The Jacobian

Department of Mathematics and Statistics

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Integration by Substitution

In integration by substitution we alter the integral by letting $u = g(x)$.

$$\int_a^b f(g(x))\, g'(x)\, dx \qquad \int_c^d f(u)\, du$$

If we think of this process in "reverse," we may think of the factor $g'(x)$ in the original integral, as a scaling factor when we substitute $g(x)$ for u in the integral $\int_c^d f(u)\, du$.

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Transformations in $\mathbb{R}^2$

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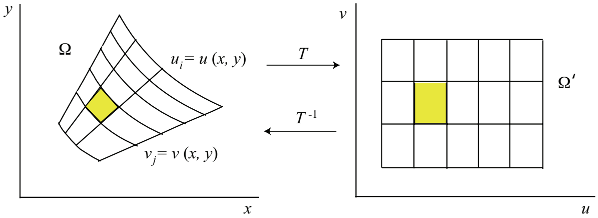
We will require the transformations that we use to be both one-to-one and differentiable at every point in the interior of Ω.



The Jacobian for a Transformation in $\mathbb{R}^2$

Definition

Let Ω and Ω' be subsets of $\mathbb{R}^2$. If the transformation $T : \Omega \to \Omega'$ has a differentiable inverse with $x = x(u, v)$ and $y = y(u, v)$, then we define the **Jacobian** of the transformation T, denoted by $\frac{\partial(x,y)}{\partial(u,v)}$, to be the determinant of the matrix

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

In an integral we have:

$$\iint_{\Omega} g(x, y)\, dA = \iint_{\Omega'} g(x(u, v), y(u, v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

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Definition

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