Surfaces and Surface Integrals

Department of Mathematics and Statistics

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Surface Area

Definition

The **surface area** of a smooth surface $\mathcal{S}$ is

$$\int_{\mathcal{S}} 1 \, dS.$$

(a) If $\mathcal{S}$ is given by $z = f(x, y)$ for $(x, y) \in D \subseteq \mathbb{R}^2$, then

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dA.$$

(b) If $\mathcal{S}$ is parametrized by $\mathbf{r}(u, v)$ for $(u, v) \in D$, then

$$\begin{aligned} dS &= \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA \\ &= \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv. \end{aligned}$$

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Surface Integral of a Multivariate Function

Definition

The **integral of** $f(x, y, z)$ **over a smooth surface** $\mathcal{S}$ is

$$\int_{\mathcal{S}} f(x,y,z)\,dS = \iint_D f\left(x(u,v), y(u,v), z(u,v)\right) \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA$$
$$= \iint_D f\left(x(u,v), y(u,v), z(u,v)\right) \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv,$$

where D is the domain of the smooth parametrization of $\mathcal{S}$ by $\mathbf{r}(u,v)$.

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The Flux of a Vector Field Across a Surface

Definition

If $\mathbf{F}(x, y, z)$ is a continuous vector field defined on an oriented surface $\mathcal{S}$ given by $\mathbf{r}(u, v)$ for $(u, v) \in D$ with unit normal vector $\mathbf{n}$, then the **flux** of $\mathbf{F}(x, y, z)$ through $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{F}(x, y, z) \cdot \mathbf{n}\, dS.$$

(a) If $\mathcal{S}$ is the graph of $z = z(x, y)$, then

$$\int_{\mathcal{S}} \mathbf{F}(x, y, z) \cdot \mathbf{n}\, dS = \iint_D (\mathbf{F}(x, y, z) \cdot \mathbf{n}) \left\| z_x \times z_y \right\| dy\, dx.$$

(b) If $\mathcal{S}$ is parametrized by u and v, then

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