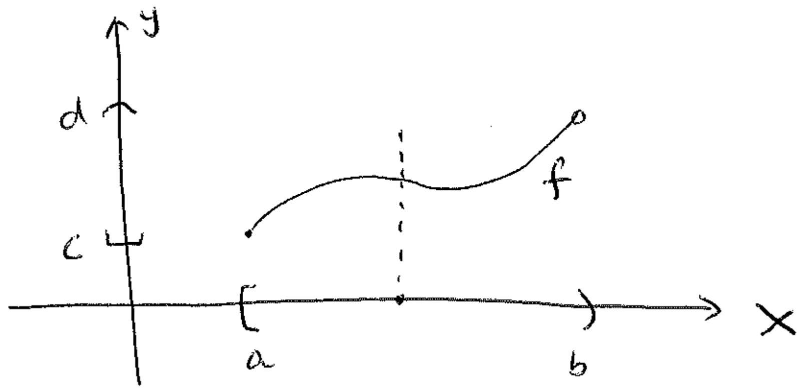
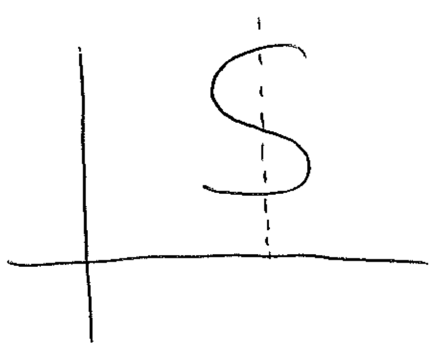
Chapter 1. Review of Functions



If a curve f satisfies "the vertical line test", then it defines a __function__, with __domain__ $[a, b)$ and __range__ $[c, d)$.

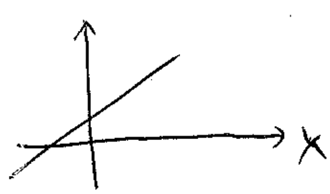
__Notation__: $y = f(x)$, $x \in [a, b)$.



Not a function

"money is not a function".

Ex. __Straight line functions__ $y = mx + b$.

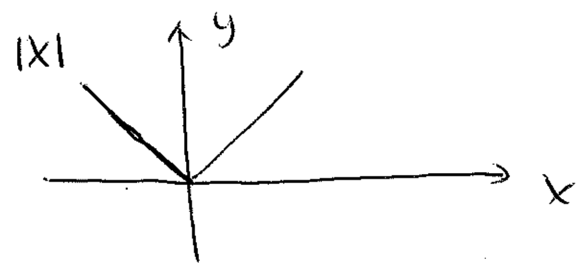


m : slope

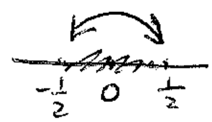
b : y-intercept

Ex. <u>Absolute Value function</u>

$$f(x) = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

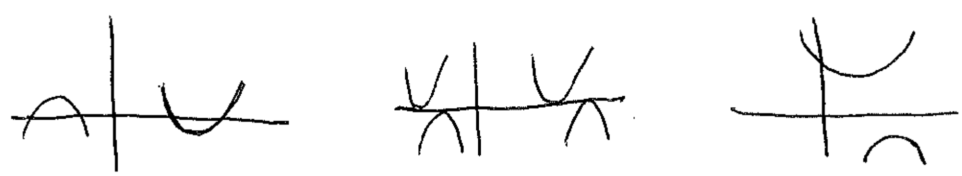


Ex. Solve $|2x - 4| < 1$.

sol: $|x - 2| < \frac{1}{2} \Rightarrow$  $\Rightarrow -\frac{1}{2} < x - 2 < \frac{1}{2}$

$\Rightarrow \frac{3}{2} < x < \frac{5}{2}$.

Ex. <u>Quadratic function</u> $f(x) = ax^2 + bx + c$



<u>Quadratic formula</u>:

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex. Solve $x^2 - 5x + 6 = 0$

sol: $x = \frac{5 \pm \sqrt{25 - 4(1)(6)}}{2} = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2} \begin{array}{l} \nearrow 3 \\ \searrow 2 \end{array}$

or $x^2 - 5x + 6 = (x-2)(x-3) = 0 \Rightarrow x = 2, 3.$

Ex. <u>Polynomial functions</u> $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots$

Ex. <u>Rational functions</u> $\dfrac{\text{Polynomial}}{\text{Polynomial}}$

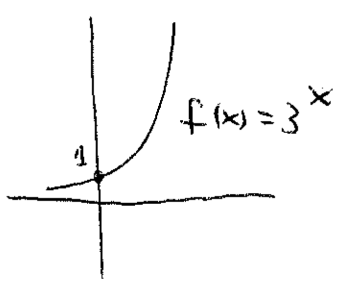
Ex. $\dfrac{x^2+1}{x^2-5x+6}$. Domain: $x \neq 2, 3$.

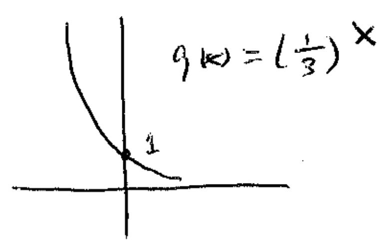
Ex. <u>Power functions</u> $f(x) = x^r$, $r \in R = (-\infty, \infty)$

Ex. $4^{-\frac{3}{2}} = \dfrac{1}{4^{3/2}} = \dfrac{1}{(4^{\frac{1}{2}})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

Ex. <u>Exponential functions</u> $f(x) = b^x$, $b > 0$, $b \neq 1$.

Ex. $f(x) = 3^x$, $g(x) = (\frac{1}{3})^x$





Ex. <u>Logarithmic functions</u> $y = \log_b x,\ x > 0.$

<u>Facts:</u> b^x and $\log_b x$ are inverses of each other:

$$\log_b(b^x) = x,\ x \in \mathbb{R}\ ;\quad b^{\log_b x} = x,\ x > 0.$$

Ex. Solve $2^x = 3$.

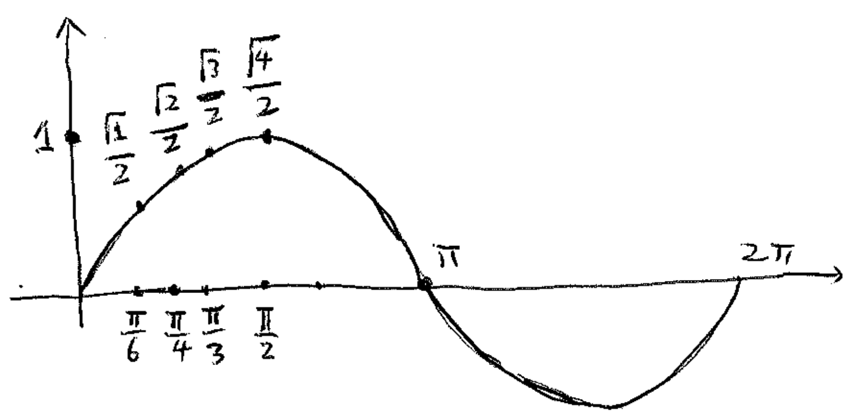
Sol: $\log_2(2^x) = \log_2 3 \Rightarrow x = \log_2 3$.

Ex. Solve $\log_5 x = 2$.

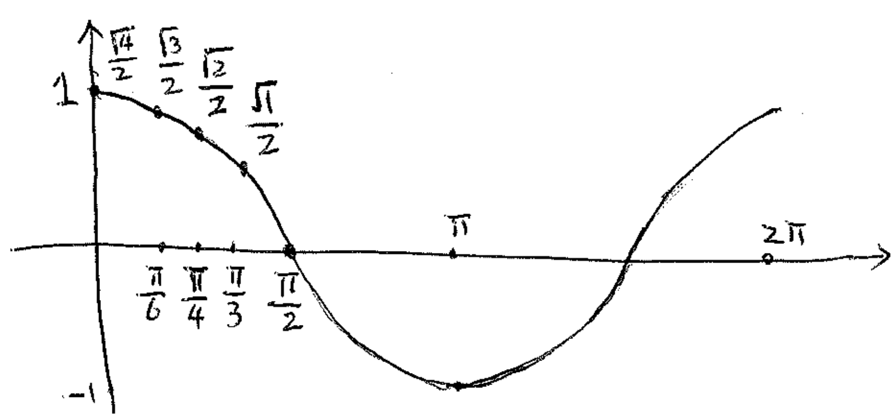
Sol: $5^{\log_5 x} = 5^2 \Rightarrow x = 5^2 = 25$.

Ex. <u>Trigonometric functions</u>

$\sin x$



$\cos x$

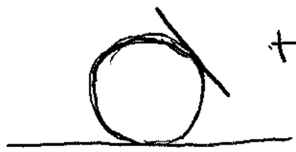


$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}.$$

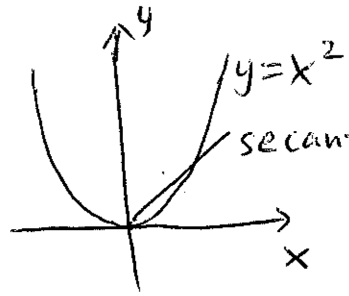
Chapter 2. Limits and Derivatives

§ 2.1. The Tangent and Velocity Problems

Ex. 

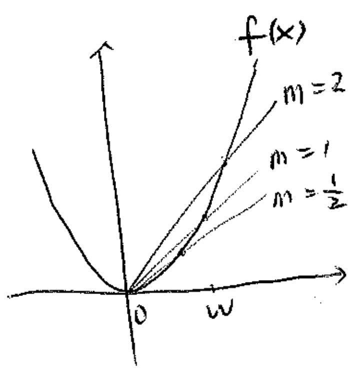
tangent line (touching line)

We will learn how to use tangent lines to approximate complicated curves.

Ex. 

secant line (cutting line)

($y=0$ is a tangent line at $x=0$)

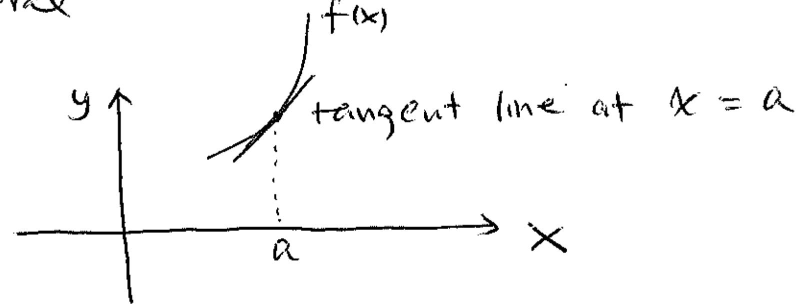
Ex. 

slope of the secant line on $[0,w]$:

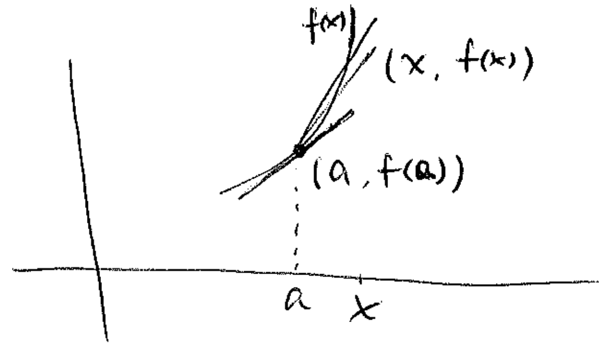
$$\frac{y_2-y_1}{x_2-x_1} = \frac{f(w)-f(0)}{w-0}$$

We see: the slopes of the secant lines approach the slope of the tangent line (which is zero) as w approaches zero.

In general



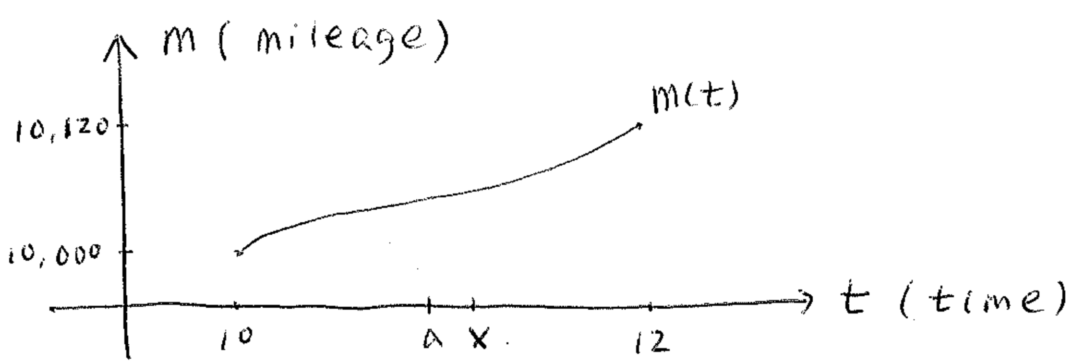
to find the tangent line at $x = a$, we need to find its slope in the following way:



the slopes of the secant lines, $\dfrac{f(x) - f(a)}{x - a}$, approach the slope of the tangent line at $x = a$ as x approaches a.

<u>Notation</u>: $\dfrac{f(x) - f(a)}{x - a} \to$ "slope" as $x \to a$.

Ex. Velocity (speed)



$$\text{Average velocity} = \frac{120}{2} = 60 \text{ m/h} = \frac{m(12) - m(10)}{12 - 10}.$$

On interval $[a, x]$, the average velocity is

$$\frac{m(x) - m(a)}{x - a}.$$

In Physics, the average velocity on $[a, x]$ approaches the "instantaneous velocity" (or "velocity") as x approaches a.

<u>Notation:</u> $\dfrac{m(x) - m(a)}{x - a} \longrightarrow$ "inst. vel." as $x \to a$.

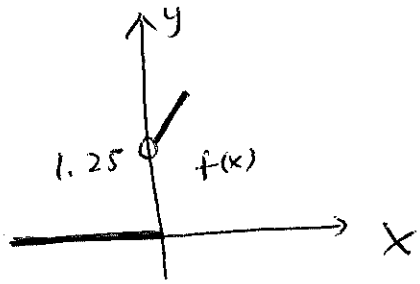
We need to understand these procedures.

§2.2. The Limits

The procedures of something approaching something are called "finding limits".

<u>Definition:</u> Limit $\overset{\text{def.}}{=\!=}$ unique destination.

Ex.
$$f(x) = \begin{cases} 0, & x \le 0 \\ 2.5x + 1.25, & x > 0 \end{cases}$$



We have: $\lim_{x\to 0^+} f(x) = 1.25$; $\lim_{x\to 0^-} f(x) = 0$;

$\lim_{x\to 0} f(x)$ DNE (Does not exist)

($x \to 0$ means $x\to 0$ from both sides)

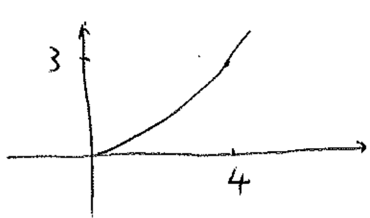
Remark : ① In general, by $x \to a$, we mean x is close to a but $x \neq a$.

② Limits may or may not be on the curve.

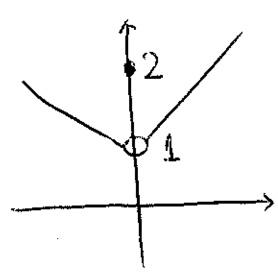
Thm : $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

More examples :

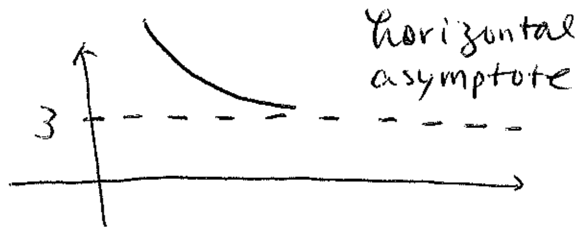
Ex.



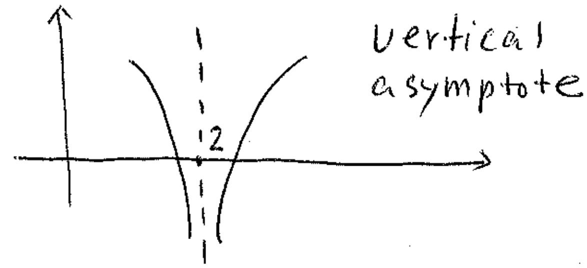
$\lim_{x\to 4} f(x) = 3$



$\lim_{x\to 0} f(x) = 1$.



$\lim_{x\to\infty} f(x) = 3$



$\lim_{x\to 2} f(x) = -\infty$

Ex. Find $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ using a table.

sol:

n	$\left(1+\frac{1}{n}\right)^n$
1	2
2	2.25
10	2.59374
1000	2.71692
1,000,000	2.71828

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\cdots = e$$

(Euler 1707–1783)

<u>Notation:</u> e^x : natural exponential function.

$\ln x = \log_e x$: natural logarithmic function.

$$\begin{cases} \ln(e^x) = x, & x \in \mathbb{R}, \\ e^{\ln x} = x, & x > 0. \end{cases}$$

Ex. Find $\lim_{x\to\pm\infty} \frac{c}{x^k}$, c any constant, $k>0$ a const.

sol: Use a table to get $\lim_{x\to\pm\infty} \frac{c}{x^k} = 0$.

Ex. Find $\lim_{x\to 0^+} \frac{1}{x}$, $\lim_{x\to 0^-} \frac{1}{x}$, $\lim_{x\to 0} \frac{1}{x^2}$,

$\lim_{x\to 1^+} \frac{x+1}{x-1}$, $\lim_{x\to 1^-} \frac{x+1}{x-1}$.

Sol: Use tables to get

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty, \quad \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

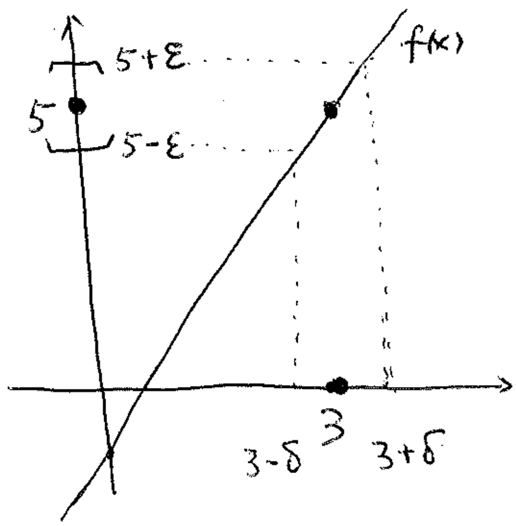
$$\lim_{x\to 1^+} \frac{x+1}{x-1} = \infty, \quad \lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty.$$

Rules ① $\dfrac{\text{the numerator is or approaches a constant}}{\text{the denominator approaches } \pm\infty} \to 0.$

② $\dfrac{\text{the numerator is or approaches a constant}}{\text{the denominator approaches } 0} \to \pm\infty.$

§2.4 The Precise Definition of a Limit

Ex. (page 105) $f(x) = 2x - 1$.



For any $\varepsilon > 0$, we can find $\delta > 0$, such that

if $0 < |x-3| \le \delta$ then $|f(x) - 5| \le \varepsilon$.

<u>Case 1:</u> If $\varepsilon = 0.1$, then

$|f(x) - 5| = |2x - 1 - 5| = |2x - 6| = 2|x-3| \le 2\delta = \varepsilon = 0.1$

if let $\delta = 0.1/2 = 0.05$.

i.e. if $0 < |x-3| \le 0.05$, then $|f(x) - 5| \le 0.1$.

<u>Case 2:</u> $|f(x) - 5| \le 0.01$ if $0 < |x-3| \le 0.005$

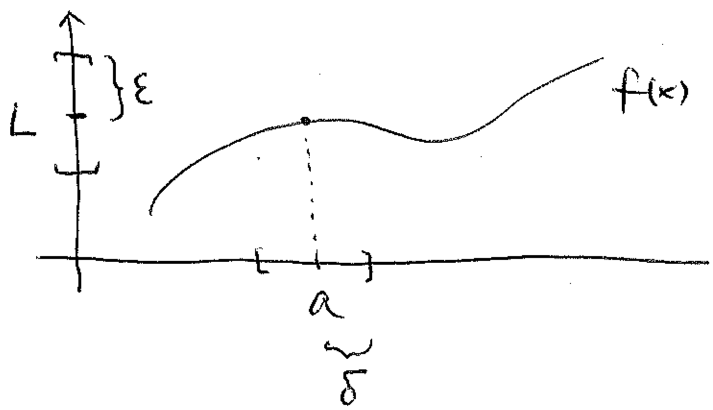
<u>Case 3:</u> $|f(x) - 5| \le 0.001$ if $0 < |x-3| \le 0.0005$.

$\varepsilon - \delta$ definition of a limit (p.106):

$\lim_{x \to a} f(x) = L$ if $\forall$ (for any) $\varepsilon > 0$, $\exists$ (there is)

$\delta > 0$ such that if $0 < |x-a| \leq \delta$, then

$|f(x) - L| \leq \varepsilon$.



Rule 1: Try to get $|f(x) - L| \leq (\text{constant})\,\delta \overset{\text{then set}}{=\!=} \varepsilon$

and let $\delta = \varepsilon / \text{const}$.

Ex. 2 (P.108) Prove ($\varepsilon - \delta$) that $\lim_{x \to 3} (4x-5) = 7$.

pf: $\forall \varepsilon > 0$, let $\delta = ?$ ($\varepsilon/4$) s.t. if $0 < |x-3| \leq \delta$,

then $|f(x) - 7| = |4x-5-7| = |4x-12| = 4|x-3|$

$\leq 4\delta = \varepsilon$. ▨

Ex. 4. (P. 110) Prove that $\lim_{x\to 3} x^2 = 9$.

pf: $\forall \varepsilon > 0$, let $\delta = ?\ (\varepsilon/7)$, s.t. if $0 < |x-3| \le \delta$,

then $|f(x) - 9| = |x^2 - 9| = |x+3||x-3| \le |x+3|\delta$

(assume $2 \le x \le 4$) $\le 7\delta = \varepsilon$. ▨

Ex. 1. (P. 107) Prove that $\lim_{x\to 1} (x^3 - 5x + 6) = 2$.

pf: $\forall \varepsilon > 0$, let $\delta = ?\ (\varepsilon/10)$, s.t. if $0 < |x-1| \le \delta$,

then $|f(x) - 2| = |x^3 - 5x + 6 - 2| = |x^3 - 5x + 4|$

$= |x^3 - x - 4x + 4| = |x(x^2-1) - 4(x-1)|$

$= |x(x+1)(x-1) - 4(x-1)| = |[x(x+1) - 4](x-1)|$

$\le |x(x+1) - 4|\delta \le [|x(x+1)| + 4]\delta$

(assume $0 < x < 2$) $\le (6+4)\delta = 10\delta = \varepsilon$. ▨

Rule 2: Try to get $|f(x) - L| \le$ a function in $\delta \overset{\text{then set}}{=\!=} \varepsilon$

and solve δ in ε.

Ex. 3. (P 110) Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

pf: $\forall \varepsilon > 0$, let $\delta = ?\ (\varepsilon^2)$, s.t. if $0 < |x - 0| \le \delta$,

then $|f(x) - 0| = \sqrt{x} \le \sqrt{\delta} = \varepsilon$. ▨

Thm (p. 111) If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$\lim_{x\to a} [f(x) + g(x)] = L + M.$$

pf: $\forall \varepsilon > 0$, let $\delta = ?$ s.t. if $0 < |x-a| \le \delta$,

then $|[f(x)+g(x)] - [L+M]| = |f(x) - L + g(x) - M|$

$\le |f(x) - L| + |g(x) - M| \le \cdots \le \varepsilon.$

As $\lim_{x\to a} f(x) = L$, for the $\varepsilon > 0$, we get $\varepsilon/2 > 0$,

then $\exists\, \delta_1 > 0$ s.t. if $0 < |x-a| \le \delta_1$, then

$$|f(x) - L| \le \varepsilon/2.$$

Same for $\lim_{x\to a} g(x) = M$: For $\varepsilon/2 > 0$, $\exists\, \delta_2 > 0$,

s.t. if $0 < |x-a| \le \delta_2$ then

$$|g(x) - M| \le \varepsilon/2.$$

Now, let $\delta = \min\{\delta_1, \delta_2\}$ s.t. if $0 < |x-a| \le \delta$,

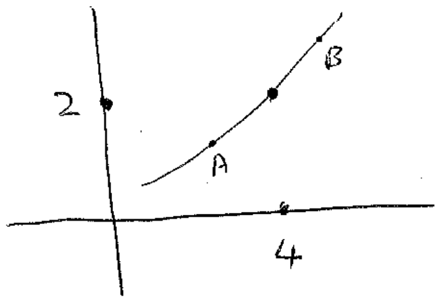
then $|[f(x)+g(x)] - (L+M)| \le |f(x) - L| + |g(x) - M|$

$\le \varepsilon/2 + \varepsilon/2 = \varepsilon.$ ∎

§2.3 and §2.5 Continuity and Limit Laws

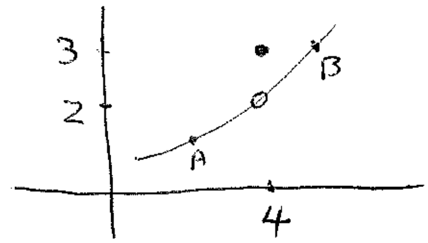
Ex.

(a)



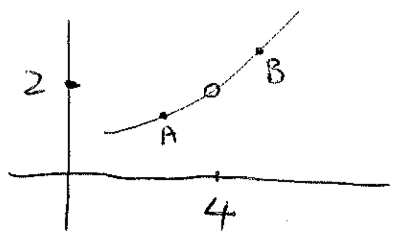
$\lim_{x \to 4} f(x) = 2 = f(4)$

(b)



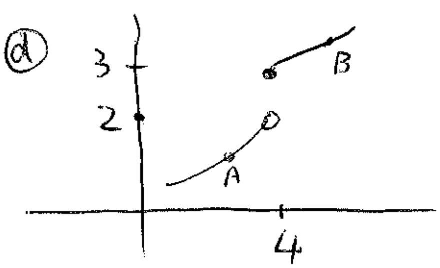
$\lim_{x \to 4} f(x) = 2 \neq f(4)$

(c)



$\lim_{x \to 4} f(x) = 2$, $f(4)$ Not defined

(d)



$\lim_{x \to 4} f(x)$ DNE

From point A to point B, only the first figure is "continuous", and "plug in" works.

<u>Definition</u> (P. 115-116) $f(x)$ is continuous at $x = a$ if

① $\lim_{x \to a} f(x)$ exists.

② $f(a)$ is defined.

③ $\lim_{x \to a} f(x) = f(a)$ (plug in).

case ⓐ : jump discontinuity
case ⓑ and ⓒ : removable discontinuity.

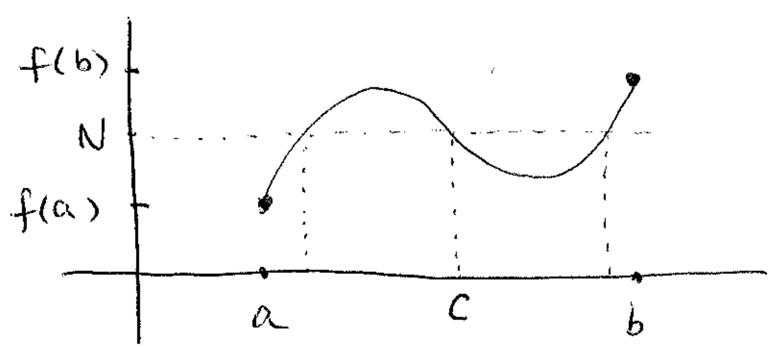
<u>Def.</u> (P. 117) $f(x)$ is continuous on an interval if it is continuous at every point in the interval.

<u>Thm</u> (P. 117) If f and g are continuous, then $f \pm g$, cf (c a constant), fg, $\frac{f}{g}$ ($g(a) \neq 0$), $f(g(x))$ are all continuous.

<u>Thm</u> (P. 118) "Almost all" non-piecewise functions are continuous on their domains.

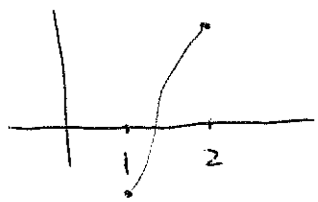
<u>Thm</u> <u>The Intermediate Value Theorem</u>

Let f be continuous on $[a, b]$ with $f(a) \neq f(b)$. If N is any number between $f(a)$ and $f(b)$, then there is at least a number $c \in (a, b)$, s.t. $f(c) = N$. (See Math 410 - 411 for a proof)



Ex. 10 (P.123) Show that $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$ has a root in $(1, 2)$.

Pf: $f(1) = -1 < 0$, $f(2) = 12 > 0$. Use Int. Val. Thm with $N = 0$, $\Rightarrow f(c) = N = 0$ for some $c \in (1, 2)$. ▨



Limit Laws (Finding limits using algebra)

If $f(x)$ and $g(x)$ are continuous at $x = a$, then

① $\lim_{x \to a} [f(x) \pm g(x)] = f(a) \pm g(a)$.

② $\lim_{x \to a} [c f(x)] = c f(a)$ (c a constant).

③ $\lim_{x \to a} [f(x) g(x)] = f(a) g(a)$.

④ $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$ if $g(a) \neq 0$.

⑤ $\lim_{x \to a} [f(x)]^r = [f(a)]^r$ for any constant r that results in a real number.

Ex. 2 (p 97)

ⓐ $\lim_{x \to 5} (2x^2 - 3x + 4) = 2(5^2) - 3(5) + 4 = 39$.

(b) $\lim\limits_{x\to -2} \dfrac{x^3+2x^2-1}{5-3x} = \dfrac{(-2)^3+2(-2)^2-1}{5-3(-2)} = -\dfrac{1}{11}$.

Ex. 3 (P. 98)

$\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x+1)(x-1)}{x-1} \overset{x-1\neq 0}{=\!=} \lim\limits_{x\to 1} (x+1) = 2$.

Ex. 5. (P. 99)

$\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h} = \lim\limits_{h\to 0} \dfrac{h^2+6h+9-9}{h} = \lim\limits_{h\to 0} \dfrac{h^2+6h}{h}$

$\overset{h\neq 0}{=\!=} \lim\limits_{h\to 0} (h+6) = 6$.

Ex. $\lim\limits_{x\to 0} \dfrac{3x^5+7x^2}{4x^6+3x^2} \overset{x\neq 0}{=\!=} \lim\limits_{x\to 0} \dfrac{3x^3+7}{4x^4+3} = \dfrac{7}{3}$.

Ex. $\lim\limits_{x\to 1} \dfrac{x^2+4x-5}{x^2-3x+2} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+5)}{(x-1)(x-2)} \overset{x-1\neq 0}{=\!=}$

$\lim\limits_{x\to 1} \dfrac{x+5}{x-2} = -6$.

Ex. 6 (P. 99) (use $(a-b)(a+b) = a^2-b^2$)

$\lim\limits_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2} = \lim\limits_{t\to 0} \dfrac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)}$

$$= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{t^2}{t^2(\sqrt{t^2+9}+3)}$$

$$\overset{t\neq 0}{=} \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{6}.$$

For piecewise functions, use left and right limits.

Ex. 7 (P.100) Find $\lim_{x\to 0} |x|$.

Sol: $|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$ Then

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$; $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0.$ $\Rightarrow$

$$\lim_{x\to 0} |x| = 0.$$

Ex. 8 (P.100) Find $\lim_{x\to 0} \frac{|x|}{x}$.

Sol: $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$; $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$

$\Rightarrow$ $\lim_{x\to 0} \frac{|x|}{x}$ DNE.

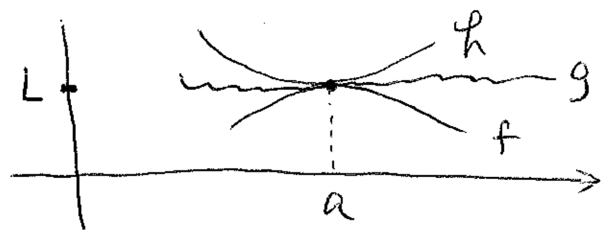
Ex. 9. (P.100) Find $\lim_{x\to 4} f(x)$, $f(x) = \begin{cases} \sqrt{x-4}, & x > 4, \\ 8-2x, & x < 4. \end{cases}$

Sol: $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = 0$; $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x) = 0.$

$\Rightarrow$ $\lim_{x\to 4} f(x) = 0.$

The Squeeze Thm (P. 101). If $f(x) \le g(x) \le h(x)$ and

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.



Ex. 11. (P. 101) Find $\lim_{x \to 0} x^2 \sin\frac{1}{x}$.

Sol: $-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -x^2 \le x^2 \sin\frac{1}{x} \le x^2$

and $\lim_{x \to 0} \pm x^2 = 0 \Rightarrow \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

§ 2.7 — 2.8 Derivatives

From § 2.1,

$$\frac{f(x) - f(a)}{x - a} \to \text{slope},$$

$$\frac{m(x) - m(a)}{x - a} \to \text{inst. velocity}.$$

Next, look at this special type of limits.

Definition (P. 144) The derivative of f at $x=a$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \overset{x=a+h}{=\!=\!=} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Applications:

Derivative
- slope of the tangent line
- (inst.) velocity, or rate of change.

Ex. 4. (P. 144) Find $f'(a)$ for $f(x) = x^2 - 8x + 9$.

Sol:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{(x^2 - 8x + 9) - (a^2 - 8a + 9)}{x - a}$$

$$= \lim_{x \to a} \frac{x^2 - a^2 - 8(x-a)}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a) - 8(x-a)}{x - a}$$

$$= \lim_{x \to a} \left[(x+a) - 8 \right] = 2a - 8. \quad \left(\text{see book for } \frac{f(a+h) - f(a)}{h} \right)$$

Ex. 5 (P. 145) Find the tangent line to $f(x) = x^2 - 8x + 9$ at $x = 3$.

Sol: Point $= (3, f(3)) = (3, -6)$.

slope $= f'(3) \overset{\text{Ex 4}}{\underset{a=3}{=\!=\!=}} 2(3) - 8 = -2$.

Tangent line: $y = -6 - 2(x-3)$. $(y = y_0 + m(x - x_0))$

Derivative function $f'(x)$: Change a to x.

Def (P. 152)

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{d}{dx} f(x).$$

Ex. 2. (P. 154) Find $f'(x)$ for $f(x) = x^3 - x$.

Sol:
$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x} = \lim_{w \to x} \frac{(w^3 - w) - (x^3 - x)}{w - x}$$

$$= \lim_{w \to x} \frac{w^3 - x^3 - (w - x)}{w - x} = \lim_{w \to x} \frac{(w-x)(w^2 + wx + x^2) - (w - x)}{w - x}$$

$$= \lim_{w \to x} (w^2 + wx + x^2 - 1) = x^2 + x^2 + x^2 - 1 = 3x^2 - 1.$$

Ex. 3 (P. 154) Find $f'(x)$ and domain for $f(x) = \sqrt{x}$, $x \geq 0$.

Sol:
$$f'(x) = \lim_{w \to x} \frac{\sqrt{w} - \sqrt{x}}{w - x} = \lim_{w \to x} \frac{(\sqrt{w} - \sqrt{x})(\sqrt{w} + \sqrt{x})}{(w - x)(\sqrt{w} + \sqrt{x})}$$

$$= \lim_{w \to x} \frac{w - x}{(w - x)(\sqrt{w} + \sqrt{x})} = \lim_{w \to x} \frac{1}{\sqrt{w} + \sqrt{x}} = \frac{1}{2\sqrt{x}},$$

domain of $f'(x)$: $x > 0$.

Ex. 4. (P. 155) Find $f'(x)$ for $f(x) = \frac{1 - x}{2 + x}$.

Sol: $f'(x) = \lim\limits_{w \to x} \dfrac{f(w) - f(x)}{w - x} = \lim\limits_{w \to x} \dfrac{\dfrac{1-w}{2+w} - \dfrac{1-x}{2+x}}{w - x}$

$= \lim\limits_{w \to x} \dfrac{(1-w)(2+x) - (1-x)(2+w)}{(2+w)(2+x)} \cdot \dfrac{1}{w-x}$

$= \lim\limits_{w \to x} \dfrac{2 + x - 2w - wx - 2 - w + 2x + xw}{(2+w)(2+x)} \cdot \dfrac{1}{w-x}$

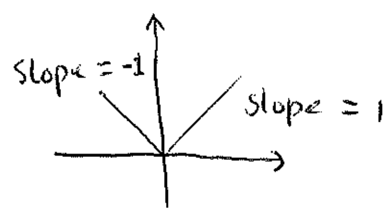
$= \lim\limits_{w \to x} \dfrac{-3w + 3x}{(2+w)(2+x)} \cdot \dfrac{1}{w-x} = \lim\limits_{w \to x} \dfrac{-3(w-x)}{(2+w)(2+x)(w-x)}$

$= \lim\limits_{w \to x} \dfrac{-3}{(2+w)(2+x)} = -\dfrac{3}{(2+x)^2}$.

Ex. 5 (P. 156) $f(x) = |x|$ has No derivative at 0.

Sol: $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{|h|}{h}$ DNE.

(P. 100, Ex. 8).



<u>Thm</u> (P. 156) If $f(x)$ is differentiable (has derivative) at a, then f is continuous at a.

Pf: $\lim\limits_{x \to a} [f(x) - f(a)] = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} \cdot (x - a) = [f'(a)][0] = 0$

$\Rightarrow \lim\limits_{x \to a} f(x) = f(a)$. ∎

Differentiable $\underset{\not\Leftarrow}{\Rightarrow}$ continuous

<u>Higher derivatives</u>:

$$f''(x) = [f'(x)]', \quad f'''(x) = [f''(x)]', \quad \ldots$$

<u>Notation</u>: $f'(x) = \frac{d}{dx} f(x)$, $f''(x) = \frac{d^2}{dx^2} f(x)$, $f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$.

<u>Application</u>: $f'(x) = $ velocity

$f''(x) = $ acceleration

$f'''(x) = $ jerk .

Chapter 3. Derivative Rules

§3.1. Derivatives of Polynomials and Exponential Functions

<u>Thm</u> (Power Rule) (p. 173. p. 221) For any $n \in R$,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

"pf" (See §3.6 for a full proof) Here, $n \geq 0$ is an integer.

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x} = \lim_{w \to x} \frac{w^n - x^n}{w - x}$$

$$= \lim_{w \to x} \frac{(w-x)(w^{n-1} + w^{n-2}x + \cdots + wx^{n-1} + x^{n-1})}{w - x}$$

$$= \lim_{w \to x} (w^{n-1} + w^{n-1}x + \cdots + wx^{n-1} + x^{n-1})$$

$$= nx^{n-1}. \qquad \blacksquare$$

Ex. 2. (P. 174)

(a) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx} x^{-2} = -2x^{-3}$.

(b) $\frac{d}{dx} \sqrt[3]{x^2} = \frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-\frac{1}{3}}$.

Thm (P. 175 - 176)

① $\frac{d}{dx}[cf(x)] = c \frac{d}{dx} f(x)$, c a constant.

② $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$.

Ex. 6 (P. 177) For $f(x) = x^4 - 6x^2 + 4$, find where the tangent line is horizontal.

Sol: $f'(x) = 4x^3 - 12x$. set $f'(x) = 0 \Rightarrow$

$4x^3 - 12x = 0 = 4x(x^2 - 3) \Rightarrow x = 0, \pm\sqrt{3}$.

Ex. 7. (p. 177) The motion of a particle is

$m(t) = 2t^3 - 5t^2 + 3t + 4$ (cm, centimeter).

Find the acceleration after 2 seconds.

Sol: $V(t) = \frac{d}{dt} m(t) = 6t^2 - 10t + 3$

$a(t) = \frac{d^2}{dt^2} m(t) = \frac{d}{dt}(6t^2 - 10t + 3) = 12t - 10.$

$\Rightarrow \quad a(2) = 12(2) - 10 = 14 \quad cm/s^2.$

Thm (p. 177) $\frac{d}{dx} e^x = e^x.$

Pf: $\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$

$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$ because $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$;

h	$\frac{e^h - 1}{h}$
0.1	1.0517
0.001	1.0005
⋮	
−0.1	0.9516
−0.001	0.9995

Ex. 9 (p. 179) At what point on $f(x) = e^x$ is the tangent line parallel to $y = 2x$?

sol: $f'(x) = (e^x)' = e^x$. Parallel $\Rightarrow$ $f'(x) = 2$

$\Rightarrow e^x = 2 \Rightarrow \ln e^x = \ln 2 \Rightarrow x = \ln 2$.

$\Rightarrow$ point $= (\ln 2, e^{\ln 2}) = (\ln 2, 2)$.

§3.2 The Product and Quotient Rules

Thm (p. 184 - 186)

① $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$ (Product)

② $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ (Quotient)

Pf ②:

$$\frac{\frac{f(w)}{g(w)} - \frac{f(x)}{g(x)}}{w - x} = \frac{f(w)g(x) - f(x)g(w)}{g(w)g(x)} \frac{1}{w-x}$$

$$= \frac{1}{g(w)g(x)} \left[\frac{f(w)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(w)}{w - x} \right]$$

$$= \frac{1}{g(w)g(x)} \left[\frac{f(w) - f(x)}{w - x} g(x) - f(x) \frac{g(w) - g(x)}{w - x} \right]$$

$$\longrightarrow \frac{f'g - fg'}{g^2}$$ ▨

Ex. 1. (P. 184)

$$\frac{d}{dx}(xe^x) = x' e^x + x(e^x)' = e^x + xe^x.$$

Ex. 4. (P. 186)

$$\frac{d}{dx}\frac{x^2+x-2}{x^3+6} = \frac{(x^2+x-2)'(x^3+6)-(x^2+x-2)(x^3+6)'}{(x^3+6)^2}$$

$$= \frac{(2x+1)(x^3+6)-(x^2+x-2)(3x^2)}{(x^3+6)^2}$$

Ex. 5. (P. 187) Find the tangent line to $f(x) = \frac{e^x}{1+x^2}$ at $x=1$.

Sol: point $= (1, f(1)) = (1, \frac{e}{2})$.

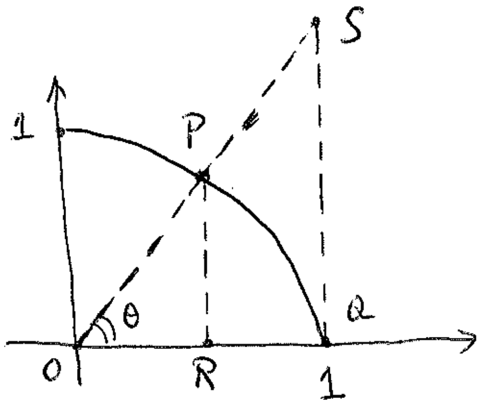
$$f'(x) = \frac{e^x(1+x^2)-e^x(2x)}{(1+x^2)^2}, \quad m = f'(1) = 0$$

$$\Rightarrow \quad y = \frac{e}{2} + 0(x-1) = \frac{e}{2}.$$

§ 3.3. Derivative of Trigonometric Functions

Ex. (P. 191) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Sol.



$P = (\cos\theta, \sin\theta)$, $Q = (1, 0)$

$R = (\cos\theta, 0)$, $S = (1, \tan\theta)$.

area of $\angle OPR \le$ area of sector $OPQ \le$ area of $\angle OSQ$

$$\frac{1}{2}(\cos\theta)(\sin\theta) \le \frac{1}{2}\theta(1^2) \le \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\overset{\theta>0}{\Longrightarrow} \quad \frac{\cos\theta}{\sin\theta} \le \frac{1}{\theta} \le \frac{1}{\cos\theta\sin\theta}$$

$$\Rightarrow \quad \cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}.$$

As $\lim_{\theta\to 0}\cos\theta = 1 \Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. (Squeeze Thm)

Ex. 5. (P. 195)

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \lim_{x\to 0}\frac{\sin 7x}{7x}\frac{7}{4} = \frac{7}{4}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{4}.$$

Ex. (P. 192)

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}\frac{\cos\theta + 1}{\cos\theta + 1} = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

$$= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\cos\theta + 1} = 1\left(\frac{0}{2}\right) = 0.$$

Thm (p. 191 – 192) $\frac{d}{dx}\sin x = \cos x$.

Pf: $\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cosh + \cos x\sin h - \sin x}{h}$

$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$

$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} = 0 + \cos x.$ ∎

Thm (p. 193) $\frac{d}{dx}\cos x = -\sin x$.

Thm (p. 193) $\frac{d}{dx}\tan x = \sec^2 x$.

Pf: $\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$ ∎

Derivatives of Trigonometric Functions (p. 193)

$\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\frac{d}{dx}\tan x = \sec^2 x$,

$\frac{d}{dx}\csc x = -\csc x\cot x$, $\frac{d}{dx}\sec x = \sec x\tan x$,

$\frac{d}{dx}\cot x = -\csc^2 x$.

Ex. 1. (P. 193)

$$\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x.$$

Ex. 2. (P. 194)

$$\frac{d}{dx}\frac{\sec x}{1+\tan x} = \frac{\sec x \tan x(1+\tan x) - \sec x \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.$$

Ex. 4. (P. 195) Find $\frac{d^{27}}{dx^{27}} \cos x$.

Sol: $\frac{d}{dx}\cos x = -\sin x$, $\frac{d^2}{dx^2}\cos x = -\cos x$,

$\frac{d^3}{dx^3}\cos x = \sin x$, $\frac{d^4}{dx^4}\cos x = \cos x$

$$\Longrightarrow \frac{d^{27}}{dx^{27}}\cos x = \frac{d^3}{dx^3}\frac{d^{24}}{dx^{24}}\cos x = \frac{d^3}{dx^3}\cos x = \sin x.$$

§3.4. The Chain Rule

Ex. $\frac{d}{dx}\sqrt{x^2+1} = ?$

<u>Note</u>: If let $y = f(u) = \sqrt{u}$, $u = g(x) = x^2+1$,

$\Rightarrow f(g(x)) = \sqrt{x^2+1}$.

In general, for $y = f(u)$, $u = g(x)$, and $f(g(x))$,

$$\frac{d}{dx} f(g(x)) = ?$$

<u>Idea</u>: Write $y(x) = f(g(x))$. then

$$\frac{d}{dx} f(g(x)) = \frac{dy}{dx} = \frac{dy}{*}\,\frac{*}{dx} = \frac{dy}{du}\,\frac{du}{dx}.$$

Ex. x = your money. Put x in bank A (return rate 200%) $\Rightarrow u = 2x$. Then, bank B offers return rate 300%. Take your money from bank A and put into bank B (composition). Then,

$$\text{total return } y = 3u = 3(2x) = 6x.$$

$$\text{i.e. } \frac{dy}{dx} = 6 = 3 \cdot 2 = \frac{dy}{du}\,\frac{du}{dx}.$$

$$\begin{tikzcd}[column sep=huge] y & u \arrow[l, "y=f(u)"'] \arrow[l, bend left, "\frac{dy}{du}\ \checkmark"] & x \arrow[l, "u=g(x)"'] \arrow[l, bend left, "\frac{du}{dx}\ \checkmark"] \arrow[ll, bend right, "\frac{dy}{dx}"'] \end{tikzcd}$$

Thm (Chain Rule) (P.198) For $y = f(u)$ and $u = g(x)$,

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x),$$

i.e. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Thm (General Power Rule) (P.200)

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

pf: Let $y = u^n$, $u = g(x)$. $\blacksquare$

Ex. 1. (P.199)

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{\frac{1}{2}} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

Ex. 2. (P.200)

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)(2x)$$

$$\frac{d}{dx}(\sin x)^2 = 2\sin x\cos x.$$

Ex. 3. (P 201)

$$\frac{d}{dx}(x^3-1)^{100} = 100(x^3-1)^{99} \cdot 3x^2.$$

Ex. 5. (p. 201)

$$\frac{d}{dx}\left(\frac{x-2}{2x+1}\right)^9 = 9\left(\frac{x-2}{2x+1}\right)^8 \frac{(2x+1)-(x-2)\cdot 2}{(2x+1)^2}$$

Ex. 6. (p. 201)

$$\frac{d}{dx}(2x+1)^5(x^3-x+1)^4 = 5(2x+1)^4(2)(x^3-x+1)^4$$
$$+ (2x+1)^5\, 4(x^3-x+1)^3(3x^2-1)$$

$\underline{\text{Thm}}$: $\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$.

pf: Let $y = e^u$. $u = g(x)$. ▨

Ex. 7. (p. 202)

$$\frac{d}{dx}e^{\sin x} = e^{\sin x}\cos x.$$

Ex. 8. (p. 202)

$$\frac{d}{dx}\sin(\cos(\tan x)) = \cos(\cos(\tan x))\frac{d}{dx}\cos(\tan x)$$
$$= \cos(\cos(\tan x))\left[-\sin(\tan x)\right]\frac{d}{dx}\tan x$$
$$= \cos(\cos(\tan x))\left[-\sin(\tan x)\right]\sec^2 x.$$

§ 3.5. Implicit Differentiation

Sometimes, we know that y is a function of x, but the explicit formula of $y = f(x)$ is hard to get. In this case, we can still find $\frac{d}{dx}y$ (in terms of x and y, thus implicit) be the chain rule.

Ex. 2. (P 210) Find the tangent line to $x^3 + y^3 = 6xy$ at $(3, 3)$.

Sol: Assume $y = f(x)$ and $\frac{d}{dx}$:

$$3x^2 + 3y^2 y' = 6y + 6xy' \Rightarrow (y^2 - 2x)y' = 2y - x^2$$

$$\Rightarrow \quad y' = \frac{2y - x^2}{y^2 - 2x}$$ (also in y, thus implicit).

At $(3, 3)$, $y' = \frac{6-9}{9-6} = -1$.

$\Rightarrow$ tangent line: $y = 3 - (x - 3)$.

Ex. 3. (P. 211) Find y' if $\sin(x+y) = y^2 \cos x$.

Sol: $\frac{d}{dx}$: $\cos(x+y)(1+y') = 2yy' \cos x - y^2 \sin x$

$$\Rightarrow \quad y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}.$$

Ex. (P. 213) Find $\frac{d}{dx}\sin^{-1}x$.

Sol: Let $y=\sin^{-1}x \Rightarrow \sin y = x$.

$\frac{d}{dx}$: $\cos y\cdot y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$.

<u>Derivatives of Inverse Trigonometric Functions</u>

$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$,

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$, $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$.

Ex. 5 (P. 214)

(a) $\frac{d}{dx}\frac{1}{\sin^{-1}x} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}\sin^{-1}x$

$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$.

(b) $\frac{d}{dx}x\arctan\sqrt{x} = \arctan\sqrt{x} + x\frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \arctan\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$.

§3.6. Derivatives of Logarithmic Functions

Thm (p. 218) $\frac{d}{dx} \ln x = \frac{1}{x}$, $(x > 0)$.

pf: $e^{\ln x} = x$. $\frac{d}{dx}$: $e^{\ln x} \frac{d}{dx} \ln x = 1 \Rightarrow$

$x \frac{d}{dx} \ln x = 1 \Rightarrow \frac{d}{dx} \ln x = 1/x$. $\blacksquare$

Thm (p. 219) $\frac{d}{dx} \ln (g(x)) = \frac{g'(x)}{g(x)}$.

pf: Let $y = \ln u$, $u = g(x)$. $\blacksquare$

Ex. 2. (p. 219)

$$\frac{d}{dx} \ln (\sin x) = \frac{\cos x}{\sin x} = \cot x.$$

Ex. 3 (p. 219)

$$\frac{d}{dx} \sqrt{\ln x} = \frac{1}{2} (\ln x)^{-\frac{1}{2}} \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}.$$

Ex. 6. (p. 220) Find $\frac{d}{dx} \ln |x|$. $x \neq 0$.

Sol: For $x > 0$, $\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln x = \frac{1}{x}$.

For $x < 0$, $\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln (-x) = \frac{-1}{-x} = \frac{1}{x}$.

$\Rightarrow \frac{d}{dx} \ln |x| = \frac{1}{x}$, $x \neq 0$.

Ex. 8 (p. 222) Find $\frac{d}{dx} x^{\sqrt{x}}$. (Logarithmic differentiation)

Sol: Let $y = x^{\sqrt{x}} \Rightarrow \ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$.

$\frac{d}{dx}$: $\frac{y'}{y} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x} \Rightarrow$

$y' = y\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right)$.

<u>Thm</u> (Power Rule) (p. 221). For $n \in \mathbb{R}$,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

pf: Let $y = x^n$. $\Rightarrow \ln|y| = n\ln|x|$.

$\frac{d}{dx}$: $\frac{y'}{y} = \frac{n}{x} \Rightarrow y' = y\frac{n}{x} = x^n\frac{n}{x} = nx^{n-1}$. ▨

§3.8. Exponential Growth and Decay

Around 1800: The rate of change of some quantity $P(t)$ is proportional to $P(t)$ with some proportional constant k.

$\Rightarrow P'(t) = kP(t)$ (Differential Equations)

<u>Thm</u> (p. 237) $y(t) = y(0)e^{kt}$ are the only solutions to $y'(t) = ky(t)$. ($y(0)$ = initial value)

pf: First, $y'(t) = y(0)e^{kt}\cdot k = k[y(0)e^{kt}] = ky(t)$

$\Rightarrow y(t) = y(0)e^{kt}$ are solutions.

Next, if $p(t)$ is a solution to $y'(t) = ky(t)$

$\Rightarrow p'(t) - kp(t) = 0 \Rightarrow p'(t)e^{-kt} - kp(t)e^{-kt} = 0$

$\Rightarrow \frac{d}{dt}[p(t)e^{-kt}] = 0 \Rightarrow p(t)e^{-kt} = c$ (a constant)

$\Rightarrow p(t) = ce^{kt} = p(0)e^{kt}$. ▨

Ex 1. (P. 238) The world population was 2560 million in 1950 and 3040 million in 1960. (Assume that the growth rate is proportional to the population.) Estimate the world population in 1993.

Sol: 1950: set $t=0$. $\Rightarrow P(0) = 2560$ and

$P(10) = 3040$. $\Rightarrow P(t) = P(0)e^{kt} = 2560e^{kt}$.

Need to find k: $P(10) = 2560e^{k(10)} = 3040$.

$\Rightarrow e^{10k} = \dfrac{3040}{2560} \Rightarrow k = \frac{1}{10}\ln\frac{304}{256} \approx 0.017185$

$\Rightarrow P(t) = 2560e^{0.017185t}$

1993: $P(43) = 2560e^{0.017185(43)} \approx 5360$ (million)

(See book: quite reliable)

$P'(t) = kP(t)$ can also be applied to radioactive decay. Half-life t_H : $P(t_H) = \frac{1}{2}P(0)$.

Ex. 2. (P239) The half-life of radium-226 is 1590 years. A sample of radium-226 has a mass of 100 mg. Find the mass after 1000 years. When will the mass be reduced to 30 mg?

Sol : $P(t) = P(0)e^{kt} = 100e^{kt}$, $P(1590) = \frac{1}{2}P(0) = 50$.

$$\Rightarrow 100e^{k(1590)} = 50 \Rightarrow k = -\frac{\ln 2}{1590}$$

$$\Rightarrow P(1000) = 100e^{k(1000)} \approx 65 \text{ mg}.$$

Next, find t such that $P(t) = 30 \Rightarrow$

$$100e^{kt} = 30 \Rightarrow e^{kt} = 0.3$$

$$\Rightarrow t = \frac{\ln 0.3}{k} \approx 2762 \text{ years}.$$

Newton's Law of Cooling (P. 240)

$$T'(t) = k(T - T_s)$$

$T(t)$: temperature of an object at time t,

k : a constant,

T_s : surrounding constant temperature.

Ex.3 (p240) A soda of temperature 72°F is placed in a refrigerator of temperature 44°F. After 30 minutes the soda cooled to 61°F. What is the temperature of the soda after one hour? How long does it take for the soda to cool to 50°F?

Sol: $T'(t) = k[T(t) - 44] \Rightarrow [T(t)-44]' = k[T(t)-44]$

$(y = T - 44 \Rightarrow y' = ky \Rightarrow y(t) = y(0)e^{kt})$

$\Rightarrow T(t) - 44 = (T(0) - 44)e^{kt} = 28e^{kt}$.

We know $T(30) = 61 \Rightarrow 61 - 44 = 28e^{k(30)}$

$\Rightarrow e^{30k} = \frac{17}{28} \Rightarrow k = \frac{1}{30}\ln\frac{17}{28} \approx -0.01663$

$\Rightarrow T(t) = 44 + 28e^{-0.01663t}$.

After one hour: $T(60) = 44 + 28e^{-0.01663(60)} \approx 54$ °F.

Next find t such that $T(t) = 50$:

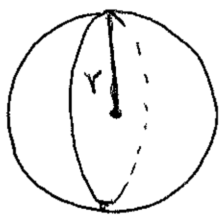
$44 + 28e^{-0.01663t} = 50 \Rightarrow t \approx \frac{\ln\frac{6}{28}}{-0.01663} \approx 93$ min.

§ 3.9. Related Rates

Sometimes, two rates are related. So, if we know one rate, we can find another rate.

Ex 1. (P. 245) Air is being pumped into a spherical balloon such that its volume increases at a rate of $100\ cm^3/s$. How fact is the radius of the balloon increasing when the diameter is 50 cm ?

Sol:



$$V = \frac{4}{3}\pi r^3 \Rightarrow V(t) = \frac{4}{3}\pi [r(t)]^3.$$

$$\Rightarrow V'(t) = \frac{4}{3}\pi 3 r^2(t) r'(t).$$

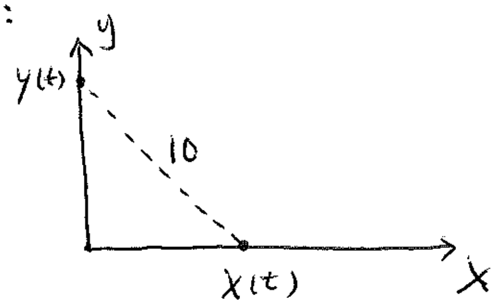
we know $V'(t) = 100\ cm^3/s \Rightarrow$

$$100 = 4\pi r^2(t) r'(t) \Rightarrow$$

$$r'(t) = \frac{100}{4\pi r^2(t)} = \frac{25}{\pi (25)^2} = \frac{1}{25\pi}\ cm/s.$$

Ex. 2 (P. 246) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away at a rate of 1 ft/s, how fast is the top sliding down when the bottom of the ladder is 6 ft from the wall ?

Sol:



$$x^2 + y^2 = 10^2 = 100$$

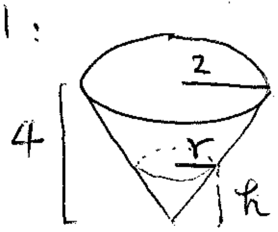
$$\frac{d}{dt}: \ 2xx' + 2yy' = 0$$

$$\Rightarrow \quad 2(6)(1) + 2\sqrt{100-36}\, y'(t) = 0$$

$$\Rightarrow \quad y'(t) = -\frac{6}{8} = -\frac{3}{4} \text{ ft/s}.$$

Ex. 3 (P.246) A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Sol:



$$V = \frac{1}{3}\pi r^2 h \quad (3 \text{ unknowns})$$

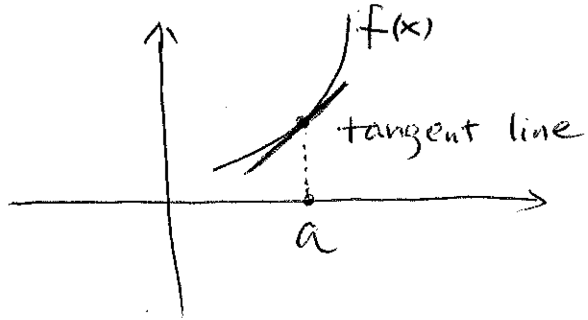
$$\frac{r}{h} = \frac{2}{4} \Rightarrow r = \frac{h}{2} \Rightarrow$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

$$\Rightarrow \quad V' = \frac{1}{12}\pi 3 h^2 h' \Rightarrow 2 = \frac{\pi}{4}(3)^2 h'(t)$$

$$\Rightarrow \quad h'(t) = \frac{8}{9\pi} \text{ m/min}.$$

§3.10. Linear Approximations



When $x \approx a$, $f(x) \approx$ tangent line at a.

<u>Linear Approximation (Tangent line approximation)</u>

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$\left(y = y_0 + m(x-x_0) \right)$$

Ex. 1. (P. 252) Approximate $\sqrt{4.05}$ using a tangent line.

Sol: Let $f(x) = \sqrt{x}$, $a = 4 \Rightarrow f(a) = \sqrt{4} = 2$.

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

$\Rightarrow$ tangent line $y = 2 + \frac{1}{4}(x-4)$.

$\Rightarrow$ $\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$, $x \approx 4$.

Now, $4.05 \approx 4 \Rightarrow \sqrt{4.05} \approx 2 + \frac{1}{4}(4.05-4) = 2.0125$

(From calculator: 2.01246)

(Math 236: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$)

Chapter 4. Applications of Derivatives

§4.4. Indeterminate Forms and L'Hospital's Rule

Ex $\lim_{x\to 1} \frac{\ln x}{x-1} = ?\ \left(\frac{0}{0}\right)$ $\lim_{x\to\infty} \frac{\ln x}{x-1} = ?\ \left(\frac{\infty}{\infty}\right)$

<u>Thm</u> (L'Hospital's Rule)(LPTR) (P. 305)

If $\left[\lim f(x) = 0 \text{ and } \lim g(x) = 0\right]$ $\left(\frac{0}{0}\right)$ or

$\left[\lim f(x) = \pm\infty \text{ and } \lim g(x) = \pm\infty\right]$ $\left(\frac{\infty}{\infty}\right)$

then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$ (if exists or $\pm\infty$).

(See Appendix F for a proof.)

Ex. 5 (P. 308) (LPTR does not apply.)

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{1-(-1)} = \frac{0}{2} = 0,$$

but $$\lim_{x\to\pi^-} \frac{(\sin x)'}{(1-\cos x)'} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = \frac{-1}{\approx 0^+} = -\infty.$$

Ex. 1. (P306) $\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\frac{0}{0}}{=\!=} \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$

Ex. 2. (P. 306)

$$\lim_{x\to\infty}\frac{e^x}{x^2}\overset{\frac{\infty}{\infty}}{=\!=}\lim_{x\to\infty}\frac{e^x}{2x}\underset{\text{LPTR again}}{\overset{\frac{\infty}{\infty}}{=\!=\!=\!=}}\lim_{x\to\infty}\frac{e^x}{2}=\infty.$$

Ex. 3. (P. 307)

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\overset{\frac{\infty}{\infty}}{=\!=}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0.$$

Ex. 4. (P. 307)

$$\lim_{x\to 0}\frac{\tan x-x}{x^3}\overset{\frac{0}{0}}{=\!=}\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}\overset{\frac{0}{0}}{=\!=}\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$$

$$=\frac{1}{3}\left(\lim_{x\to 0}\sec^2 x\right)\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\frac{1}{3}.$$

<u>Other forms</u>: <u>$0\cdot\infty$; $\infty-\infty$; 0^0, ∞^0, $1^\infty \Rightarrow \frac{0}{0}$ or $\frac{\infty}{\infty}$.</u>

Ex. 6. (P. 308)

$$\lim_{x\to 0^+}x\ln x\overset{0\cdot\infty}{=\!=}\lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}\overset{\frac{\infty}{\infty}}{=\!=}\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0}(-x)=0.$$

Ex. 7. (P. 309)

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)\overset{\infty-\infty}{=\!=\!=}\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}\overset{\frac{0}{0}}{=\!=}\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}$$

$$=\lim_{x\to 1^+}\frac{x-1}{x\ln x+x-1}\overset{\frac{0}{0}}{=\!=}\lim_{x\to 1^+}\frac{1}{\ln x+1+1}=\frac{1}{2}.$$

Ex. 8. (P. 309)

$$\lim_{x\to\infty}(e^x - x) \overset{\infty-\infty}{=\!=} \lim_{x\to\infty} x\left[\frac{e^x}{x} - 1\right] = \infty\cdot\infty = \infty.$$

Ex. 9. (P. 310) Find $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$ (1^∞).

Sol: Let $y = (1+\sin 4x)^{\cot x} \Rightarrow \ln y = \cot x \ln(1+\sin 4x)$

or $\ln y = \dfrac{\ln(1+\sin 4x)}{\tan x}$ $\left(\frac{0}{0}\right)$

$$\Rightarrow \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\frac{4\cos x}{1+\sin 4x}}{\sec^2 x} = 4.$$

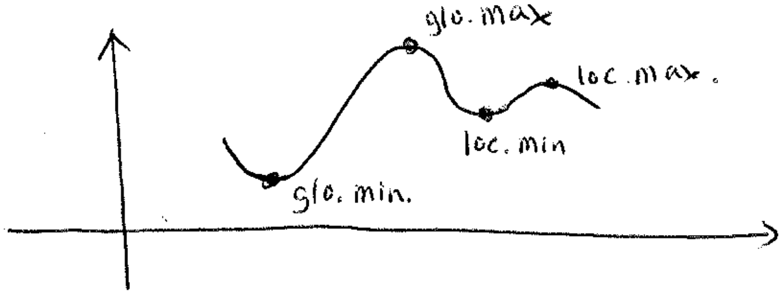
$$\Rightarrow \lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4.$$

Ex. 10. (P. 310) Find $\lim_{x\to 0^+} x^x$. (0^0)

Sol: Let $y = x^x \Rightarrow \ln y = x\ln x \to 0$ (Ex. 6)

$$\Rightarrow \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

§4.1. Maximum and Minimum Values



Def. (P 276) Let $c \in$ domain D. $f(c)$ is the

① global (or absolute) maximum value of f on D
if $f(c) \geq f(x)$ for all $x \in D$,

② global (or absolute) minimum value of f on D
if $f(c) \leq f(x)$ for all $x \in D$,

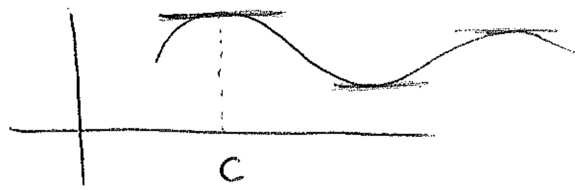
③ local maximum value of f on D
if $f(c) \geq f(x)$ for x near c,

④ local minimum value of f on D
if $f(c) \leq f(x)$ for x near c.

Extreme Values = global max or global min.

The Extreme Value Thm (P. 278) If f is continuous on $[a,b]$, then f attains a global max value $f(c)$ and a global min. value $f(d)$ for some $c, d \in [a,b]$.

<u>Fermat's Thm</u> (P. 279) If f has a global/local max/min at c and if $f'(c)$ exists, then $f'(c)=0$.



Pf: Say, $f(c)$ is a max. value. Then

$$f(c+h) \le f(c), \quad h \approx 0.$$

For $h>0$: $\dfrac{f(c+h)-f(c)}{h} \le 0$,

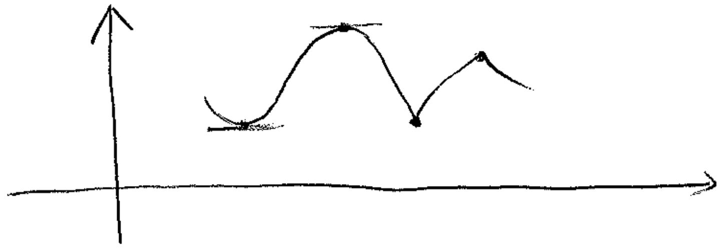
for $h<0$: $\dfrac{f(c+h)-f(c)}{h} \ge 0$.

$$\Rightarrow \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0 \text{ and } \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0.$$

Now, $f'(c)$ exists $\Rightarrow$

$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$$

$$\Rightarrow \quad f'(c)=0. \quad \blacksquare$$



Def. (p. 280) A critical point of f is a point $c \in$ domain s.t. $f'(c) = 0$ or $f'(c)$ DNE.

Ex. 7. (p. 280) Find the critical points of

$f(x) = x^{3/5}(4-x)$.

Sol: $f'(x) = \frac{3}{5}x^{-2/5}(4-x) - x^{3/5} = \frac{3(4-x) - 5x}{5x^{2/5}}$

$= \frac{12 - 8x}{5x^{2/5}} \Rightarrow$ criti. pts: $x = \frac{3}{2}, 0$.

To find the global max/min values of a continuous function f: compare f at all critical points. On closed interval, also compare at endpoints.

Ex. 8. (p. 281) Find the global max/min of

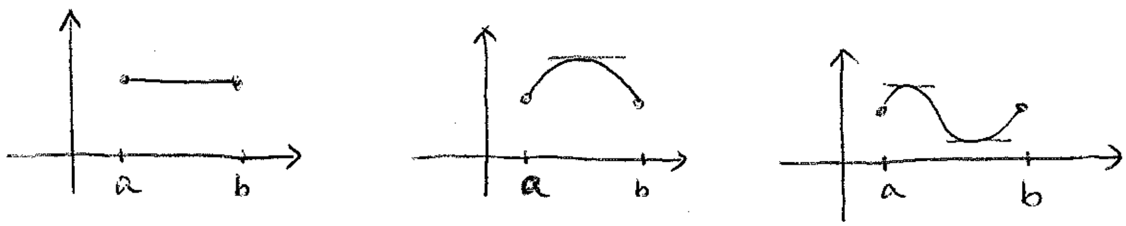
$f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$.

Sol: $f'(x) = 3x^2 - 6x = 3x(x-2) \Rightarrow x = 0, 2$.

Next, $f(-\frac{1}{2}) = \frac{1}{8}$, $f(0) = 1$, $f(2) = -3$, $f(4) = 17$.

$\Rightarrow$ global max: $f(4) = 17$; global min: $f(2) = -3$.

§4.2. The Mean Value Theorem



Rolle's Thm (p. 287) Assume that

① f is continuous on $[a, b]$,

② f is differentiable on (a, b),

③ $f(a) = f(b)$,

then there is $c \in (a, b)$, such that $f'(c) = 0$.

Pf: From the Extreme Value Thm (p. 278), f attains a global max value M_1 at $x_1 \in [a, b]$ and a global min value M_2 at $x_2 \in [a, b]$.

If $x_1 \in (a, b)$ or $x_2 \in (a, b)$, then by Fermat's Thm (p279), we are done.

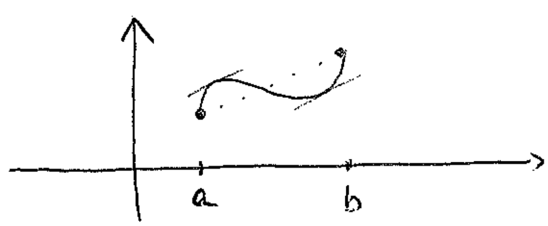
Otherwise, x_1 and x_2 are all endpoints of $[a,b]$. But $f(a) = f(b)$, then $M_1 = M_2$, thus f is a constant on $[a, b]$ and hence $f'(x) = 0$ for $x \in (a, b)$. ▨

Ex. 2. (p. 287) Prove that $f(x) = x^3 + x - 1 = 0$ has exactly one real root.

pf: $f(\infty) = \infty$ and $f(-\infty) = -\infty \Rightarrow f = 0$ has a real root. (Intermediate Value Thm)

If f has two (or more) roots a and b, then $f(a) = f(b) = 0$. By Rolle's Thm, $f'(c) = 0$ for some c. But $f'(c) = 3c^2 + 1 \geq 1$. A contradiction. $\blacksquare$

Rolle's Thm can be generalized.



The Mean Value Theorem (p. 288) Assume that

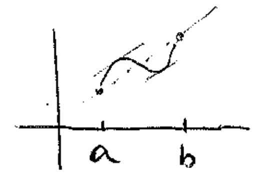
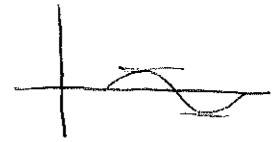
① f is continuous on $[a, b]$,

② f is differentiable on (a, b),

then there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{or}$$

$$f(b) - f(a) = f'(c)(b - a).$$

pf : (Idea : Change  to )

The straight line passing $(a, f(a))$ and $(b, f(b))$ is $y = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$.

Define $h(x) = f(x) - \left[f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)\right]$.

$\Rightarrow h(a) = h(b) = 0$.

By Rolle's Thm, there is $c \in (a,b)$ s.t. $h'(c) = 0$.

But $h'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

$\Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$. ▨

MVT : Average rate of change is attained by instantaneous rate of change at some point.

Ex. 5 (P. 290) If $f(0) = -3$ and $f'(x) \le 5$. How large can $f(2)$ possibly be ?

Sol : By MVT, $f(2) - f(0) = f'(c)(2-0)$

$\Rightarrow f(2) = f(0) + f'(c)2 \le -3 + 5(2) = 7$.

Thm (P. 290) If $f'(x) = 0$ on (a,b), then f is a constant on (a,b).

pf: For any $x_1 \neq x_2$, $x_1, x_2 \in (a,b)$,

$$\text{MVT} \Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0,$$

$$\Rightarrow f(x_1) = f(x_2), \Rightarrow f = \text{a constant}. \ \blacksquare$$

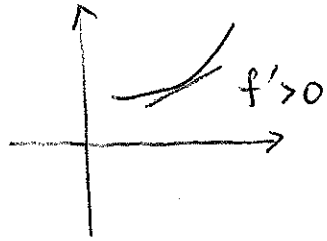
Thm (P. 291) If $f'(x) = g'(x)$ on (a,b), then $f - g$ is a constant.

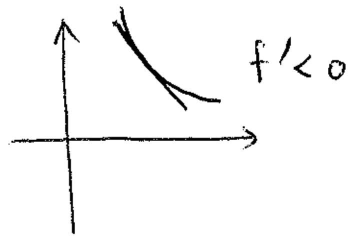
Ex. Show that $|\sin x_2 - \sin x_1| \le |x_2 - x_1|$.

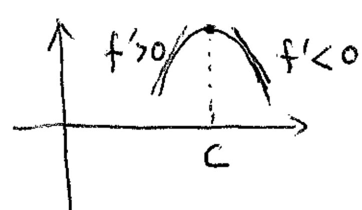
pf: By MVT, $\sin x_2 - \sin x_1 = \cos c\,(x_2 - x_1)$

$$\Rightarrow |\sin x_2 - \sin x_1| \le |x_2 - x_1|. \ \blacksquare$$

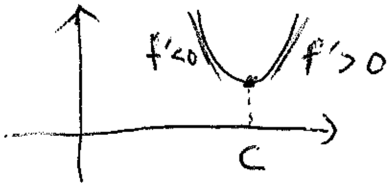
§4.3. How Derivatives Affect the Shape of a Graph







(a max. value)



(a min. value)

The First Derivative Test (P. 293 – 294)

① If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).

② If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b).

③ If f' changes from positive to negative at c, then $f(c)$ is a max. value (global or local).

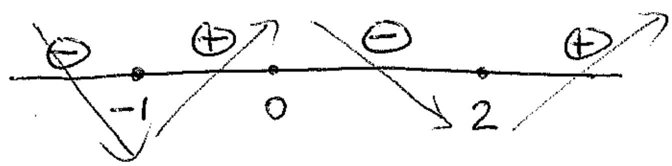
④ If f' changes from negative to positive at c, then $f(c)$ is a min. value (global or local).

Ex 1 and Ex. 2 (P. 293 – 294) Find increasing/decreasing and max./min. values for $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Sol: $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$= 12x(x-2)(x+1)$.

f'



(Sign chart)

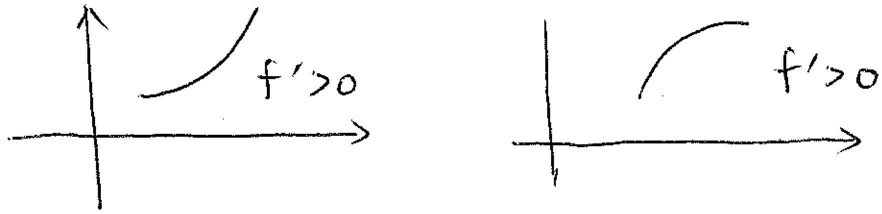
$f \uparrow$: $(-1, 0)$, $(2, \infty)$

$f \downarrow$: $(-\infty, -1)$, $(0, 2)$

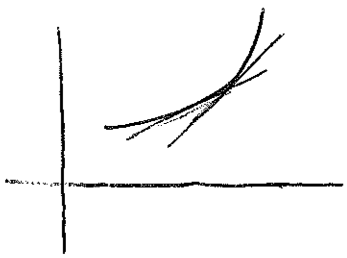
local max. $f(0) = 5$; global min $f(2) = -27$;

local min. $f(-1) = 0$.

Ex.

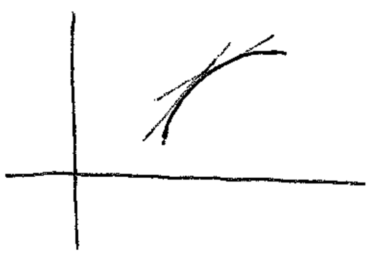


f' fails to tell the difference.



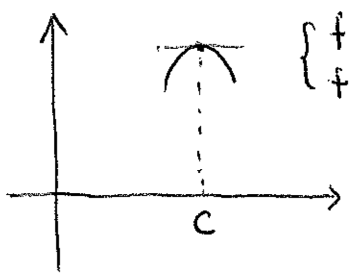
<u>concave up</u>: curve is above (up) its tangent lines.

$\Leftrightarrow f' \uparrow \Leftrightarrow (f')' > 0 \Leftrightarrow f'' > 0.$



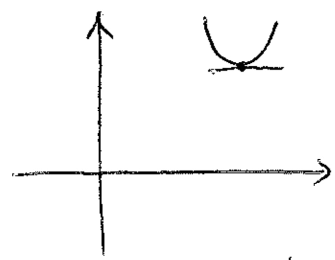
<u>concave down</u>: curve is below (down) its tangent lines.

$\Leftrightarrow f' \downarrow \Leftrightarrow (f')' < 0 \Leftrightarrow f''(x) < 0.$



$\begin{cases} f'(c) = 0 \\ f''(c) < 0 \end{cases}$

a max value



$\begin{cases} f'(c) = 0 \\ f''(c) > 0 \end{cases}$

a min value

Ex. $f(x) = x^4, \ f'(x) = 4x^3, \ f''(x) = 12x^2.$

$\Rightarrow f'(0) = f''(0) = 0.$

a min value at $x = 0$.

Ex. $f(x) = -x^4$, $f'(x) = -4x^3$, $f''(x) = -12x^2$.

$\Rightarrow f'(0) = f''(0) = 0$.  a max value at $x = 0$.

<u>The Second Derivative Test</u> (p. 296 − 297)

① If $f''(x) > 0$ on (a, b), then f is concave up on (a, b).

② If $f''(x) < 0$ on (a, b), then f is concave down on (a, b).

③ If $f'(c) = 0$ and $f''(c) > 0$, then $f(c) =$ min. value.

④ If $f'(c) = 0$ and $f''(c) < 0$, then $f(c) =$ max. value.

⑤ If $f'(c) = f''(c) = 0$, then no general conclusions.

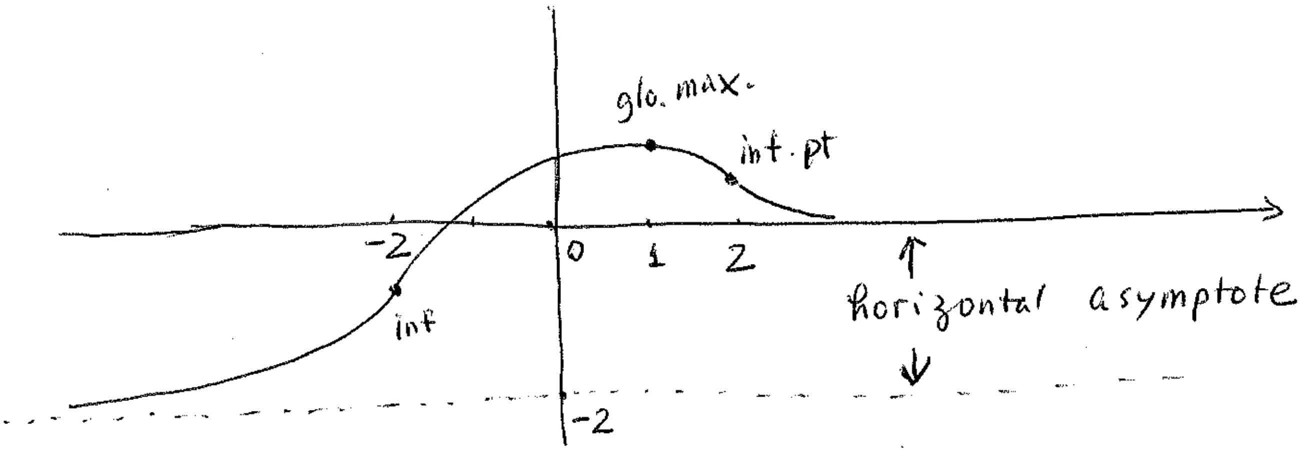
<u>Def.</u> (p. 297) $f(c)$ is an <u>inflection point</u> if f changes concavity there.

Ex. 5. (P. 297) Sketch f if

① $f' > 0$ on $(-\infty, 1)$; $f' < 0$ on $(1, \infty)$,

② $f'' > 0$ on $(-\infty, -2)$ and $(2, \infty)$; $f'' < 0$ on $(-2, 2)$,

③ $\lim_{x \to -\infty} f(x) = -2$; $\lim_{x \to \infty} f(x) = 0$.



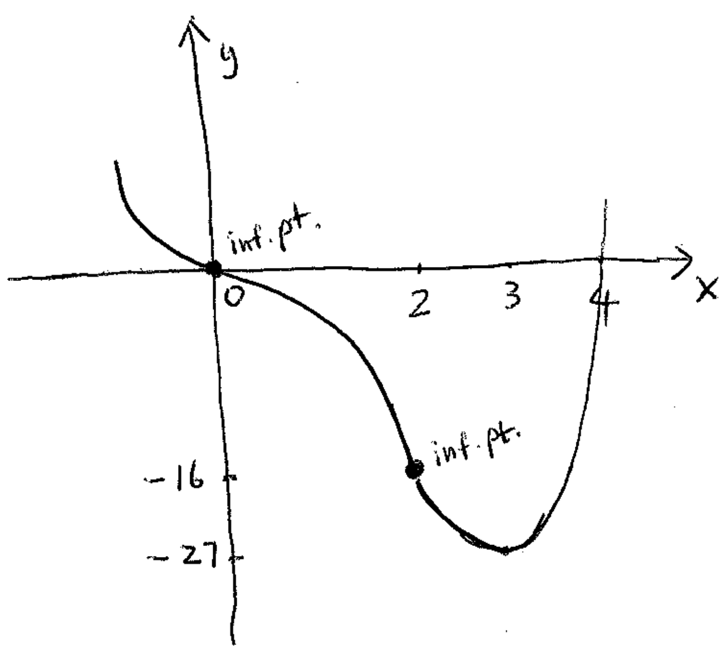
Ex. 6. (P. 298) Sketch $f(x) = x^4 - 4x^3$.

Sol: $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

f': $\ominus$ on $(-\infty, 0)$, $\ominus$ on $(0, 3)$, $\oplus$ on $(3, \infty)$, with $\searrow$, $\searrow$, $\nearrow$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

f'': $\oplus$ $\cup$ on $(-\infty, 0)$, $\ominus$ $\cap$ on $(0, 2)$, $\oplus$ $\cup$ on $(2, \infty)$



y-intercept : $f(0) = 0$.

x-intercepts : set $f(x) = 0$

$\Rightarrow x^3(x-4) = 0 \Rightarrow x = 0, 4$.

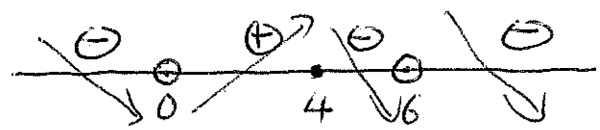
Also, $f(2) = -16$,

$f(3) = -27$.

Ex. 7. (p. 298) sketch $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

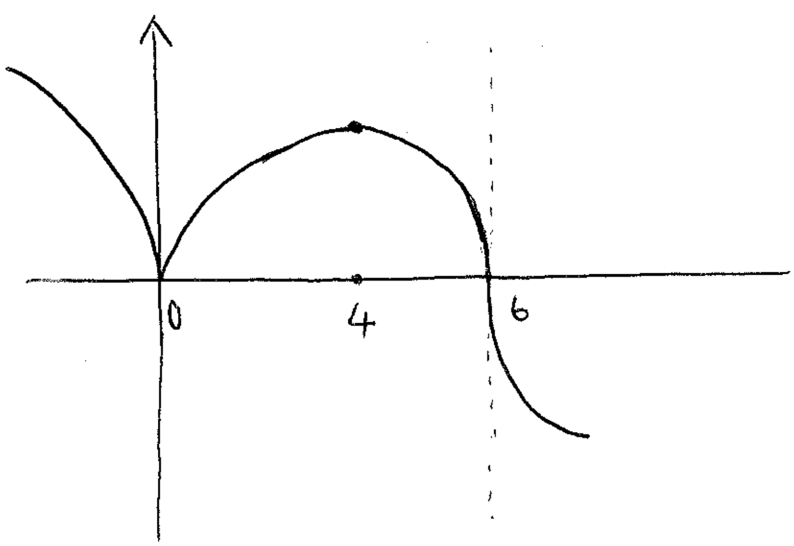
sol: $f'(x) = \dfrac{4-x}{x^{\frac{1}{3}}(6-x)^{2/3}}$

f'



$f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$

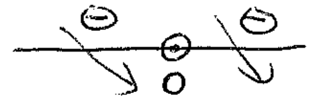
f''



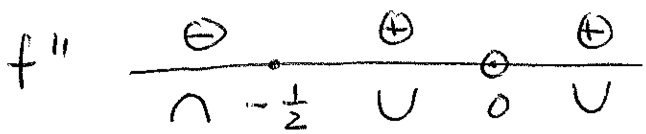
$f(0) = 0$. set $f(x) = 0 \Rightarrow x = 0, 6$.

Ex. 8. (P299) sketch $e^{\frac{1}{x}}$, $x \neq 0$.

Sol: $f'(x) = -\dfrac{e^{\frac{1}{x}}}{x^2}$



$$f''(x) = \frac{-\left[e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)x^2 - e^{\frac{1}{x}}2x\right]}{x^4} = \frac{e^{\frac{1}{x}}(2x+1)}{x^4}$$

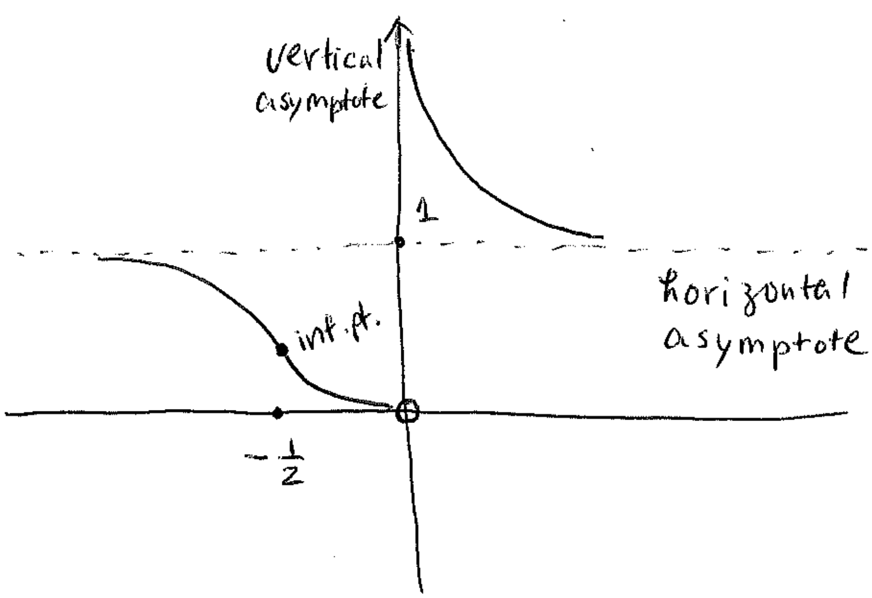


$$\lim_{x\to 0^+} e^{\frac{1}{x}} \overset{t=\frac{1}{x}}{=\!=\!=} \lim_{t\to\infty} e^t = \infty,$$

$$\lim_{x\to 0^-} e^{\frac{1}{x}} \overset{t=\frac{1}{x}}{=\!=\!=} \lim_{t\to -\infty} e^t = 0,$$

$$\lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^0 = 1.$$

No x or y intercept.

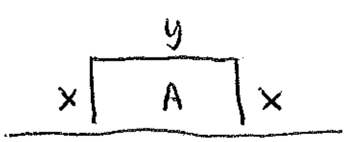


§ 4.7. Optimizations

In some applications:

$$\begin{cases} ① \text{ Objective} \\ ② \text{ Objective function} \\ ③ \text{ use derivatives to optimize.} \end{cases}$$

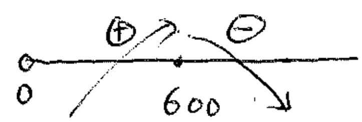
Ex. 1. (P. 331) A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. What are the dimensions of the field that has the largest area?

Sol: 

objective: maximize $A = xy$.

Objective function: $2x + y = 2400 \Rightarrow y = 2400 - 2x$.

$\Rightarrow A(x) = x(2400 - 2x) = 2400x - 2x^2$.

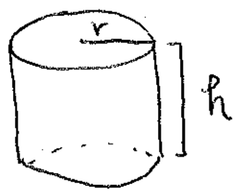
Next, $A'(x) = 2400 - 4x$ 

$\Rightarrow A(600) = \text{global max}$.

$\Rightarrow x = 600, \ y = 2400 - 2(600) = 1200$.

Ex. 2. (p332) A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that minimize the surface area of the can.

Sol:



Objective: minimize $S = 2\pi r h + 2\pi r^2$.

$$\pi r^2 h = 1\,L = 1000\ cm^3,$$

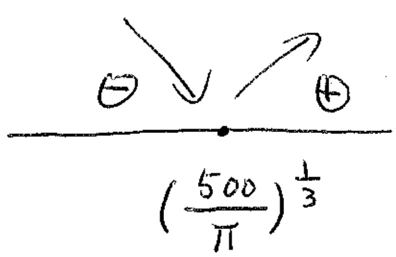
$$\Rightarrow h = \frac{1000}{\pi r^2}.$$

Objective function $S(r) = 2\pi r \dfrac{1000}{\pi r^2} + 2\pi r^2$

$$= \frac{2000}{r} + 2\pi r^2.$$

Next, $S'(r) = -\dfrac{2000}{r^2} + 4\pi r = \dfrac{-2000 + 4\pi r^3}{r^2}$

$$= \frac{4(\pi r^3 - 500)}{r^2}$$

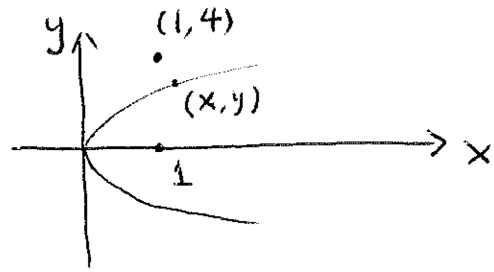


$\Rightarrow r = \left(\dfrac{500}{\pi}\right)^{\frac{1}{3}}$ gives global min.

$$\Rightarrow h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \frac{2 \cdot 500}{\pi^{\frac{1}{3}}\, 500^{2/3}} = 2\,\frac{500^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}$$

$$= 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}} = 2r.$$

Ex. 3 (P. 333) Find the point on $y^2 = 2x$ that is closest to $(1, 4)$.

Sol: 

$d = \sqrt{(x-1)^2 + (y-4)^2}$

Objective : minimize d.

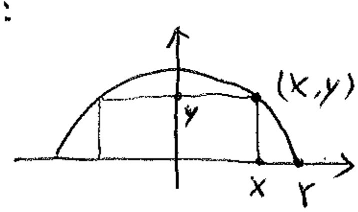
Objective function $f(y) = \left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2$.

$f'(y) = 2\left(\frac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$.

$y = 2$ gives global min.

Next. $y^2 = 2x \Rightarrow 4 = 2x. \Rightarrow x = 2, \Rightarrow$ point $= (2, 2)$.

Ex. 5 (P. 335) Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

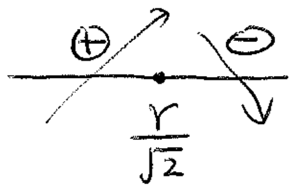
Sol: 

Objective: Maximize $A = 2xy$.

Objective function : $A(x) = 2x\sqrt{r^2 - x^2}$.

$$A'(x) = 2\sqrt{r^2-x^2} + 2x\,\frac{-2x}{2\sqrt{r^2-x^2}} = \frac{2(r^2-x^2)-2x^2}{\sqrt{r^2-x^2}}$$

$$= \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}} = \frac{2(r+\sqrt{2}x)(r-\sqrt{2}x)}{\sqrt{r^2-x^2}}$$



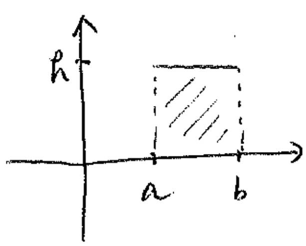
global max.

$$\Rightarrow \quad x = \frac{r}{\sqrt{2}}, \quad y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}, \quad A = 2xy = r^2.$$

Chapter 5. Integrals

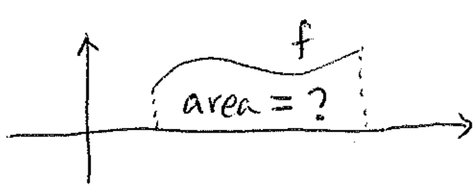
§5.2. The Definite Integral

Ex.

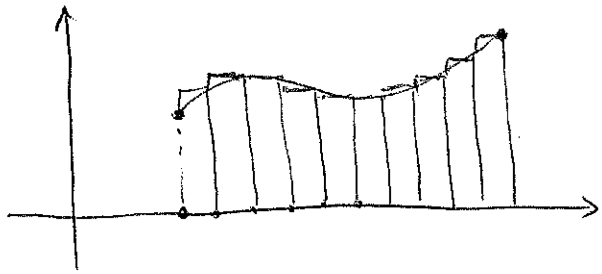


Area $= h(b-a)$.

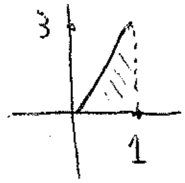
Ex.



Idea: Use summation of area of rectangles to approximate and finally take a limit (more and more rectangles).



Ex. $f(x) = 3x$ on $[0, 1]$.

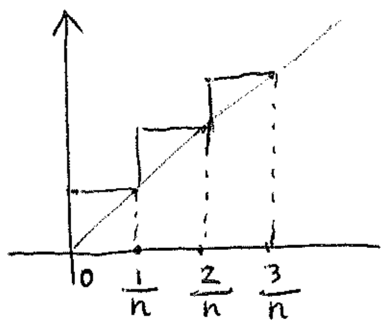


area $= (1)(3)/2 = \frac{3}{2}$.

Divide $[0, 1]$ into n equal subintervals

$[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], [\frac{2}{n}, \frac{3}{n}], \cdots, [\frac{k}{n}, \frac{k+1}{n}], \cdots, [\frac{n-1}{n}, 1]$.

First, use the biggest height:



On $[0, \frac{1}{n}]$: $h = f(\frac{1}{n}) = 3(\frac{1}{n})$, area of rectangle: $3(\frac{1}{n})(\frac{1}{n})$.

on $[\frac{1}{n}, \frac{2}{n}]$: $h = f(\frac{2}{n}) = 3(\frac{2}{n})$, area of rectangle: $3(\frac{2}{n})(\frac{1}{n})$.

$\cdots$

On $[\frac{n-1}{n}, 1]$: $h = f(1) = 3(\frac{n}{n})$, area of rectangle: $3(\frac{n}{n})(\frac{1}{n})$.

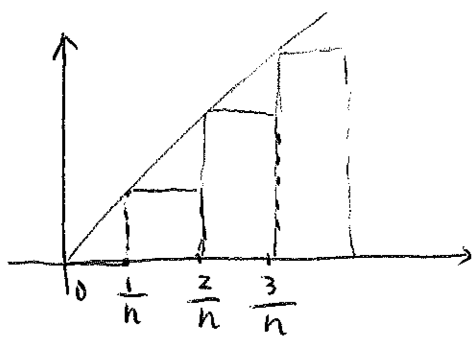
Sum the area of these rectangles :

$$\sum = 3\left(\tfrac{1}{n}\right)\left(\tfrac{1}{n}\right) + 3\left(\tfrac{2}{n}\right)\left(\tfrac{1}{n}\right) + \cdots + 3\left(\tfrac{n}{n}\right)\left(\tfrac{1}{n}\right)$$

$$= \frac{3}{n^2}(1+2+3+\cdots+n)$$

$$= \frac{3}{n^2}\frac{(1+n)n}{2} = \frac{3}{2}\frac{n+1}{n} \xrightarrow{n\to\infty} \frac{3}{2} . \quad \left(\frac{\frac{1}{n}+1}{1} \xrightarrow{n\to\infty} 1\right)$$

Next, use the smallest height :



On $[0, \frac{1}{n})$: $h = f(0) = 0$, area of rectangle : $3(0)(\frac{1}{n})$.

On $[\frac{1}{n}, \frac{2}{n})$: $h = f(\frac{1}{n}) = 3(\frac{1}{n})$, area of rectangle : $3(\frac{1}{n})(\frac{1}{n})$.

. . .

On $[\frac{n-1}{n}, 1)$: $h = f(\frac{n-1}{n}) = 3(\frac{n-1}{n})$, area of rectangle : $3(\frac{n-1}{n})(\frac{1}{n})$.

Sum the area of these rectangles :

$$\sum = 3\left(\tfrac{1}{n}\right)\left(\tfrac{1}{n}\right) + 3\left(\tfrac{2}{n}\right)\left(\tfrac{1}{n}\right) + \cdots 3\left(\tfrac{n-1}{n}\right)\left(\tfrac{1}{n}\right)$$

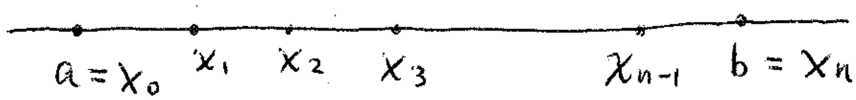
$$= \frac{3}{n^2}(1+2+3+\cdots+(n-1))$$

$$= \frac{3}{n^2}\frac{n(n-1)}{2} = \frac{3}{2}\frac{n-1}{n} \xrightarrow{n\to\infty} \frac{3}{2} .$$

Finally, if the height on any subinterval is $f(x^*) = 3x^*$ with x^* arbitrarily chosen on that subinteral, then the Squeeze Thm (p. 101) can be used to show that the corresponding sum also $\longrightarrow \frac{3}{2}$ as $n \to \infty$. (It works !!!)

<u>Definition of Riemann Sum</u> (p. 378 – 379)

For f on $[a, b]$, divide $[a, b]$ into n subintervals



let $\Delta x_i = x_i - x_{i-1}$, and let $x_i^* \in [x_{i-1}, x_i]$ be arbitrary.

<u>Riemann Sum</u>: $f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$

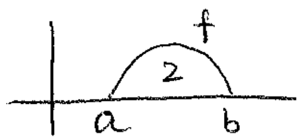
$$= \sum_{i=1}^{n} f(x_i^*)\Delta x_i \quad \text{(sum of rectangles)}.$$

<u>Definition of a (definite) integral</u> (p. 378)

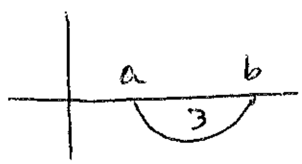
If $\displaystyle\lim_{\max.\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ exists, then f is <u>integrable</u> on $[a, b]$, and its (definite) integal is this limit, denoted by

$$\int_a^b f(x)\,dx = \lim_{\max. \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i .$$

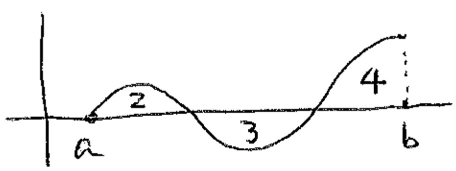
$$\left(\lim \sum \to \int_a^b ; \quad f(x_i^*) \to f(x) ; \quad \Delta x_i \to dx \right)$$

Ex. 

$\int_a^b f(x)dx = 2$.

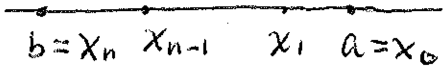


$\int_a^b f(x)dx = -3$ (negative of the area)



$\int_a^b f(x)dx = 2 + (-3) + 4 = 3$.

Thm (p. 380) If f is continuous on $[a,b]$ or has only a finite number of jump discontinuities, then f is integrable on $[a,b]$.

Note: If $a > b$: 

then $\Delta x_i = -(x_{i-1} - x_i)$.

Properties of integrals (P385 – 387)

① $\int_a^a f(x)\,dx = 0$; $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

② $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

③ $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, c a constant.

④ $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.

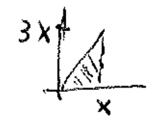
⑤ If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

⑥ If $M_1 \leq f(x) \leq M_2$ on $[a,b]$, then

$$M_1(b-a) \leq \int_a^b f(x)\,dx \leq M_2(b-a).$$

§5.3. The Fundamental Theorem of Calculus

Ex. For $f(x) = 3x$, $\int_0^1 f(x)\,dx = \int_0^1 3x\,dx = \frac{3}{2}$.

Let $g(x) = \int_0^x f(t)\,dt = \int_0^x 3t\,dt$,  $\Rightarrow g(x) = \frac{3}{2}x^2$.

Then we get $g'(x) = \left(\frac{3}{2}x^2\right)' = 3x = f(x)$.

The Fundamental Theorem of Calculus (FTC) (p394-396)

Let f be continuous on $[a,b]$.

① If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

② If $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \overset{\text{notation}}{=\!=} F\Big|_a^b .$$

Pf: ① Let $h > 0$ (the case for $h < 0$ is similar).

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$

$$= \frac{1}{h}\left[\int_a^x + \int_x^{x+h} - \int_a^x\right] = \frac{1}{h}\int_x^{x+h} f(t)\,dt .$$

Next, f is continuous on $[x, x+h]$, then by the Extreme Value Thm (P. 278),

$$f(u) \le f(t) \le f(v), \quad t \in [x, x+h],$$

for some $u, v \in [x, x+h]$. Hence,

$$f(u) = \frac{1}{h}\int_x^{x+h} f(u)\,dt \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \frac{1}{h}\int_x^{x+h} f(v)\,dt = f(v).$$

$$\Rightarrow \quad f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v).$$

Now, f is continuous, then

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x),$$

$$\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x).$$

Use Squeeze Thm,

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x), \text{ i.e. } g'(x) = f(x).$$

Pf: ② Let $F' = f$. From ①, for $g(x) = \int_a^x f(t)\,dt$, $g' = f$.

$\Rightarrow [F(x) - g(x)]' = 0 \Rightarrow F(x) - g(x) = C$ (a constant).

$\Rightarrow F(b) - F(a) = [g(b)+C] - [g(a)+C] = g(b) - g(a)$

$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx.$ ▨

§5.4 Antiderivatives (Indefinite Integrals)

Ex. $\quad 5' = 0, \quad x' = 1, \quad (x^2)' = 2x$

Antiderivative $\quad (?)' = 0, \quad (?)' = 1, \quad (?)' = 2x$

$\quad ? = C$ (const.), $\quad ? = x + C, \quad ? = x^2 + C$

Def. (p. 403) If $F'(x) = f(x)$, then $F(x)$ are called <u>antiderivatives</u> of $f(x)$, denoted by

$$F(x) = \int f(x)dx.$$

<u>Note</u>: $\displaystyle \int_a^b f(x)dx \xlongequal[\text{if } F'=f]{FTC} F\Big|_a^b = \int f(x)dx \Big|_a^b$,

this explains why $\int f(x)dx$ is used for antiderivatives.

Ex. $\int 0\,dx = C$, $\int dx = x + C$, $\int x\,dx = \frac{x^2}{2} + C$,

$\int x^2 dx = \frac{x^3}{3} + C$

<u>Table</u>: $\int cf(x)\,dx = c\int f(x)dx$, $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

$\int k\,dx = kx + C$, $\displaystyle \int x^n dx = \frac{x^{n+1}}{n+1} + C$, $(n \neq -1)$

$\int \frac{1}{x} dx = \ln|x| + C$, $\displaystyle \int e^{kx} dx = \frac{e^{kx}}{k} + C$ ($k \neq 0$, const.)

$\int \sin x\,dx = -\cos x + C$, $\int \cos x\,dx = \sin x + C$,

$\int \sec^2 x\,dx = \tan x + C$, $\int \csc^2 x\,dx = -\cot x + C$,

$\displaystyle \int \frac{1}{x^2+1} dx = \tan^{-1} x + C$, $\displaystyle \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$.

Ex. 1. (p.404) $\int (10x^4 - 2\sec^2 x)\,dx$

$= 10\int x^4 dx - 2\int \sec^2 x\,dx = 10\frac{x^5}{5} - 2\tan x + C$

$= 2x^5 - 2\tan x + C$ (No need for C_1 and C_2).

Ex. 3. (p 404)

$\int_0^3 (x^3 - 6x)\,dx = \int (x^3 - 6x)\,dx \Big|_0^3 = \left(\frac{x^4}{4} - 6\frac{x^2}{2}\right)\Big|_0^3$

$= \frac{81}{4} - 27$ (No need for C : $C - C = 0$)

Ex 4. (p. 405)

$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx = \int \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx \Big|_0^2$

$= \left(2\frac{x^4}{4} - 6\frac{x^2}{2} + 3\tan^{-1}x\right)\Big|_0^2 = 8 - 12 + 3\tan^{-1}2$

$= -4 + 3\tan^{-1}2$.

Ex. 5. (p. 405)

$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 \left(2 + \sqrt{t} - \frac{1}{t^2}\right)dt$

$= \left(2t + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\right)\Big|_1^9 = \left(18 + 18 + \frac{1}{9}\right) - \left(2 + \frac{2}{3} + 1\right)$

$= 32\frac{4}{9}$.

§5.5. Integration by Substitutions

Ex. $\int \cos 6x\,dx = ?$ $\quad \int 2x\sqrt{x^2+1}\,dx = ?$

<u>Thm</u> (P. 413, 416) (Integration by substitutions)

$$\int f(g(x))\,g'(x)\,dx \xlongequal[du = g'(x)dx]{u = g(x)} \int f(u)\,du ,$$

and $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Ex. $\int \cos 6x\,dx \xlongequal[du = 6dx]{u = 6x} \frac{1}{6}\int \cos u\,du = \frac{1}{6}\sin u + C$

$= \frac{1}{6}\sin 6x + C$ $\quad$ (<u>Check</u>: $\frac{d}{dx}\frac{1}{6}\sin 6x = \cos 6x$)

Ex. $\int 2x\sqrt{x^2+1}\,dx \xlongequal[du = 2x\,dx]{u = x^2+1} \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$ $\quad$ (<u>Check</u>: $\frac{d}{dx}\frac{2}{3}(x^2+1)^{\frac{3}{2}} = \sqrt{x^2+1}\cdot 2x$.)

Ex. 1 (P. 413)

$$\int x^3\cos(x^4+2)\,dx \xlongequal[du = 4x^3dx]{u = x^4+2} \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C$$

$= \frac{1}{4}\sin(x^4+2) + C$.

Ex. 2. (P. 414)

$$\int \sqrt{2x+1}\,dx \xrightarrow[du=2dx]{u=2x+1} \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

Ex. 3. (P. 414)

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx \xrightarrow[du=-8xdx]{u=1-4x^2} -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}2u^{\frac{1}{2}} + C$$

$$= -\frac{1}{4}(1-4x^2)^{\frac{1}{2}} + C.$$

Ex. 5. (P. 415)

$$\int \sqrt{1+x^2}\,x^5\,dx \xrightarrow[du=2xdx]{u=1+x^2} \frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$$

$$= \frac{1}{2}\int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du = \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - 2\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$$

Ex. 6. (P 415)

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \xrightarrow[du=-\sin x dx]{u=\cos x} -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln|\sec x| + C.$$

Ex. 7 (P. 416) Find $\int_0^4 \sqrt{2x+1}\,dx$.

Sol: $\int \sqrt{2x+1}\,dx \xlongequal[du=2dx]{u=2x+1} \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(2x+1)^{3/2} + c.$

① $\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_0^4 = \frac{1}{3}(27-1) = \frac{26}{3}.$

or ② $\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{3}u^{\frac{3}{2}}\Big|_1^9 = \frac{1}{3}(27-1) = \frac{26}{3}.$

Ex. 9. (P. 417)

$\int_1^e \frac{\ln x}{x}\,dx \xlongequal[du=\frac{1}{x}dx]{u=\ln x} \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}\Big|_1^e$

$= \frac{1}{2} - 0 = \frac{1}{2}.$