Introduction to Power Series Solutions

Department of Mathematics and Statistics

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Taylor Polynomials and Maclaurin Polynomials

Definition

Let f be a function with an nth derivative at every point in an interval I containing the point x_0. The **nth Taylor polynomial for f at x_0** is the function

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

If $x_0 = 0$ we also call this the **nth Maclaurin polynomial for f**. That is,

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The Remainder of a Function

Definition

Let f be a function with an nth order derivative at every point in an open interval I containing the point x_0, and let $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ be the nth Taylor polynomial for f at x_0. We define the **nth remainder for f**, to be

$$R_n(x) = f(x) - T_n(x).$$

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Lagrange's Form for the Remainder

Theorem

Let f be a function which can be differentiated $n+1$ times in some open interval I containing the point x_0 and let $R_n(x)$ be the nth remainder for f at $x = x_0$. For each point $x \in I$ there is at least one c between x_0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

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Power Series

Taylor series and Maclaurin series are examples of power series.

Definition

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers and let x be a variable. A **power series in** $x - x_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

If c is a real number at which the series of constants $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges, the power series is said to converge at c.

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Convergence of a Power Series in $x - x_0$

Theorem

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a power series in $x - x_0$. Exactly one of the following occurs:

1. *The series converges only at $x = x_0$.*
2. *There exists a positive real number ρ such that the series converges absolutely for every $x \in (x_0 - \rho, x_0 + \rho)$ and diverges if $x < x_0 - \rho$ or $x > x_0 + \rho$.*
3. *The series converges absolutely for every real number x.*

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The Interval of Convergence and Radius of Convergence

The three conditions in the previous theorem define both an **interval of convergence** and a **radius of convergence** for the power series in $x - x_0$:

1. The interval of convergence is the single point $x = x_0$ and the radius of convergence is 0.
2. The interval of convergence is one of:

   $$(x_0 - \rho, x_0 + \rho), \quad (x_0 - \rho, x_0 + \rho], \quad [x_0 - \rho, x_0 + \rho) \quad \text{or} \quad [x_0 - \rho, x_0 + \rho]$$

   and the radius of convergence is ρ.
3. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is infinite.

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Analytic Functions

Corollary

If $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ *has a positive radius of convergence then*

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and hence the Taylor series for $f(x)$ *expanded about* x_0 *is* $\sum_{n=0}^{\infty} a_n(x - x_0)^n$*.*

Definition

A function f is said to be **analytic** at a point x_0 if $f(x)$ is equal to a power series on some open interval $|x - x_0| < R$ about x_0.

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