Green's Theorem

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The Del Operator

Definition

The **del operator**, ∇ is the vector of operations

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$$

or, in $\mathbb{R}^2$,

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j}.$$

Divergence of a Vector Field

Definition

The **divergence** of vector field $\mathbf{F}$ is the dot product of ∇ and $\mathbf{F}$.

(a) In $\mathbb{R}^2$, if $\mathbf{F}(x,y) = F_1(x,y)\,\mathbf{i} + F_2(x,y)\,\mathbf{j}$, then

$$\text{div }\mathbf{F}(x,y) = \nabla \cdot \mathbf{F}(x,y) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

(b) In $\mathbb{R}^3$, if $\mathbf{F}(x,y,z) = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$, then

$$\text{div }\mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Curl of a Vector Field

Definition

The **curl** of a vector field $\mathbf{F}(x,y,z) = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$ is the cross product of ∇ with **F**(x,y,z):

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}(x,y,z) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

If $\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$ is a vector field in $\mathbb{R}^2$, we define the curl of **F** to be the curl of the vector field in $\mathbb{R}^3$ whose first two components are the same as **F**'s and whose third component is 0. So,

$$\begin{aligned}\text{curl } \mathbf{F} &= \text{curl } \langle F_1(x,y), F_2(x,y), 0\rangle = \nabla \times \langle F_1(x,y), F_2(x,y), 0\rangle \\ &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Divergence of Curl, Curl of a Gradient

Theorem

(a) *If* $\mathbf{F} = \langle F_1(x,y,z), F_2(x,y,z) \rangle$ *is a vector field in* $\mathbb{R}^2$ *or* $\mathbf{F} = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$ *is a vector field in* $\mathbb{R}^3$*, for which* F_1*,* F_2*, and* F_3 *have continuous second-order partial derivatives, then*

$$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

(b) *For a multivariate function* f *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *with continuous second partial derivatives,*

$$\text{curl } \nabla f = \nabla \times (\nabla f) = \mathbf{0}.$$

Green's Theorem

Theorem

Let $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$ be a vector field defined on a region R in the plane whose boundary is a smooth or piecewise smooth simple closed curve C. If $\mathbf{r}(t)$ is a parametrization of C in the counterclockwise direction (as viewed from the positive z-axis), then

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C F_1(x, y)\, dx + F_2(x, y)\, dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)\, dA.$$

Green's Theorem, Curl Expression

Theorem

Let R be a region in the plane to which Green's Theorem applies, with smooth boundary curve C oriented in the counterclockwise direction by $\mathbf{r}(t) = \langle (x(t), y(t) \rangle$, with vector field $\mathbf{F}(x,y) = \langle (F_1(x,y), F_2(x,y) \rangle$ defined on R.

(a) *Green's Theorem, Curl Form:*
A unit vector perpendicular to the xy-plane and thus to the region R in the positive direction is just $\mathbf{n} = \mathbf{k}$. So we can rewrite Green's Theorem as

$$\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \iint_R curl \, \mathbf{F} \cdot \mathbf{k} \, dA.$$

Green's Theorem, Divergence Expression

Theorem

(b) *Green's Theorem, Divergence Form:*
If we restrict our attention to the plane, we see that a unit vector that lies in the xy-plane and is perpendicular to the curve C is given by

$$\mathbf{n} = \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}\mathbf{i} + \frac{-x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}}\mathbf{j}.$$

Then Green's Theorem is equivalent to the statement

$$\int_C \mathbf{F}(x,y) \cdot \mathbf{n}\, ds = \iint_R \mathit{div}\, \mathbf{F}\, dA.$$