Initial Value Problems and Laplace Transform

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The Laplace Transform of a Derivative

$$\mathcal{L}\left(y'\right) = \int_0^\infty e^{-st}\, y'(t)\, dt = \lim_{b\to\infty} e^{-st} y(t)\Big|_0^b + s\int_0^\infty e^{-st}\, y(t)\, dt$$

$$= -y(0) + s\,\mathcal{L}\left(y\right).$$

Replacing y by y':

$$\mathcal{L}\left(y''\right) = -y'(0) + s\,\mathcal{L}\left(y'\right),$$

which in turn is:

$$\mathcal{L}\left(y''\right) = -y'(0) + s\left(-y(0) + s\,\mathcal{L}\left(y\right)\right) = -y'(0) - s\,y(0) + s^2\mathcal{L}\left(y\right).$$

The Laplace Transform of the nth Derivative

Using mathematical induction, we can show:

$$\mathcal{L}\left(y^{(n)}\right) = -y^{(n-1)}(0) - s\,y^{(n-2)}(0) - \cdots - s^{n-1}y(0) + s^n\mathcal{L}\left(y\right).$$

We can apply this result to each side of a constant coefficient linear differential equation,

$$a_n\,y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_1\,y' + a_0\,y = g(t),$$

to turn the differential equation into an algebra problem:

$$a_n\,\mathcal{L}\left(y^{(n)}\right) + a_{n-1}\,\mathcal{L}\left(y^{(n-1)}\right) + \cdots + a_1\,\mathcal{L}\left(y'\right) + a_0\,\mathcal{L}\left(y\right) = \mathcal{L}\left(g(t)\right).$$