The Jacobian

Department of Mathematics and Statistics

November 2, 2012

Integration by Substitution

In integration by substitution we alter the integral by letting $u = g(x)$.

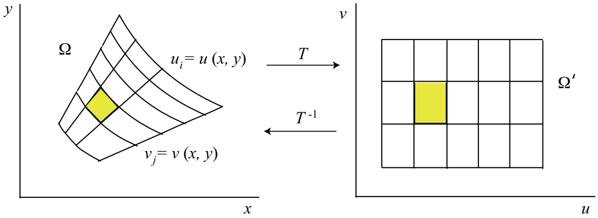
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_c^d f(u)\, du$$

If we think of this process in "reverse," we may think of the factor $g'(x)$ in the original integral, as a scaling factor when we substitute $g(x)$ for u in the integral $\int_c^d f(u)\, du$.

Transformations in $\mathbb{R}^2$

Typically we start with some region Ω in $\mathbb{R}^2$. We wish to find a function a $T : \Omega \to \Omega'$ where Ω' is another, and hopefully simpler, subset of $\mathbb{R}^2$. Such functions are called **transformations**.

We will require the transformations that we use to be both one-to-one and differentiable at every point in the interior of Ω.



The Jacobian for a Transformation in $\mathbb{R}^2$

Definition

Let Ω and Ω' be subsets of $\mathbb{R}^2$. If the transformation $T : \Omega \to \Omega'$ has a differentiable inverse with $x = x(u, v)$ and $y = y(u, v)$, then we define the **Jacobian** of the transformation T, denoted by $\frac{\partial(x,y)}{\partial(u,v)}$, to be the determinant of the matrix

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

In an integral we have:

$$\iint_{\Omega} g(x, y)\, dA = \iint_{\Omega'} g(x(u, v), y(u, v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

The Jacobian for a Transformation in $\mathbb{R}^3$

Definition

Let Ω and Ω' be subsets of $\mathbb{R}^3$. If the transformation $T : \Omega \to \Omega'$ has a differentiable inverse with $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$, then we define the **Jacobian** of the transformation T, denoted by $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, to be the determinant of the matrix

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian in an Integral

In an integral we have:

$$\iiint_{\Omega} g(x, y, z)\, dV =$$

$$\iiint_{\Omega'} g(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$