses have the same color, then you must conclude the same about sets of $n+1$ horses. But are we now justified in saying that we have proved, by induction, that all horses have the same color? Certainly not. Our implication is true enough, but it is not true that in any set of two horses both horses have the same color. Consequently, our proof never gets off the ground.

As a practical matter the base case of an induction proof is often easy to prove. But it is an essential ingredient nonetheless.

4.7.3 Working out the Details

The final stumbling block comes after we have decided to proceed by induction. Most students are capable of carrying out the sometimes tedious algebraic manipulations required in such a proof. The difficulty lies in determining what $P(n+1)$ actually says. This difficulty is especially acute in problems involving elaborate summation formulas.

Typically you are given the proposition $P(n)$. To decide what $P(n+1)$ says, you look for every appearance of n and replace it with $n+1$. For example, if $P(n)$ is

$$\sum_{k=1}^{n}(2k+1) = n^2,$$

then $P(n+1)$ is

$$\sum_{k=1}^{n+1}(2k+1) = (n+1)^2.$$

On the other hand, if $P(n)$ is

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

then $P(n+1)$ is

$$\sum_{k=1}^{n+1} k^2 = \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6}.$$

Do not try to simplify the expression in the same line in which you replace n by $n+1$. There will be plenty of time for that after you properly carry out the substitution.

4.8 Problems

In the problems below, n is always a positive integer. The symbols f_n refer to the Fibonacci numbers defined in the previous section. Recall that $n!$ (read: n ***factorial***) means the product of all the integers between one and n. So $3! = 3 \times 2 = 6$ and $5! = 5 \times 4 \times 3 \times 2 = 120$. Also notice that $(n+1)! = (n+1)n!$.

1. Let $P(n)$ be the proposition, "$2n+3$ is a perfect square."
   (a) List three values of n for which this proposition is true and three values for which it is false.
   (b) What is $P(n+1)$? What is $P(n^3+2)$?
2. Let $P(n)$ be the proposition, "If f_n is even then f_{n+1} is odd." What are $P(n+1)$ and $P(2n+3)$?
3. Let $P(n)$ be the proposition, "$\sum_{k=1}^{n}(2k-1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$." What are $P(n+1)$ and $P(2n-5)$

Prove each of the following statements for all integers n.

4. $1+2+3+\ldots+(n-1)+n=\frac{n(n+1)}{2}$.

5. $1+8+27+64+\ldots+(n-1)^3+n^3=\frac{n^2(n+1)^2}{4}$.

6. $f_n < 2^n$.

7. $\sum_{k=1}^{n} k(k!) = (n+1)! - 1$.

8. Let a be a positive integer and let r be a positive integer other than one. Then $\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1}$.

9. $\sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n-1}{2^n}$.

10. $\sum_{k=1}^{n} f_{2k-1} = f_{2n}$.

11. The number $4^n - 1$ is a multiple of three.

12. $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

13. The numbers $2^n + 3^n$ and 5^n leave the same remainder when divided by six. For example, if $n = 3$ then $2^3 + 3^3 = 35$ which leaves a remainder of five when divided by six. On the other hand $5^3 = 125$ which also leaves a remainder of five when divided by six.

14. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$.

15. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

16. Suppose you have n lines drawn through a circle in such a way that no two of the lines are parallel and no three of the lines intersect at a common point. Then these n lines divide the circle into $\frac{n^2+n+2}{2}$ regions. For example, when we have $n = 1$ line the circle is divided into two regions, and we have $\frac{1^2+1+2}{2} = 2$. If we have $n = 2$ lines we will have four regions, and $\dfrac{2^2+2+2}{2} = 4$. You may find it helpful to draw a picture for this one.

17. For all integers n, it is impossible to have a solution to the equation

$$4^n = a^2 + b^2 + c^2.$$

where a, b and c are all positive integers. (Hint: Notice that $4^n = 2^{2n}$ is a perfect square. Show that if $m^2 = a^2 + b^2 + c^2$, then we must have that a, b and c are all even. This can be done without induction; just think about what remainders a perfect square can leave when divided by four.)

In each of the next three problems an erroneous proof by induction is presented. Give a careful explanation of what the error is.

18. Let n be an integer satisfying $n \geq 1$. Then $3^n - 2$ is even.

    Proof. Suppose the theorem is true for an integer n, where $n \geq 1$. Then $3^n - 2$ is even. We must show this implies that $3^{n+1} - 2$ is even as well. To do that, we carry out the following computation:

    $$3^{n+1} - 2 = 3^n(3) - 2 = 3^n(2+1) - 2 = (3^n - 2) + 2(3^n).$$

    Notice that $2(3^k)$ is even, since it is a multiple of two. By our assumption, $3^n - 2$ is also even. Since the sum of two even numbers is even, we conclude that 3^{n+1} is even, and the proof is complete. □

19. Let x and y be positive integers and let n be the larger of the two of them (if x and y are equal then we will have $x = y = n$). Then $x = y$.

    Proof. First suppose that $n = 1$. Since n is the larger of x and y, and x, y are both positive, we have $x = y = n = 1$ in this case. So the proposition is true when $n = 1$.

    Now assume the proposition is true for some particular integer k. Then whenever k is the larger of x and y, then $x = y$. We must show that the same holds true when we replace k by $k + 1$.

    Suppose that $k + 1$ is equal to the larger of x and y. We must show this implies $x = y$. Since $k+1$ is the larger of x and y, we know that k must be the larger of $x - 1$ and $y - 1$. By our assumption, this implies that $x - 1 = y - 1$. Since that implies $x = y$, the proof is complete. □

20. Let a be a positive integer. Then $a^n = 1$ for all nonnegative integers n.

    Proof. When $n = 0$ our proposition says $a^0 = 1$. This is true by the definition of a^0. (Remember that anything to the zero power is equal to one.)

We will now proceed by strong induction. Suppose that $a^j = 1$ whenever j is a nonnegative integer smaller than k. we must show that $a^{k+1} = 1$ as well. Notice that:

$$a^{k+1} = \frac{(a^k)(a^k)}{a^{k-1}} = \frac{(1)(1)}{1} = 1,$$

as desired. This completes the proof. $\square$