Step Functions, Impulse Functions, and the Delta Function

Department of Mathematics and Statistics

September 7, 2012

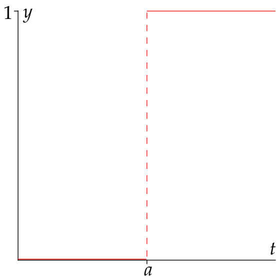
The Unit Step Function

Definition

The **unit step function** or **Heaviside function** is a function of the form

$$u_a(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a, \end{cases}$$

where $a > 0$.



The Laplace Transform of the Unit Step Function

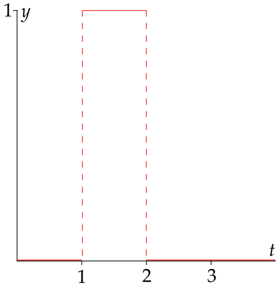
The Laplace transform of u_a is

$$\mathcal{L}(u_a(t)) = \int_0^a e^{-st} \cdot 0 \, dt + \int_a^\infty e^{-st} \, dt = \int_a^\infty e^{-st} \, dt$$

$$= \lim_{b \to \infty} -\frac{1}{s} e^{-st} \Big|_a^b = \frac{1}{s} e^{-as}.$$

The Difference Between Two Unit Step Functions

$$u_1(t) - u_2(t)$$



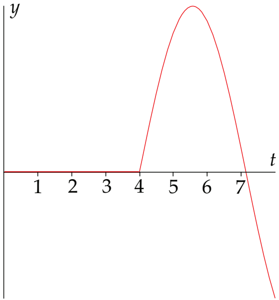
$$\mathcal{L}(u_1(t) - u_2(t)) = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}.$$

The Unit Step Function Combined with Other Functions

Consider

$$g(t) = \begin{cases} 0, & t < a, \\ f(t-a), & t \geq a, \end{cases}$$

We may write $g(t) = u_a(t)f(t-a)$.



$\mathcal{L}(g(t))$

$$\mathcal{L}(g(t)) = \mathcal{L}(u_a(t)f(t-a)) = \int_0^\infty e^{-st} u_a(t) f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt.$$

Substituting $\tau = t - a$:

$$\int_0^\infty e^{-s(\tau+a)} f(\tau)\,d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau = e^{-as}\mathcal{L}(f(t)).$$

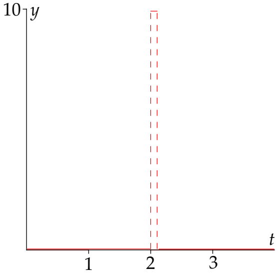
The kth Unit Impulse Function

Definition

The k**th unit impulse function** is

$$\delta_{a,k}(t) = \begin{cases} k, & a < t < a + \frac{1}{k}, \\ 0, & 0 \leq t \leq a \text{ or } t \geq a + \frac{1}{k}, \end{cases}$$

where $a > 0$.



The Laplace Transform of the Unit Impulse Function

Note that $\int_0^\infty \delta_{a,k}(t)\,dt = 1$, and that

$$\mathcal{L}(\delta_{a,k}(t)) = \int_a^{a+\frac{1}{k}} k\,e^{-st}\,dt = -k \cdot \frac{e^{-s\left(a+\frac{1}{k}\right)} - e^{-sa}}{s}$$

$$= e^{-sa} \cdot \frac{1 - e^{-s/k}}{s/k}.$$

By L'Hôpital's rule: $\lim_{k\to\infty} \mathcal{L}(\delta_{a,k}(t)) = e^{-sa}$.

The Dirac Delta Function

Note that $\lim_{k\to\infty} \mathcal{L}(\delta_{0,k}(t)) = 1$.

We let $\lim_{k\to\infty} \mathcal{L}(\delta_{0,k}(t)) = \mathcal{L}(\delta(t))$, where $\delta(t)$ is called the **Dirac delta function** or **delta function**.

So $\mathcal{L}(\delta(t)) = 1$ and $\mathcal{L}^{-1}(1) = \delta(t)$.