The Laplace Transform

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The Laplace Transform

Definition

Let f be an integrable function on $[0, \infty)$. The **Laplace transform** of f, denoted by $\mathcal{L}(f)$ is the improper integral

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

We let the domain of $\mathcal{L}$ to be the set of all functions defined on the interval $[0, \infty)$ for which the improper integral, above, exists.

Properties of the Laplace Transform

Theorem

If V is the domain of the Laplace transform, then the Laplace transform is a linear transformation on V.

Theorem

If $\mathcal{L}(f) = F(s)$, then

1. $\mathcal{L}(e^{at} f(t)) = F(s-a)$.
2. $\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s)$.
3. $F'(s) = -\mathcal{L}(t\,f(t))$.

Exponentially Bounded Functions

Definition

A function $f : [0, \infty) \to \mathbb{R}$ is said to be **exponentially bounded** if there is a positive constant M and a constant a such that $|f(t)| \leq M\, e^{at}$.

Theorem

If f is an exponentially bounded function, then f is in the domain of the Laplace transform.

Inverse Laplace Transforms

Theorem

If f and g are continuous on $[0, \infty)$ with $\mathcal{L}(f(t)) = \mathcal{L}(g(t))$, then $f(t) = g(t)$ for $t \in [0, \infty)$.

We will be using Laplace transforms (and their inverses) to solve initial value problems.