

Euler Type Equations

Department of Mathematics and Statistics

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Euler Type Equations

Definition

A second order differential equation of the form

$$(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0,$$

where α and β are constants, is called an **Euler type** equation or **equidimensional** equation.

Euler type differential equations of other orders may also be constructed, in the analogous way.

Note that x_0 is *not* an ordinary point of the differential equation.

Solving a Second Order Euler Type Equation

To solve a second order Euler type equation

$$x^2 y'' + \alpha x y' + \beta y = 0,$$

we guess that the solutions are of the form $y = x^r$.

Using this power function in the differential equation, we obtain

$$r(r - 1) + \alpha r + \beta = 0,$$

the **Euler indicial equation**.

The Euler Indicial Equation

The Euler indicial equation has roots:

$$r = \frac{1 - \alpha \pm \sqrt{(1 - \alpha)^2 - 4\beta}}{2}.$$

There are three cases to consider, depending upon the discriminant, $(1 - \alpha)^2 - 4\beta$, of the indicial equation:

- ① $(1 - \alpha)^2 - 4\beta > 0$, there are two distinct real roots.
- ② $(1 - \alpha)^2 - 4\beta = 0$, there is one repeated real root.
- ③ $(1 - \alpha)^2 - 4\beta < 0$, there are two distinct complex roots.

Solutions of Euler Type Equations

- ① When there are two distinct real roots, the solutions are of the form:

$$y = x^{r_1} \text{ and } y = x^{r_2}.$$

- ② When there is a single repeated real root, the solutions are of the form:

$$y = x^r \text{ and } y = x^r \ln x.$$

- ③ When there are two complex roots, the solutions are of the form:

$$y = x^a \cos(b \ln x) \text{ and } y = x^a \sin(b \ln x).$$