Step Functions, Impulse Functions, and the Delta Function

Department of Mathematics and Statistics

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The Unit Step Function

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Definition

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$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a, \end{cases}$$

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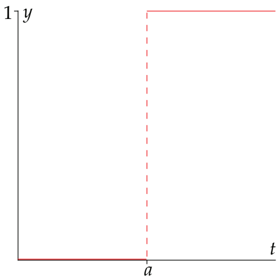

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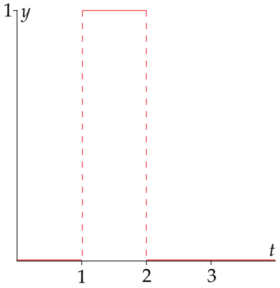

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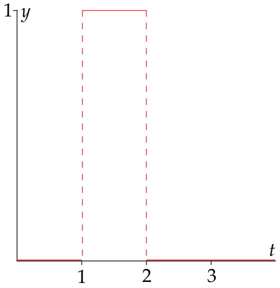

$$\mathcal{L}(u_1(t) - u_2(t)) = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}.$$

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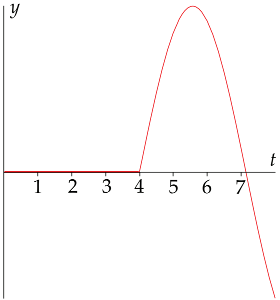

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$$\delta_{a,k}(t) = \begin{cases} k, & a < t < a + \frac{1}{k}, \\ 0, & 0 \leq t \leq a \text{ or } t \geq a + \frac{1}{k}, \end{cases}$$

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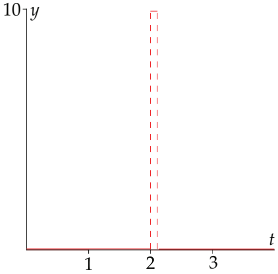

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So $\mathcal{L}(\delta(t)) = 1$ and $\mathcal{L}^{-1}(1) = \delta(t)$.