Series Solutions for Second Order Equations

Department of Mathematics and Statistics

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Series Solutions for Second Order Differential Equations

We may also use series to find solutions to second (and higher) order differential equations.

We will consider second order differential equations of the form

$$y'' + q(x)y' + r(x)y = g(x); \quad y(x_0) = k_0, \quad y(x_0) = k_1,$$

where the functions q, r, and g are analytic at x_0.

Definition

When q and r are analytic at x_0, then x_0 is said to be an **ordinary point** of the homogeneous differential equation

$$y'' + q(x)y' + r(x)y = 0.$$

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Second Order Linear Differential Equations

Theorem

If x_0 is an ordinary point of the differential equation

$$y'' + q(x)y' + r(x)y = 0,$$

then the solution to the initial value problem

$$y'' + q(x)y' + r(x)y = 0; \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

is given by

$$y = \sum_{k=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are power series in $x - x_0$ and $a_0 = k_0$ and $a_1 = k_1$.

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Radius of Convergence

Theorem

Under the hypotheses and notation of the previous theorem, the radius of convergence of each power series $y_1(x)$ and $y_2(x)$ is at least R, where R is the smaller of the radii of convergence of the Taylor series of q and r about x_0.

Theorem

If x_0 is an ordinary point of the differential equation

$$y'' + q(x)y' + r(x)y = 0$$

and if $y_1(x)$, $y_2(x)$, and R are as in the previous theorems, then $y_1(x)$ and $y_2(x)$ are linearly independent on the interval $|x - x_0| < R$ and the general solution of the differential equation is $a_0 y_1(x) + a_1 y_2(x)$.

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Legendre Equations

Definition

A differential equation of the form

$$\left(1-x^2\right)y'' - 2xy' + v(v+1)y = 0,$$

where v is a constant is called a **Legendre equation**.

Note that $x_0 = 0$ is an ordinary point of the Legendre equation.

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