

# Introduction to Power Series Solutions

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# Taylor Polynomials and Maclaurin Polynomials

## Definition

Let  $f$  be a function with an  $n$ th derivative at every point in an interval  $I$  containing the point  $x_0$ . The  $n$ th **Taylor polynomial for  $f$  at  $x_0$**  is the function

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

If  $x_0 = 0$  we also call this the  $n$ th **Maclaurin polynomial for  $f$** . That is,

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k.$$

# The Remainder of a Function

## Definition

Let  $f$  be a function with an  $n$ th order derivative at every point in an open interval  $I$  containing the point  $x_0$ , and let  $T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$  be the  $n$ th Taylor polynomial for  $f$  at  $x_0$ . We define the  **$n$ th remainder for  $f$** , to be

$$R_n(x) = f(x) - T_n(x).$$

# Lagrange's Form for the Remainder

## Theorem

*Let  $f$  be a function which can be differentiated  $n + 1$  times in some open interval  $I$  containing the point  $x_0$  and let  $R_n(x)$  be the  $n$ th remainder for  $f$  at  $x = x_0$ . For each point  $x \in I$  there is at least one  $c$  between  $x_0$  and  $x$  such that*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

# Taylor Series and Maclaurin Series

## Definition

Let  $f$  be a function with derivatives of all orders on an interval  $I$  containing the point  $x_0$ . The **Taylor series for  $f$  at  $x_0$**  is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

If  $x_0 = 0$  we also call this the **Maclaurin series for  $f$** . That is,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

# Power Series

Taylor series and Maclaurin series are examples of power series.

## Definition

Let  $\{a_n\}_{n=0}^{\infty}$  be a sequence of real numbers and let  $x$  be a variable. A **power series in**  $x - x_0$  is a series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

If  $c$  is a real number at which the series of constants  $\sum_{n=0}^{\infty} a_n(c - x_0)^n$  converges, the power series is said to **converge** at  $c$ .

# Convergence of a Power Series in $x - x_0$

## Theorem

Let  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$  be a power series in  $x - x_0$ . Exactly one of the following occurs:

- 1 The series converges only at  $x = x_0$ .
- 2 There exists a positive real number  $\rho$  such that the series converges absolutely for every  $x \in (x_0 - \rho, x_0 + \rho)$  and diverges if  $x < x_0 - \rho$  or  $x > x_0 + \rho$ .
- 3 The series converges absolutely for every real number  $x$ .

# The Interval of Convergence and Radius of Convergence

The three conditions in the previous theorem define both an **interval of convergence** and a **radius of convergence** for the power series in  $x - x_0$ :

- 1 The interval of convergence is the single point  $x = x_0$  and the radius of convergence is 0.
- 2 The interval of convergence is one of:

$$(x_0 - \rho, x_0 + \rho), \quad (x_0 - \rho, x_0 + \rho], \quad [x_0 - \rho, x_0 + \rho) \quad \text{or} \quad [x_0 - \rho, x_0 + \rho]$$

and the radius of convergence is  $\rho$ .

- 3 The interval of convergence is  $(-\infty, \infty)$  and the radius of convergence is infinite.



# Analytic Functions

## Corollary

If  $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$  has a positive radius of convergence then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

and hence the Taylor series for  $f(x)$  expanded about  $x_0$  is  $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ .

## Definition

A function  $f$  is said to be **analytic** at a point  $x_0$  if  $f(x)$  is equal to a power series on some open interval  $|x - x_0| < R$  about  $x_0$ .