Line Integrals

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November 9, 2012

Line Integrals of Multivariate Functions

Definition

Let C be a curve in $\mathbb{R}^3$ with a smooth parametrization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in [a, b]$. Then the **integral of** $f(x, y, z)$ **along** C is

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \|\mathbf{r}'(t)\|\, dt.$$

Line Integral of a Vector Field

Definition

Suppose that C is a smooth curve in $\mathbb{R}^3$ with a smooth parametrization $\mathbf{r}(t)$ for $t \in [a, b]$ and with a vector field $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ whose component functions are each continuous on C and whose domain is open, connected, and simply connected. Then the **line integral of $\mathbf{F}(x, y, z)$ along** C is

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_a^b \left(F_1(x, y, z)\, x'(t) + F_2(x, y, z)\, y'(t) + F_3(x, y, z)\, z'(t)\right)\, dt.$$

Line Integral of a Vector Field, continued

Two alternative ways of writing this integral are

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_C (F_1\ dx + F_2\ dy + F_3\ dz)$$
$$= \int_C \mathbf{F} \cdot \mathbf{T}\ ds,$$

where $\mathbf{T}$ is the unit tangent vector.

The Fundamental Theorem of Line Integrals

Theorem

Let C be a smooth curve that is the graph of the vector function $\mathbf{r}(t)$ defined on the interval $[a, b]$ with $P = \mathbf{r}(a)$ and $Q = \mathbf{r}(b)$. If $\mathbf{F}(x, y, z)$ is a conservative vector field with $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$ on an open, connected, and simply connected domain containing the curve C, then

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(Q) - f(P).$$