BRIEF CALCULUS

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Contents

Introductory Remarks

Question 1: *Do you know how to approximate* $\sqrt{81.34}$ *without using a calculator? Next, if you do use a calculator to approximate, then do you know what happens after you press some buttons?*

Question 2: *When you are driving a car, do you understand what the speedometer is telling you?*

Question 3: *If you throw a stone with a certain velocity, then can you locate the position of the stone?*

Question 4: *When you play basketball, do you know how to figure out the surface area and the volume of the ball?*

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One of our students said that when he pressed buttons on his calculator, there was a little man with a big brain working inside the calculator......

In fact, that is exactly what we will do here: we will equip you with a big brain, so you can answer all those questions.

To be more specific, we will study two important subjects in this book: the differential calculus and the integral calculus.

The differential calculus is related to the study of rate of change of certain quantities, such as how fast you are driving, or the rate of change in the population of a college.

The integral calculus is related to the study of areas and volumes, such as the area of a circle, and the surface area and the volume of a basketball.

These two important subjects of differential and integral calculus are very closely related, and their relationship is described by the so-called *fundamental theorem of calculus.*

The differential and integral calculus have wide applications in many areas. Here, we will study some of their applications in biology, business, chemistry, physics, and population dynamics.

Chapter 1

Functions

Many important ideas in calculus can be derived from straight lines. Therefore, in this chapter, we start with straight lines, which lead in a natural way to the notions of functions and slopes of functions.

For functions, we introduce the functions one may encounter in an elementary calculus course. The reason to put these functions in Chapter 1 is to give students a chance to work with more algebra operations at the beginning of such a course. This will improve and enhance their overall algebra skill, which is vitally important in learning the subjects in a calculus course and, of course, other related courses.

Another reason to put these functions in Chapter 1 is that it enables us to introduce some applications earlier in the course, such as the applications of exponential and logarithmic functions in business, biology, and population dynamics. This will demonstrate the usefulness of calculus in applications.

1.1 Straight Lines

Paul's story: I attended a private school during 8th and 9th grade. This was by choice having survived several disconcerting cockroach moments during the 7th grade in our early 1900's public school gym shower, and I naively believed that teachers at a "cash driven" private school wouldn't fail a consumer. My brother quickly joined me at my new academic home upon hearing that he could eat lunch each day from vending machines. One afternoon during my eighth grade year I got the word from my buddies that my brother had just set the unofficial school record for number of swats received in a single day at the lower school. The infraction that sent my brother to the office had to do with the cleaning of his classroom's erasers. He had kindly pounded out the room's erasers over the seat of the chair of his 6th grade teacher, which unfortunately resulted in an embarrassing white stain on the rear of a dark skirt. Once in the dim chambers of the office, mind you this was not an initial visit, a furious principal informed my brother of his impending strike from the wooden paddle bouncing in his hand. My brother I guess reasoned that since a swat was inevitable why not take the opportunity to verbally critique the principal face to face. His evaluation ended with the word "jerk". A further enraged red-faced administrator, see **Figure 1.1**, informed my brother he was now facing three swats instead of the original one for his disrespect. My brother inadvisably snapped out a second, "jerk", to which the principal growled "five". Ignorance brought a third "jerk" followed by a shout of "seven!". My brother claimed the pattern continued until the principal just started swinging. My brother proudly maintains that the principal made contact over ten times that afternoon.

Figure 1.1: A Furious Principal

We use this story to motivate the mathematical concept of a "linear relationship" between two quantities. Man has repeatedly exploited the fact that even if only briefly a straight line can be used to approximate most "real world" applications. A linear relationship between two quantities is one in which any sequence of identical changes made in one of the quantities produces a corresponding sequence of identical changes in the other quantity. Examining the relationship between the quantity representing the number of times my brother called the principal a "jerk" and the quantity representing the number of "swats" he was going to receive, we see that at least for a period it was a linear relationship, as each increase of a "jerk" produced an identical increase of two "swats". **End of the story**.

Now, we start this section by briefly introducing the real number system. It is imperative that you at LEAST leave this section with ability to:

1. Recognize what expressions represent or do NOT represent real num-

bers.

2. Determine the number that is the slope of a line either from two points on the line or from the equation of the line itself.

3. Use formulas to determine the equation of a line.

In this course we are ONLY interested in quantities that can be represented by REAL numbers. Recall however that certain mathematical operations involving real numbers can produce results that are NOT real numbers. Two common examples that you are already familiar with are division by zero and the square root of a negative real number. While this is naively thought of as being BAD it is exactly these types of results that have helped lead mathematicians to new vital branches of mathematics like calculus and complex analysis. All real numbers are complex numbers, but not all complex numbers are real numbers. For example any even root of a negative real number, like $\sqrt[4]{-3}$, is a complex number but not a real number, while the expression $\frac{5}{0}$ is not defined in any number system. Also ∞ (meaning very large without bound) and $-\infty$ (negative without bound) are not complex numbers which in turn implies that they are NOT real numbers. It is imperative in this course that you are able to identify whether expressions represent or do not represent real numbers. **Figure 1.2** is meant to help visualize the real number system.

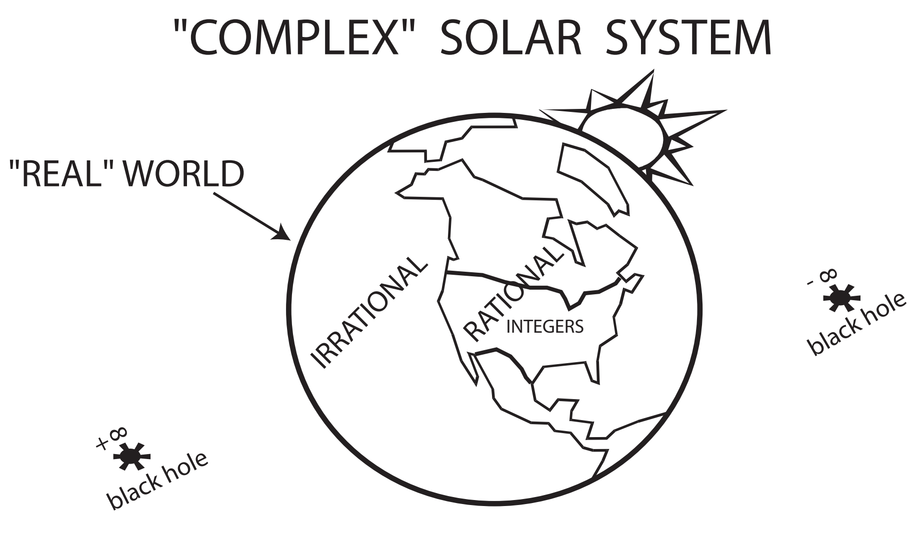

Figure 1.2: "The Real World"

The "world" is divided distinctly into dry land and water and in the same way the real numbers are divided into the rational and irrationals. The dry land in the U.S. is part of all dry land, and similarly the collection of integers is part of the rational numbers. Our world lies in a larger solar system, and likewise the real number system is part of the larger complex number system. Thank goodness black holes are NOT in our solar system and in the same way ∞ and $-\infty$ are NOT part of the complex number system.

A common way to visualize the real number system is to use a horizontal line with an arrow to denote all real numbers, which is called the x-axis, see **Figure 1.3**.

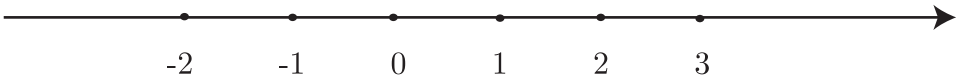

Figure 1.3: The real number line

We select a place and call it the *origin*, or *zero*, denoted by 0, and call the numbers on the right-hand (left-hand) side of 0 as *positive* (*negative*) numbers. In particular, $1, 2, 3, \cdots$ are called *positive integers* and $-1, -2, -3, \cdots$ are called *negative integers.* Positive and negative integers, together with 0, are called *integers.*

If a number can be written as $\frac{p}{q}$ with p and q integers and $q \neq 0$, such as $\frac{4}{3}$ or $\frac{-5}{2}$, then it is called a *rational number.* Accordingly, integers are also rational numbers, because, for example, $3 = \frac{3}{1}$. Some numbers, such as $\sqrt{3}$ and π, cannot be written as $\frac{p}{q}$ with p and q integers, so they are called *irrational numbers.* Putting all rational and irrational numbers together, we get all real numbers, or the x-axis.

Example 1.1.1 For each of the following mathematical expressions identify in which of the subsequent number systems, *Complex*, *Real*, *Rational*, *Irrational*, and *Integer*, they exist, if any.

1. $\frac{5}{x-2}$ evaluated at $x = 3$.
2. $\sqrt{x-7}$ evaluated at $x = 9$.
3. The number π.
4. $\frac{x+1}{x^2-4}$ evaluated at $x = 2$.
5. $\sqrt{x-7}$ evaluated at $x = 6$.
6. $\frac{x-2}{x+4}$ evaluated at $x = 2$.
7. $\frac{x-1}{x+6}$ evaluated at $x = -1$.
8. ∞.

Solution.

1. The expression $\frac{5}{x-2}$ evaluated at $x = 3$ simplifies to $\frac{5}{3-2} = 5$. Since 5 is an integer, it is also a rational, real and complex.

2. The expression $\sqrt{x-7}$ evaluated at $x = 9$ simplifies to $\sqrt{9-7} = \sqrt{2}$. The Pythagoreans proved over a millennium ago that there DOES NOT exist a ratio of integers that is equivalent to $\sqrt{2}$. The $\sqrt{2}$ is a classic irrational number, which makes it also both real and complex.

3. The number π which is the ratio of the circumference to the diameter for any circle is again a classic irrational number, which makes it also both real and complex.

4. The expression $\frac{x+1}{x^2-4}$ evaluated at $x = 2$ simplifies to $\frac{2+1}{4-4} = \frac{3}{0}$. This does not represent a number in any number system and as such we say it *Does Not Exist*.

5. The expression $\sqrt{x-7}$ evaluated at $x = 6$ simplifies to $\sqrt{6-7} = \sqrt{-1}$. This IS NOT a real number but it IS a classic complex number. Since it is not real it is also clearly neither rational nor irrational.

6. The expression $\frac{x-2}{x+4}$ evaluated at $x = 2$ simplifies to $\frac{2-2}{2+4} = \frac{0}{6} = 0$. Since 0 is an integer, it is also a rational, real and complex.

7. The expression $\frac{x-1}{x+6}$ evaluated at $x = -1$ simplifies to $\frac{-1-1}{-1+6} = \frac{-2}{5}$. While $\frac{-2}{5}$ is NOT an integer it is the ratio of two integers which makes it rational, and then also real and complex.

8. ∞ means very large without bound, so it is NOT a complex number, which of course implies it is not a real number either.

♠

The *intervals* can be defined accordingly. For example,

$$\begin{aligned}
[3,5] &\text{ denotes all real values } x \text{ such that } 3 \le x \le 5,\\
(3,5] &\text{ denotes all real values } x \text{ such that } 3 < x \le 5,\\
(3,\infty) &\text{ denotes all real values } x \text{ such that } 3 < x < \infty,\\
(-\infty,5] &\text{ denotes all real values } x \text{ such that } -\infty < x \le 5,
\end{aligned}$$

see **Figure 1.4**.

In the same way, we use two perpendicular real number lines to construct a *plane*, called the *Cartesian coordinate system* or xy-plane, in which a point in the plane is given by (x, y), where x is the horizontal coordinate and y is the vertical coordinate. For example, points $P_1 = (1, 2), P_2 = (-3, 1)$, and $P_3 = (-2, -3)$ are plotted in **Figure 1.5**. For a plane, the origin is $(0, 0)$.

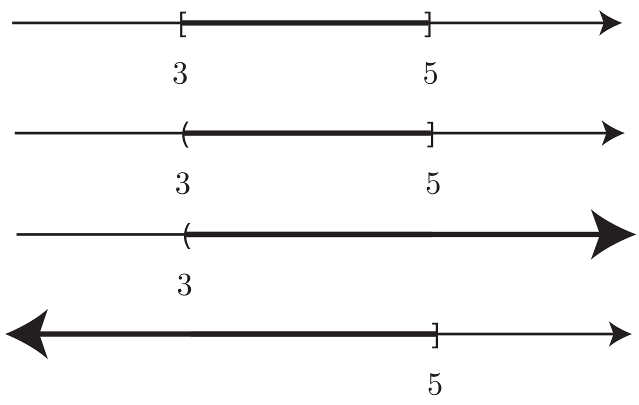

Figure 1.4: Intervals on the real line

Note that when we deal with a real number line, (a, b) denotes an interval, but when we deal with a plane, (a, b) denotes a point. It should not cause confusion in different situations.

In this elementary calculus course, the xy-plane will be our "playground", and many "activities" will take place there. Among them, a very important one is to learn how to deal with complicated curves in the plane. Here, let's start with the simplest curves: straight lines and their slopes. And to motivate the concept of "slope", let's consider a standard staircase, see **Figure 1.6**, consisting of a number of identical treads and risers. This describes a linear relationship, as each time one moves right by running the length of the tread it produces an identical increase in the height by the length of a riser. In fact it is not difficult to visualize a line that basically describes the stair case, see **Figure 1.7**. Here the number found by dividing the riser length by the tread length is the **slope** of the line, which measures the angle or the steepness of the staircase. The slope of a line is extremely important and can kind of be viewed as the "DNA" for the line, as this number alone almost completely describes the line. Actually starting with any two random points on this line the right triangle of the form in **Figure 1.8** is such that the "rise" (here the length of the vertical leg of the triangle) divided by the "run" (here the length of the horizontal leg of the triangle) also equals the number that is the slope.

Now, let's provide some details for straight lines and their slopes. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points with $x_1 \neq x_2$ in an xy-plane. Then, using plane geometry, we can draw a unique straight line, L, passing through P_1 and P_2, given in **Figure 1.9**.

From geometry we see that if a point $P = (x, y)$ is also on the straight line L, then there must be some relationship between x and y. Because if x and y are arbitrary, then the point (x, y) may not be on the straight line L. So, we ask the following

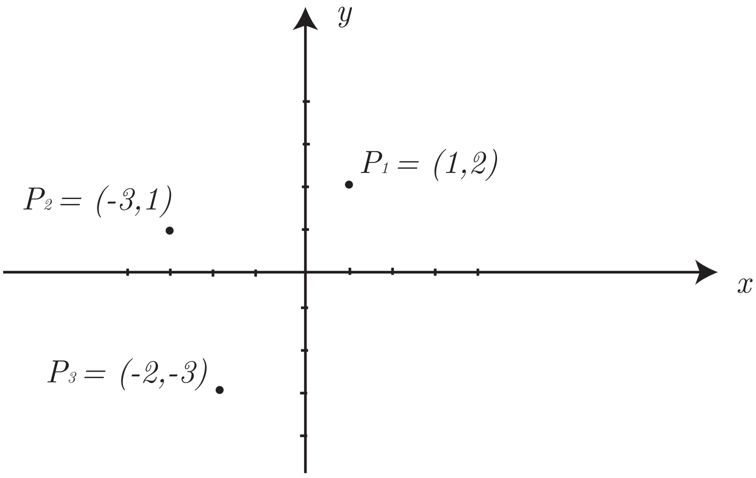

Figure 1.5: The xy-plane

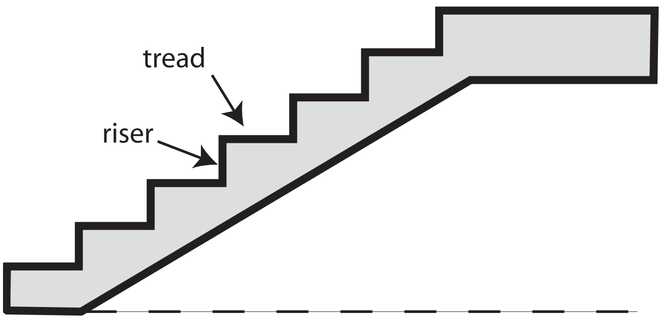

Figure 1.6: A Standard Staircase

Question: *If* (x, y) *is on the straight line* L*, then what is the relationship between* x *and* y*? Or, how do we describe the straight line* L*?*

To answer this question, let's make two right triangles P_1P_2Q and P_1PR as shown in **Figure 1.10**, with $\overline{P_1R}$ a horizontal segment. Then, from plane geometry, we see that the three corresponding angles of the two triangles are the same. Thus, the two triangle are *similar*, and hence the ratios of the corresponding sides are the same. Therefore,

$$\frac{|PR|}{|P_1R|} = \frac{|P_2Q|}{|P_1Q|},$$

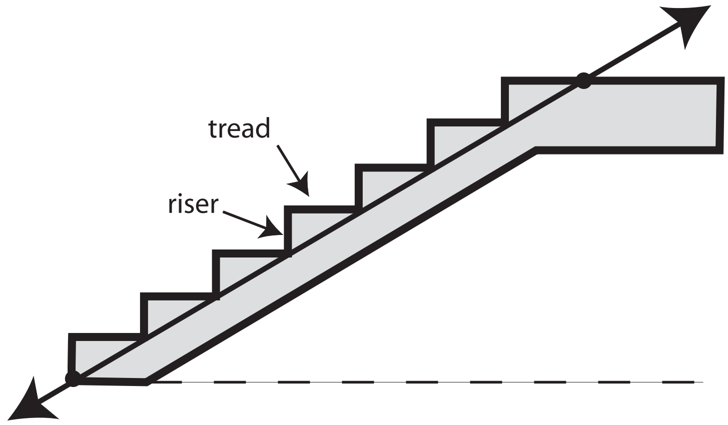

Figure 1.7: Line Representing the Standard Staircase

or

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}.$$

Now, we multiply $x - x_1$ on both sides and then move y_1 to the other side (and change the sign), this gives

$$y = y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1). \tag{1.1}$$

The ratio $\frac{y_2-y_1}{x_2-x_1}$ is based on the triangle P_1P_2Q. But applying the knowledge of similar triangles again, you will see that for any two points on the straight line L, the corresponding ratio will be the same as $\frac{y_2-y_1}{x_2-x_1}$. And in plane geometry, $\frac{y_2-y_1}{x_2-x_1}$ measures

$$\frac{rise}{run}$$

of the straight line L. Thus, we call

$$m = \frac{y_2-y_1}{x_2-x_1} \tag{1.2}$$

the **slope** of the straight line L, which gives the **direction** of the straight line L. In geometry, if a straight line "climbs up" (like climbing up a hill), then the slope is positive (see **Figure 1.11**). If a straight line "slides down" (like sliding down a hill), then the slope is negative (see Figure 1.11), because now the "rise" is negative and "run" is positive.

Note that bigger slopes mean that the straight lines are more steep. Negative slopes can be explained similarly. See **Figure 1.12**.

For a horizontal straight line, such as $y = 3$ in **Figure 1.13**, for any two different points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on the straight line $y = 3$, we must have $y_1 = y_2 = 3$ and $x_1 \neq x_2$.

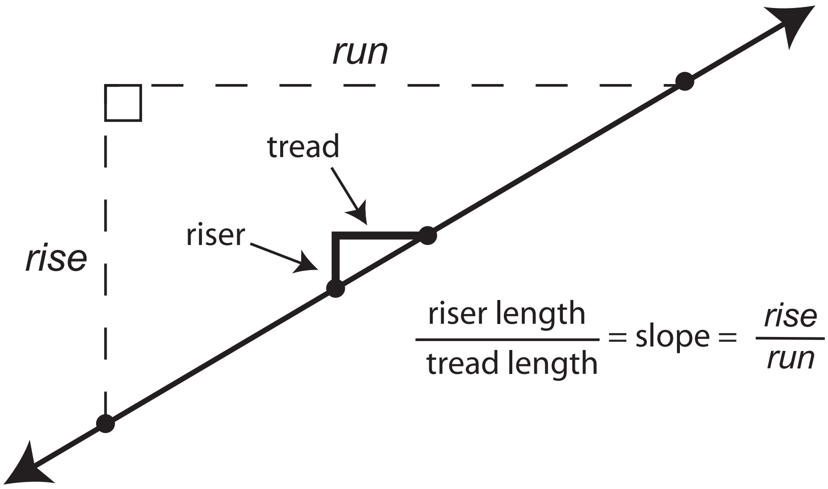

Figure 1.8: Slope of the Staircase

Thus, the m in the formula (1.2) must be zero: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{x_2 - x_1} = 0$. Therefore, the slope of every horizontal straight line must be zero. It means a smooth drive on a flat road.

For a vertical straight line, such as $x = 3$ in **Figure 1.14**, for any two different points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on the straight line $x = 3$, we must have $x_1 = x_2 = 3$. Thus, the formula (1.2) cannot be defined now since the denominator would be $x_2 - x_1 = 3 - 3 = 0$. Therefore, we do not define slopes for vertical straight lines. For example, in front of a vertical cliff, what is the slope?

Now, (1.1) can be written as

$$y = y_1 + m(x - x_1), \tag{1.3}$$

which is called the **slope-point form** for the straight line L. The formula (1.3) answers the question raised above, that is, (x, y) must satisfy (1.3) in order for (x, y) to be on the straight line L. Thus, (1.3) describes the relationship between x and y if the point (x, y) is on the straight line L. In geometry, the formula (1.3) means that given a point (x_1, y_1) and a slope (direction) m, one can uniquely determine the straight line L, given by (1.3).

Example 1.1.2 (a). Find the straight line passing through $(1, -2)$ and with slope -3. (b). For $x = 5$, find the corresponding y value so that the point (x, y) is on the straight line.

Solution. (a). We have $m = -3$ and $(x_1, y_1) = (1, -2)$, so by using the slope-point form (1.3), the straight line is given by

$$y = y_1 + m(x - x_1) = -2 - 3(x - 1).$$

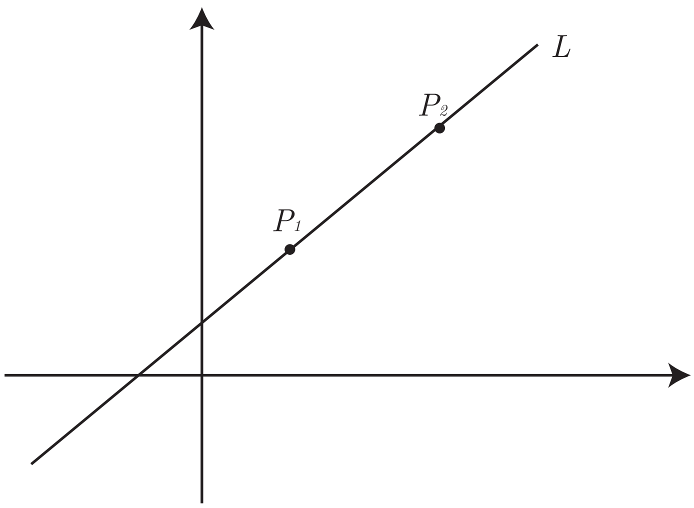

Figure 1.9: The straight line L passing through P_1 and P_2

(b). For $x = 5$, the corresponding y value is uniquely determined by

$$y = -2 - 3(5 - 1) = -14,$$

that is, $(5, -14)$ is on the straight line. ♠

Next, from Figure 1.9, we see that if we replace the point (x_1, y_1) in (1.3) by (x_2, y_2), then we must obtain the same straight line. Therefore, (x_1, y_1) in (1.3) can be replaced by (x_2, y_2), as can be seen from the following example.

Example 1.1.3 Find the straight line passing through $(1, -2)$ and $(-3, 4)$.

Solution. Let $(x_1, y_1) = (1, -2)$ and $(x_2, y_2) = (-3, 4)$, then the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{(-3) - 1} = \frac{6}{-4} = -\frac{3}{2}.$$

If we use the point $(x_1, y_1) = (1, -2)$, then we get, using (1.3),

$$y = -2 - \frac{3}{2}(x - 1) = -\frac{3}{2}x - 2 + \frac{3}{2} = -\frac{3}{2}x - \frac{1}{2}.$$

If we use the point $(-3, 4)$ for (x_1, y_1) in (1.3), then we get

$$y = 4 - \frac{3}{2}(x + 3) = -\frac{3}{2}x + 4 - \frac{9}{2} = -\frac{3}{2}x - \frac{1}{2}.$$

That is, after simplifying them, we see that no matter what point we use, we get the same straight line. ♠

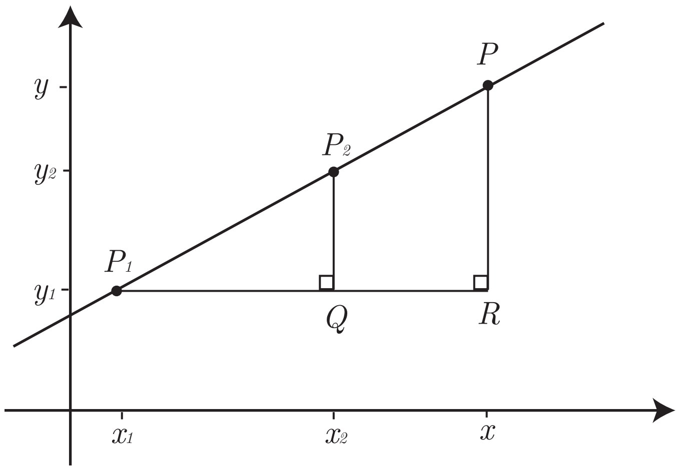

Figure 1.10: The two right triangles are similar

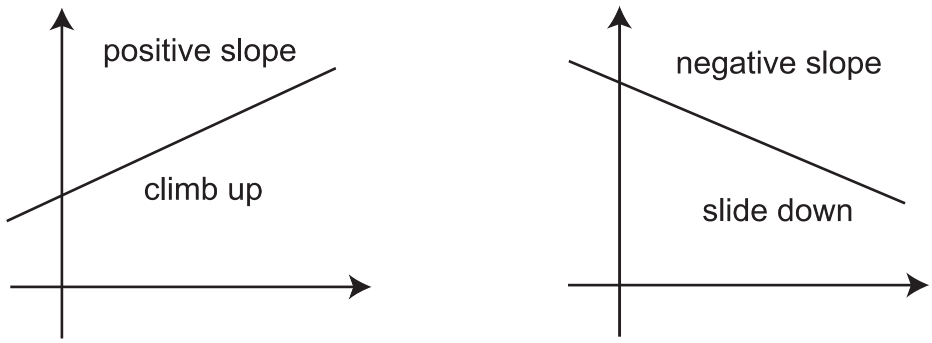

Figure 1.11: Straight lines with positive and negative slopes

In Example 1.1.3, we simplified $y = -2 - \frac{3}{2}(x-1)$ as $y = -\frac{3}{2}x - \frac{1}{2}$. In general, we can rewrite (1.3) as

$$y = mx + (y_1 - mx_1),$$

or

$$y = mx + b, \tag{1.4}$$

where $b = y_1 - mx_1$. Now, in (1.4), when $x = 0$, $y = b$. The geometry explanation is that the straight line L intercepts the y-axis at b. Thus, (1.4) is called the **slope-intercept form** for the straight line L. Note that for a given straight line, the formulas in slope-point and slope-intercept forms must describe the same straight line. Note also that to find b, we can plug any given point into (1.4), as can be seen from the following example.

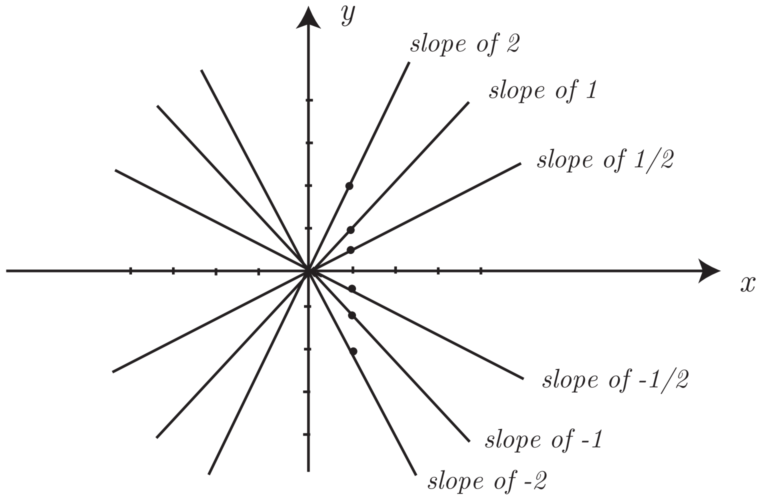

Figure 1.12: Geometry explanation of bigger slopes

Example 1.1.4 Redo Example 1.1.3 using the slope-intercept form.

Solution. The slope is

$$m = \frac{4-(-2)}{(-3)-1} = \frac{6}{-4} = -\frac{3}{2},$$

so the straight line is given by

$$y = mx + b = -\frac{3}{2}x + b.$$

To find b, if we plug $(x, y) = (1, -2)$, then we get $-2 = -\frac{3}{2}(1) + b$, so $b = -\frac{1}{2}$. If we plug $(x, y) = (-3, 4)$, then we get $4 = -\frac{3}{2}(-3) + b$, so $b = -\frac{1}{2}$ also. ♠

Remark 1.1.5 After seeing Examples 1.1.3 and 1.1.4, we conclude that to find a straight line, we can use either the slope-point form or the slope-intercept form, and we can use any given point to complete the form. But note that these forms do not apply to vertical straight lines as we do not define slope then. ♠

Note that for constants a, b, and c with $a \neq 0$ or $b \neq 0$, the form

$$ax + by = c \tag{1.5}$$

represents a straight line: if $b \neq 0$, then (1.5) becomes $y = -\frac{a}{b}x + \frac{c}{b}$; if $b = 0$ but $a \neq 0$, then (1.5) becomes a vertical straight line $x = \frac{c}{a}$. Note that (1.5) represents all straight lines, including vertical straight lines, and is called the **general form** of straight lines.

To introduce other notions, we look at the following examples.

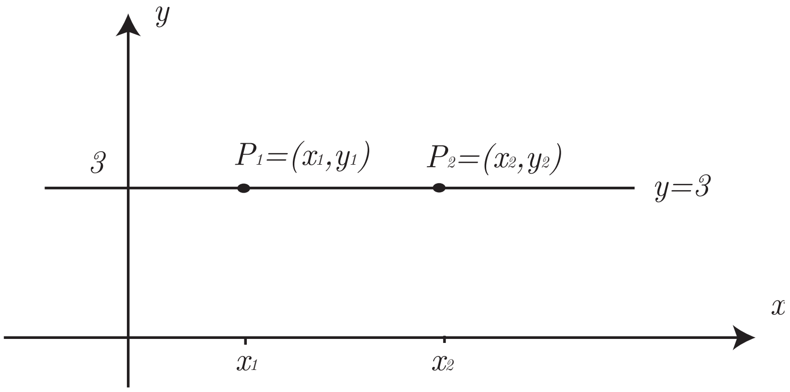

Figure 1.13: The horizontal straight line $y = 3$

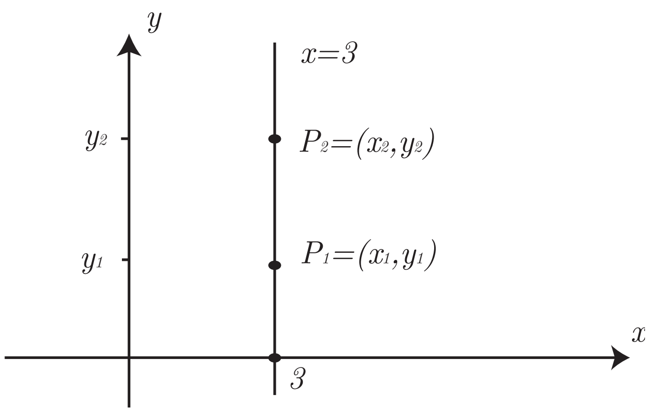

Figure 1.14: The vertical straight line $x = 3$

Example 1.1.6 Draw the straight lines $L_1 : y = 2x$ and $L_2 : y = 2x + 1$.

Solution. Since L_1 is a straight line, we only need two points in order to draw L_1. For example, we can use $(0, 0)$ and $(1, 2)$. Similarly, for L_2, we can use $(0, 1)$ and $(1, 3)$. See **Figure 1.15**. ♠

From Figure 1.15, we see that the two straight lines L_1 and L_2 are **parallel** to each other, which is expected because the two straight lines have the same slope (direction), $m = 2$.

Example 1.1.7 Draw the straight lines $L_1 : y = x$ and $L_2 : y = -x$.

Solution. We can use $(0, 0)$ and $(1, 1)$ for L_1, and use $(0, 0)$ and $(1, -1)$ for L_2. So we obtain **Figure 1.16**. ♠

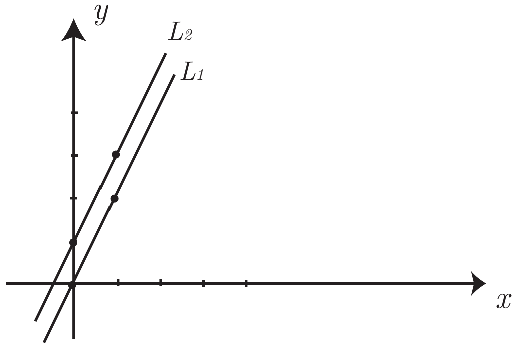

Figure 1.15: L_1 and L_2 are parallel to each other

From Figure 1.16, we see that the two straight lines L_1 and L_2 are **perpendicular** to each other, that is, the angle formed by L_1 and L_2 is 90^0. Now, the slope of L_1 is $m_1 = 1$, and the slope of L_2 is $m_2 = -1$, so we get

$$m_1 m_2 = -1. \tag{1.6}$$

If you check other cases, such as $y = 2x$ and $y = -\frac{1}{2}x$, you will see that the same thing will happen, that is, the two straight lines are perpendicular and the product of the two slopes is -1.

Note that a vertical straight line and a horizontal straight line are perpendicular to each other, but the slope of the vertical straight line is not defined, so (1.6) does not apply in this case.

Based on these, we state the following

Property 1.1.8 *Let L_1 and L_2 be two non-vertical and non-horizontal straight lines with slopes m_1 and m_2 respectively.*

1. *L_1 is parallel to L_2 if and only if $m_1 = m_2$.*
2. *L_1 is perpendicular to L_2 if and only if $m_1 m_2 = -1$.* ♠

Example 1.1.9 Find the straight line passing through $(1, 2)$ that is

1. parallel to $3x + 4y = 5$.
2. perpendicular to $3x + 4y = 5$.

Solution. (1). We first rewrite $3x + 4y = 5$ as $y = -\frac{3}{4}x + \frac{5}{4}$, which is a straight line with slope $m_1 = -\frac{3}{4}$. To find the straight line passing through $(1, 2)$ that is parallel to $y = -\frac{3}{4}x + \frac{5}{4}$, we use the point $(1, 2)$ and the same slope as $m_1 = -\frac{3}{4}$. So the straight line is given by

$$y = 2 - \frac{3}{4}(x - 1).$$

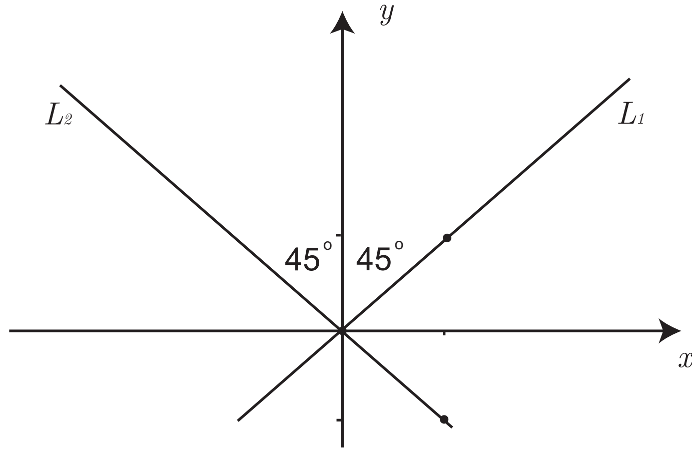

Figure 1.16: L_1 and L_2 are perpendicular to each other

(2). Now, since the two straight lines are perpendicular to each other, we use the point $(1, 2)$ and the slope $m_2 = -\frac{1}{m_1} = \frac{4}{3}$ (which is simply changing the sign and flipping $\frac{3}{4}$), so the straight line is given by

$$y = 2 + \frac{4}{3}(x - 1).$$

♠

Example 1.1.10 (Depreciation of your car) Assume that you bought a four year old car for \$15,000 and sold it after three years for \$11,100, and also assume that the value of your car depreciated in a linear fashion. Find a straight line to describe the situation. How much is depreciated each year? What does the slope mean in this case?

Solution. When the car is four years old, its value is \$15,000; when the car is seven years old, its value is \$11,100, thus we get two points: (4, 15000) and (7, 11100). Then the slope is given by

$$m = \frac{11100 - 15000}{7 - 4} = -1300,$$

so the straight line is given by

$$y = 15000 - 1300(x - 4).$$

The value of the car depreciated \$3900 in three years, so that in a linear fashion, it depreciated \$1300 each year. Accordingly, the slope means the negative of the yearly depreciation. ♠

Exercises 1.1

1. For each of the following mathematical expressions identify in which of the subsequent number systems, *Complex*, *Real*, *Rational*, *Irrational*, and *Integer*, they exist, if any.

(a) $\frac{x^2-9}{x+5}$ evaluated at $x = 3$.

(b) $\sqrt{x+4}$ evaluated at $x = -13$.

(c) The number π^3.

(d) $\frac{x+3}{x-4}$ evaluated at $x = 4$.

(e) $\sqrt{x+5}$ evaluated at $x = 7$.

(f) $\frac{x+12}{x+5}$ evaluated at $x = -10$.

(g) $\frac{8}{x+6}$ evaluated at $x = 2$.

(h) $-\infty$.

2. In **Figure 1.17**, what is the slope of the ramp?

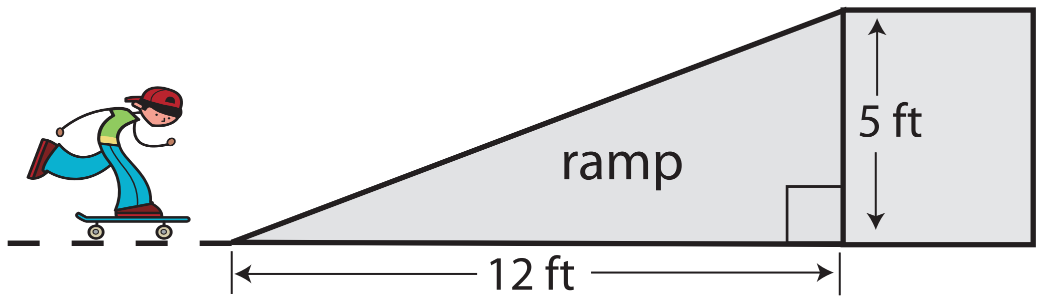

Figure 1.17: A ramp

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3. Given a point $(1, 2)$, how many straight lines can you draw to pass the point $(1, 2)$?

4. Given a slope $m = 2$, how many straight lines can you draw with the slope $m = 2$?

5. Consider a straight line $L : y = 3x - 8$. For $x = 3$, how many y values can you find so that (x, y) is on L? Find this (or these) value(s), if any.

6. Consider a straight line $L : y = 9$. For $x = 3$, how many y values can you find so that (x, y) is on L? Find this (or these) value(s), if any.

7. Consider a straight line $L : x = 8$. For $x = 3$, how many y values can you find so that (x, y) is on L? Find this (or these) value(s), if any. Next, do the same for $x = 8$.

8. Find the straight line passing through $(-3, 2)$ with slope 2.

9. Find the straight line passing through $(3, -2)$ with slope -2.

10. Find the straight line passing through $(-1, 2)$ and $(5, -4)$.

11. Find the straight line passing through $(1, -2)$ and $(-5, 4)$.

12. Find the slope for (a). $y = 3x + 4$; (b). $x = 3y + 4$; (c). $-4y = 3x + 4$; (d). $-3x - 5y + 4 = 0$; (e). $y = 4$; (f). $x = 3$.

13. Find the straight line passing through $(-2, 3)$ that is

(a) parallel to $7x - 2y = 5$.

(b) perpendicular to $7x - 2y = 5$.

14. Draw the following straight lines in one xy-plane. $L_1 : y = x$; $L_2 : y = 2x$; $L_3 : y = 3x$; $L_4 : y = \frac{1}{2}x$; $L_5 : y = -x$; $L_6 : y = -2x$; $L_7 : y = -3x$; $L_8 : y = -\frac{1}{2}x$.

15. In **Figure 1.18**, a 4-ft boy walks away from a street light which is 20 ft above the ground. The $x(t)$ is the distance between the boy and the base of the lamppost, and $s(t)$ is the shadow of the boy. Find a relationship between $x(t)$ and $s(t)$.

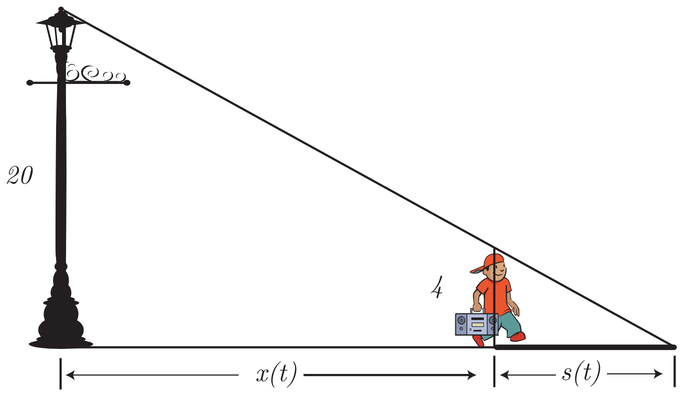

Figure 1.18: The boy walks away from a street light

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16. (Depreciation of your car) Assume that you bought a three year old car for \$17,000 and sold it after three years for \$11,600, and that the value of your car depreciated in a linear fashion, then find a straight line to describe the situation. How much is depreciated each year? What does the slope mean in this case?

17. (Depreciation of your car) Assume that you bought a three year old car for \$17,000 and sold it after three months for \$16,400, and that the value of your car depreciated in a linear fashion, then find a straight line to describe the situation. How much is depreciated each month? What does the slope mean in this case?

1.2 Functions and Inverse Functions

Question: [Payment at a gas station] *If you go to a gas station to add gas to your car and the gas price is \$2.5 per gallon, then what is the relationship between your payment and the number of gallons of gas you add?*

To answer the question, let's use x to denote the number of gallons of gas you add and use y to denote your payment. Since the price is \$2.5 per gallon, if $x = 1$, then $y = 2.5$; if $x = 2$, then $y = 2 \times 2.5 = 5$; in general, the y value (payment) is uniquely determined by the x value (gallons of gas), given by

$$y = 2.5x,$$

where x takes the values in the interval $[0, \infty)$.

We realize that $y = 2.5x$ is a non-vertical straight line for $x \geq 0$, given in **Figure 1.19**, which shows a **unique correspondence** between x and y.

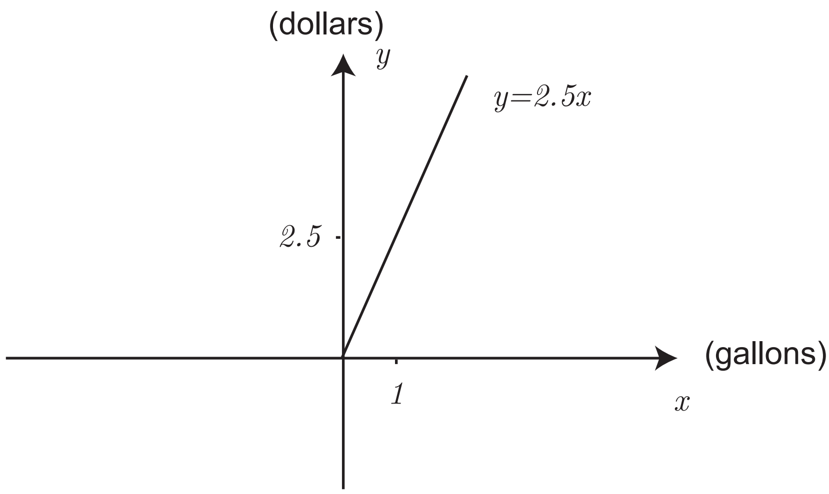

Figure 1.19: The non-vertical straight line $y = 2.5x$

In fact, from our study of straight lines in Section 1.1, we know that for any non-vertical straight line L, such as the one given in **Figure 1.20**, this *unique correspondence* holds true: for each given x value, there is a unique corresponding y value so that the point (x, y) is on the straight line L. This can be viewed in geometry as follows: for each given x value, the vertical straight line passing through x crosses the straight line L at exactly one point, as can be seen from Figure 1.20.

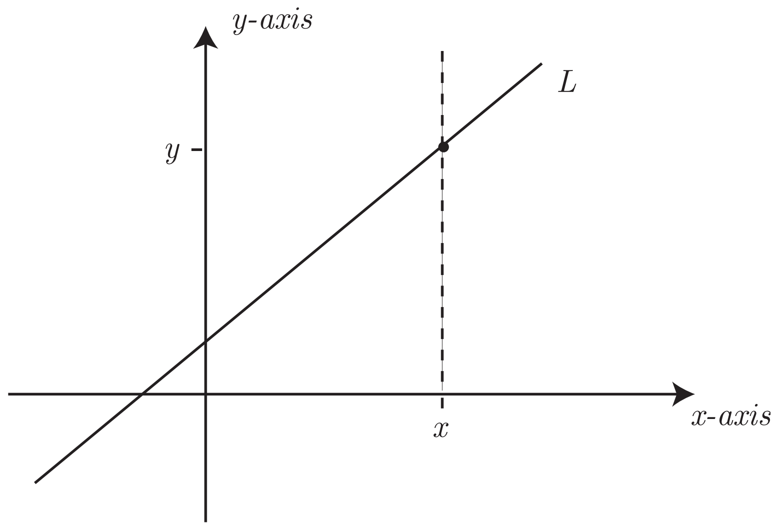

Figure 1.20: A non-vertical straight line

Now, with the above discussions as a background (which is one of the reasons we started this course with straight lines, because they should strengthen our understanding here), we are ready to extend the concept of the *unique correspondence*.

That is, in general, for a curve f given in **Figure 1.21**, we see that for any value x in the interval (a, b), the vertical straight line passing through x crosses the curve of f at exactly one point. Or, for any value x in (a, b), the curve determines a unique value y such that the point (x, y) is on the curve of f. In this sense, we say that the curve f passes the **rule of vertical line test** and hence defines a **function** y in terms of x on the interval (a, b), denoted by

$$y = f(x), \quad x \in (a, b),$$

where "$\in$" means "inside". Here, x is called the **independent variable** and y is called the **dependent variable** or **function value**, and the notation $y = f(x)$ means that f is a *rule* such that for a given value x, the rule f assigns a unique value y corresponding to x.

For a function $y = f(x)$, the collection of all x values that can be plugged in $f(x)$ (meaning that $f(x)$ gives a real number) is called the **domain** of the function $f(x)$. When x takes all values of the domain of $y = f(x)$, the collection of all y values is called the **range** of the function $f(x)$. For example, for $f(x) = 2.5x$ in the payment at the gas station, the domain is

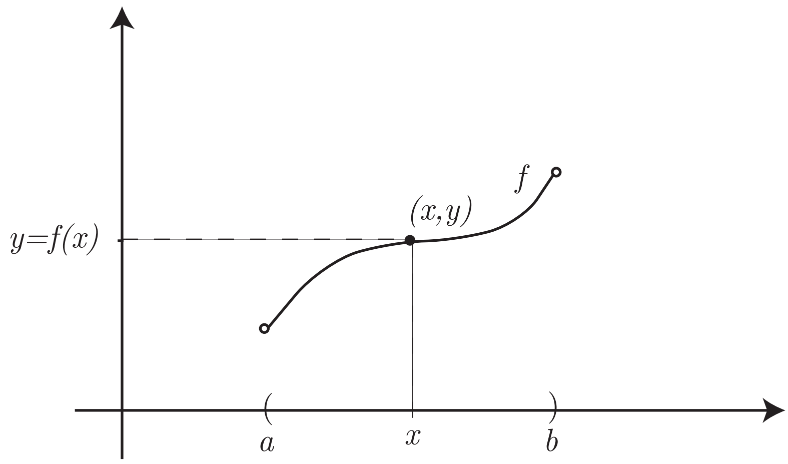

Figure 1.21: The curve f defines a function

$[0, \infty)$ and the range is also $[0, \infty)$ in that application. See **Figure 1.22** for a demonstration of a domain and a range of a function.

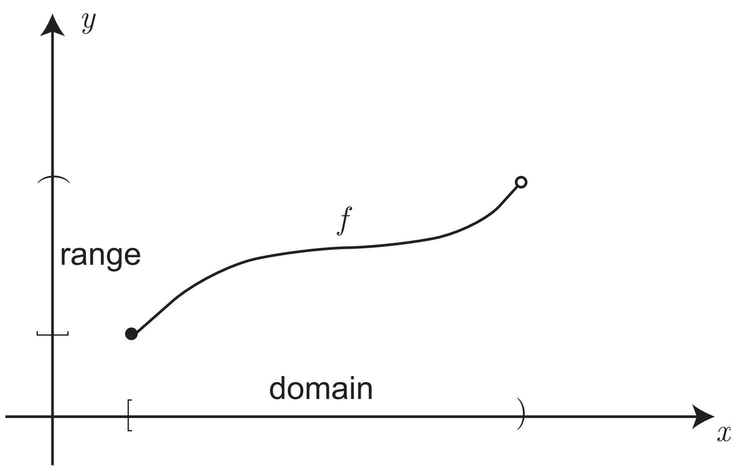

Figure 1.22: Domain and range of a function f

The above rule for a function f can also be explained in the following way: we treat the independent variable x as an **input**, and treat the function f as a **machine**, and then treat the dependent variable y as an **output**. Thus, a function f means that if you put an input into this machine f, then the machine f will produce a unique corresponding output. See **Figure 1.23**.

We know now that non-vertical straight lines define functions, and we call them **linear functions** since they are straight lines. The functions defined

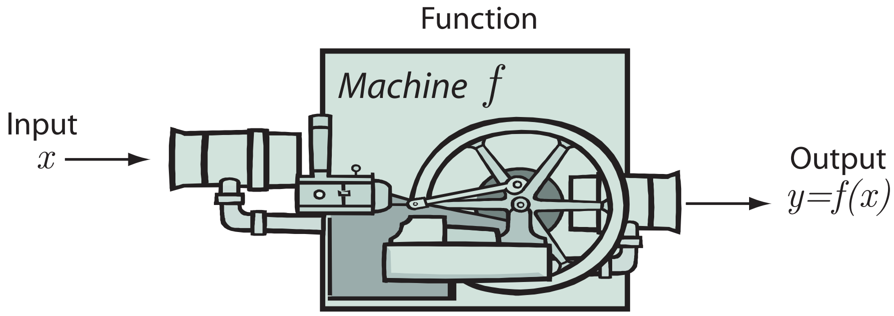

Figure 1.23: A function works like a machine

by horizontal straight lines are called **constant functions**.

Accordingly, a curve that looks like an "s" given in **Figure 1.24** does not define a function because the second picture indicates that the vertical line test fails. The second picture looks like the dollar sign \$, so a good way to understand functions is to say that

"money is not a function."

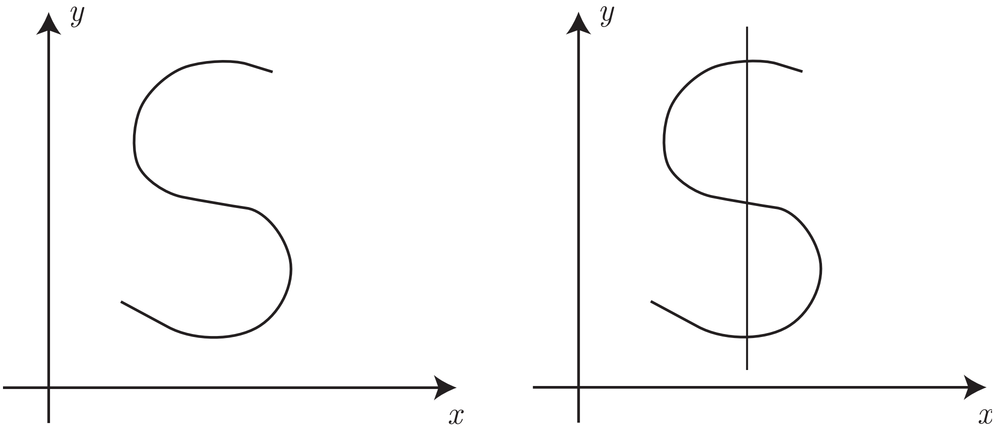

Figure 1.24: The curve "s" does not define a function

Similarly, vertical straight lines cannot define functions.

Example 1.2.1 (Payment at a gas station) Continue with the payment at the gas station, if the gas station charges \$1.25 service fee, then find the relationship between your payment and the number of gallons of gas you add.

Solution. Now, if you do not add any gas, then your payment is zero. But as long as you add gas, your payment will be the money for the gas plus the $1.25 service fee. Therefore, your payment will be

$$y = f(x) = \begin{cases} 0, & x = 0, \\ 2.5x + 1.25, & x > 0, \end{cases} \tag{2.1}$$

which is given in **Figure 1.25**, where the open hole indicates that the function is not defined there. ♠

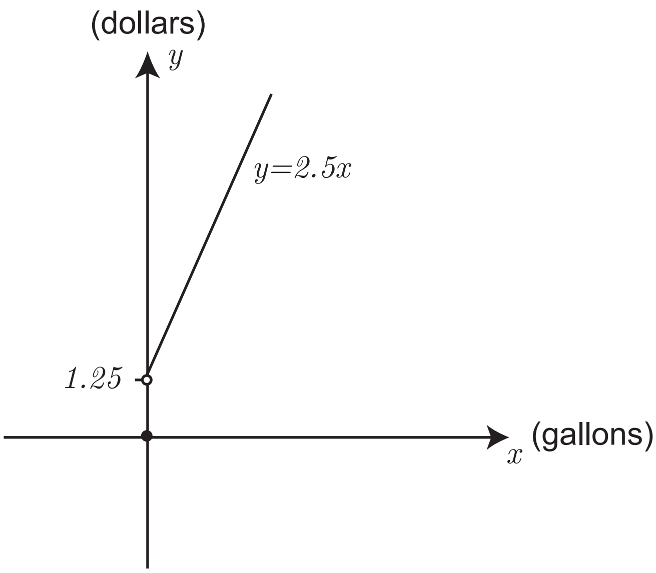

Figure 1.25: A piecewise defined function

The function in Example 1.2.1 is called a **piecewise defined function**, in the sense that it is defined *piecewisely*. The key to understand piecewise defined functions is that they may be given by different fomulas (defined differently) on different intervals. The following is such an example.

Example 1.2.2 For the following piecewise defined function

$$f(x) = \begin{cases} 2x + 1, & x \leq -1, \\ -3x + 2, & x > -1, \end{cases} \tag{2.2}$$

find $f(-3), f(-2), f(-1), f(0)$ and $f(1)$. Then draw the curve for the function.

Solution. Since $-3 < -1$, we must use $f(x) = 2x + 1$ to evaluate $f(-3) =$

$2(-3)+1 = -5$. Similarly, we get $f(-2) = 2(-2)+1 = -3$. Since -1 is still included in $x \leq -1$, we get $f(-1) = 2(-1) + 1 = -1$. Next, since $0 > -1$, we must use $f(x) = -3x + 2$ to evaluate $f(0) = -3(0) + 2 = 2$. Similarly, we get $f(1) = -3(1) + 2 = -1$.

To draw the curve, we can use two methods. The first method is to draw the straight line $2x + 1$ for *all* x values (that is, $x \in (-\infty, \infty)$), for which we can choose two points that are easy to work with, such as $(0, 1)$ and $(1, 3)$. Then we realize that the straight line $2x+1$ is only defined on $(-\infty, -1]$, so we erase the part of the straight line on $(-1, \infty)$. For $-3x + 2$, we do the same. That is, we first draw the straight line $-3x + 2$ for all $x \in (-\infty, \infty)$ and then erase the part on $(-\infty, -1]$. The resulting curve for $f(x)$ is given in **Figure 1.26**.

The second method is to evaluate on different intervals. That is, we can select two x values in $(-\infty, -1]$ and use $f(x) = 2x + 1$ to get two y values and then draw the straight line on $(-\infty, -1]$. The same can be done on $(-1, \infty)$, and we get the same curve as in Figure 1.26. ♠

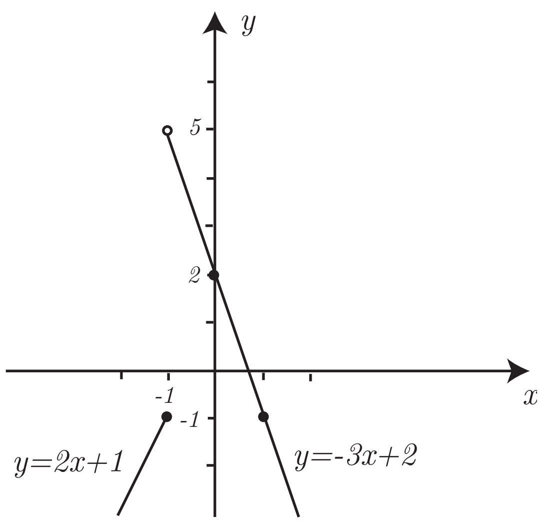

Figure 1.26: The curve of $f(x)$ in Example 1.2.2

Next, on the real number line, we define

$$|a - b| = \text{distance from a to b (or from b to a)}.$$

In particular, we define

$$|a| = |a - 0|,$$

which gives the distance from a to zero. For example, we have $|4| = 4$, $|0| =$

0, $|-5| = 5 = -(-5)$, $|7-2| = 5$, $|2-7| = 5$. From these, we conclude that

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases} \tag{2.3}$$

Therefore,

$$f(x) = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0, \end{cases} \tag{2.4}$$

defines a function, and is called the **absolute value function**. It is a special piecewise defined function, and its curve is given in **Figure 1.27**. Note that in (2.3) and (2.4), the $\geq$ and $<$ can be replaced by $>$ and $\leq$, respectively.

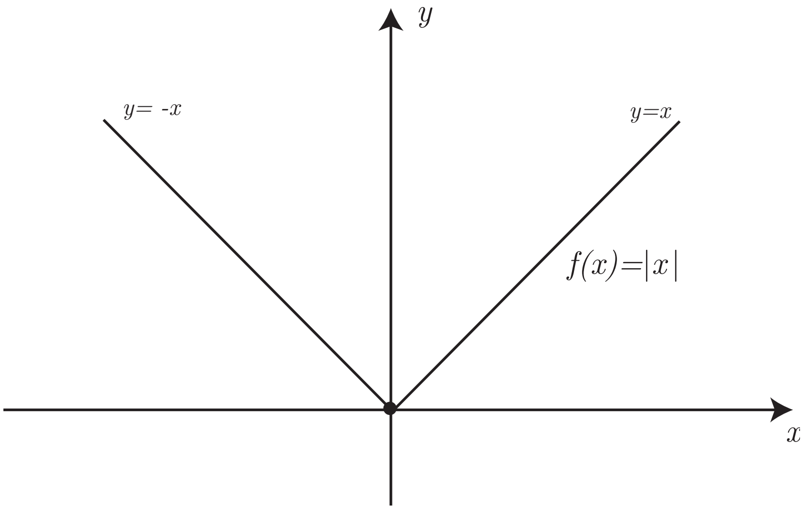

Figure 1.27: The curve of the absolute value function

Example 1.2.3 Find values of x such that

$$|2x - 3| \leq 4. \tag{2.5}$$

Solution. First, we see that $|2x-3| \leq 4$ is the same as $2|x - \frac{3}{2}| \leq 4$, or

$$|x - \frac{3}{2}| \leq 2,$$

which means that the distance from x to $\frac{3}{2}$ is less than or equal to 2, or x can go from $\frac{3}{2}$ to two sides within 2 units. Thus, x must be inside the interval with "center" $\frac{3}{2}$ and with "radius" 2. Therefore, we obtain

$$\frac{3}{2} - 2 \leq x \leq \frac{3}{2} + 2, \quad \text{or} \quad -\frac{1}{2} \leq x \leq \frac{7}{2}.$$

♠

In general,

$$|x - c| \le r$$

means

$$c - r \le x \le c + r,$$

which gives the interval where c is the center and r is the radius. See **Figure 1.28**.

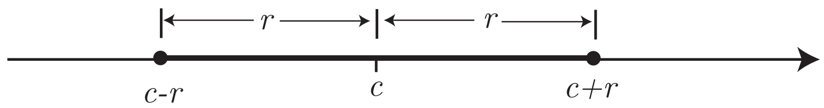

Figure 1.28: Interval $|x - c| \le r$: c is the center and r is the radius

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Example 1.2.4 (Revenue, cost, profit, and break-even) Suppose you sell certain T-shirts on campus for \$15 each. Determine the function for the money received by selling x T-shirts, called the *revenue function*, denoted by $R(x)$. If it costs you \$1.4 to sell each T-shirt (such as postage) and you paid \$110 for advertizement, then determine the function for the cost of selling x T-shirts, called the *cost function*, denoted by $C(x)$. Finally determine the function for the net income (revenue − cost) of selling x T-shirts, called the *profit function*, denoted by $P(x)$, then find when you will break even.

Solution. If you sell one T-shirt, then you get \$15; if you sell two T-shirts, then you get \$30. Thus, if you sell x T-shirts, then the money you receive, or the revenue function, is given by $R(x) = 15x$.

The \$110 paid for advertizement is called the *fixed cost*, so that the cost of selling x T-shirts, or the cost function, is given by $C(x) = 1.4x + 110$.

Next, the net income, or the profit function, is given by

$$P(x) = R(x) - C(x) = 15x - 1.4x - 110 = 13.6x - 110.$$

"Break even" means $P(x) = R(x) - C(x) = 0$, so we have

$$13.6x - 110 = 0 \quad \text{or} \quad x \approx 8,$$

that is, you need to sell 8 T-shirts to break even. ♠

Next, let $f(x)$ and $g(x)$ be functions. Then it can be verified that

$$|f(x)|,\ f(x) \pm g(x),\ f(x)g(x),\ \frac{f(x)}{g(x)},\ \sqrt[n]{f(x)}$$

are also functions if the stated operations are allowed and give real values. For example, $\frac{f(x)}{g(x)}$ defines a function when $g(x) \neq 0$; for n even, $\sqrt[n]{f(x)}$ defines a function if $f(x) \ge 0$.

Now, $f(x) = x$ and $g(x) = x$ are nonvertical straight lines, so they are functions. Then $f(x)g(x) = x \cdot x = x^2$ is also a function. This explains that for any positive integer k, x^k is a function. Then linear combinations of these x^k (when k takes different positive integers), such as

$$3x^5 + x^4 - 4x^2 + 5x + 7,$$

are also functions, and this type of functions are called **polynomial functions**. In particular, for constants a, b, and c with $a \neq 0$,

$$f(x) = ax^2 + bx + c \tag{2.6}$$

is called a **quadratic function**, and the *solutions* of $f(x) = 0$ are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.2.5 Find values of x such that

$$x^2 + x - 6 = 0. \tag{2.7}$$

Solution. Comparing with the quadratic function (2.6), we have $a = 1$, $b = 1$, and $c = -6$, so from the quadratic formula we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2} = \frac{-1 \pm 5}{2} = 2 \text{ or } -3.$$

Another way is to factor $x^2 + x - 6$ into

$$x^2 + x - 6 = (x - 2)(x + 3),$$

where $(x - 2)$ and $(x + 3)$ are called **linear factors**, and then obtain $x = 2$ or $x = -3$. ♠

A polynomial function divided by another polynomial function, such as

$$\frac{x^4 - 4x^2 + 5}{x^6 - 5x + 2},$$

is called a **rational function**.

To understand other functions and their operations, let's look at the following simple examples,

$$\begin{aligned} a^3 \cdot a^2 &= (a \cdot a \cdot a)(a \cdot a) = a^5 = a^{3+2}, \\ (a^3)^2 &= (a^3)(a^3) = (a \cdot a \cdot a)(a \cdot a \cdot a) = a^6 = a^{3 \cdot 2}, \\ \frac{a^5}{a^2} &= \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3 = a^{5-2}. \end{aligned}$$

Based on these examples, let's recall that we have the following:

Laws of Exponents. Let a and b be positive numbers that are different from 1, and let r and s be any real numbers. Then

$$\begin{aligned} &b^r b^s = b^{r+s}, \quad (b^r)^s = b^{rs}, \\ &(ab)^r = a^r b^r, \quad \frac{b^r}{b^s} = b^{r-s}, \\ &b^{-r} = \frac{1}{b^r}, \quad (\frac{a}{b})^r = \frac{a^r}{b^r}, \\ &b^r = b^s \text{ implies } r = s. \end{aligned}$$

Example 1.2.6 Find values of x such that $3^{x-1} = 27$.

Solution. Since $27 = 3^3$, we have

$$3^{x-1} = 3^3.$$

Now, we can use laws of exponents to derive $x - 1 = 3$. Thus, $x = 4$. ♠

For a real value r, if x^r defines a function, then

$$y = x^r$$

is called a **power function** with power r. For example, using laws of exponents,

$$y = x^{-3/2} = \frac{1}{(\sqrt{x})^3} \quad \text{on domain } (0, \infty)$$

defines a power function with power $-\frac{3}{2}$. For example, $4^{-3/2} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}$.

Next, let's look at some operations and evaluations of functions.

Example 1.2.7 For $f(x) = 1$, find $f(2)$, $f(w)$, $f(x+h)$, $\frac{f(w)-f(x)}{w-x}$ $(w \neq x)$ and $\frac{f(x+h)-f(x)}{h}$ $(h \neq 0)$.

Solution. Now, the function (machine) is such that for all inputs, the output is always 1. Thus, we get $f(2) = 1$, $f(w) = 1$, $f(x+h) = 1$, where $2, w$ and $(x+h)$ are regarded as inputs.

Of course, this matches with geometry because $y = 1$ is a horizontal line.

With these, we get

$$\begin{aligned} \frac{f(w) - f(x)}{w - x} &= \frac{1-1}{w-x} = 0, \\ \frac{f(x+h) - f(x)}{h} &= \frac{1-1}{h} = 0. \end{aligned}$$

Another way to evaluate $f(x+h)$ is to rewrite $f(x) = 1$ (for all real values x) as $f(w) = 1$ (for all real values w) and then regard the combination $(x+h)$ as "one value", that is, let $w = x + h$, to obtain $f(x+h) = f(w) = 1$. ♠

Example 1.2.8 For $f(x) = (3x+1)^2+4$, find $f(2)$, $f(w)$, $f(x+h)$, $\frac{f(w)-f(x)}{w-x}$ $(w \neq x)$ and $\frac{f(x+h)-f(x)}{h}$ $(h \neq 0)$.

Solution. In this case, the function (machine) is to multiply an input by 3, then add 1, then square the whole thing, and finally add 4. Thus we get

$$\begin{aligned} f(2) &= (3 \cdot 2 + 1)^2 + 4 = 53, \\ f(w) &= (3w + 1)^2 + 4. \end{aligned}$$

For $f(x + h)$, we regard $(x + h)$ as an input, so we get

$$f(x + h) = (3(x + h) + 1)^2 + 4.$$

With these, we get

$$\begin{aligned} \frac{f(w) - f(x)}{w - x} &= \frac{[(3w + 1)^2 + 4] - [(3x + 1)^2 + 4]}{w - x} \\ &= \frac{(3w + 1)^2 - (3x + 1)^2}{w - x} \\ &= \frac{9w^2 + 6w + 1 - 9x^2 - 6x - 1}{w - x} \\ &= \frac{9(w - x)(w + x) + 6(w - x)}{w - x} = 9(w + x) + 6, \end{aligned}$$

where we have used $A^2 - B^2 = (A - B)(A + B)$. Also, we get

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{[(3(x + h) + 1)^2 + 4] - [(3x + 1)^2 + 4]}{h} \\ &= \frac{(3(x + h) + 1)^2 - (3x + 1)^2}{h} \\ &= \frac{9(x + h)^2 + 6(x + h) + 1 - 9x^2 - 6x - 1}{h} \\ &= \frac{9x^2 + 18xh + 9h^2 + 6h - 9x^2}{h} \\ &= \frac{18xh + 9h^2 + 6h}{h} = 18x + 9h + 6. \end{aligned}$$

Another way to evaluate $f(x+h)$ is to rewrite $f(x) = (3x+1)^2+4$ (for all real values x) as $f(w) = (3w + 1)^2 + 4$ (for all real values w) and then regard the combination $(x + h)$ as "one value", that is, let $w = x + h$, to obtain $f(x + h) = f(w) = (3w + 1)^2 + 4 = (3(x + h) + 1)^2 + 4$. ♠

For other functions, such as $F(x) = f(x)g(x)$ and $G(x) = \frac{f(x)}{g(x)}$, we should do the same. For example, regarding $(x + h)$ as an input, we get

$$F(x + h) = f(x + h)g(x + h), \quad G(x + h) = \frac{f(x + h)}{g(x + h)}.$$

Our experience with the above functions indicates that we have another way to understand $(3x+1)^2$. That is, if we define

$$y = z^2, \quad z = 3x + 1,$$

and then plug $z = 3x + 1$ into $y = z^2$, we will get $y = z^2 = (3x + 1)^2$ back. Here, we used z as an intermediate variable to make the "plugging in" easier to understand. In fact, we can just use x as independent variable and define

$$f(x) = x^2, \quad g(x) = 3x + 1,$$

and then treat $g(x) = 3x + 1$ as "one value" and plug into x in $f(x) = x^2$ (or z in $f(z) = z^2$) and obtain the same result,

$$f(g(x)) = (g(x))^2 = (3x + 1)^2,$$

because the function (machine) f squares its input $g(x) = 3x + 1$.

In general, for two functions $f(x)$ and $g(x)$, we call

$$f(g(x))$$

the **composition** of the functions f and g, if $g(x)$ is inside the domain of the function f. Similarly, $g(f(x))$ is called the composition of g and f. Other compositions are defined in the same way.

Example 1.2.9 For $f(x) = \sqrt{x}$, $g(x) = x^3$, find $f(g(x))$, $g(f(x))$, $g(g(x))$, and $f(f(x))$.

Solution. Note that the function $f(x)$ is only defined for $x \geq 0$; g is defined for all x. For $f(g(x))$, the function (machine) f takes a square root of an input, so we treat $g(x) = x^3$ as an input to obtain

$$f(g(x)) = \sqrt{g(x)} = \sqrt{x^3} = x^{3/2}, \quad \text{for } x \geq 0.$$

Similarly, we have

$$\begin{aligned} g(f(x)) &= (f(x))^3 = (\sqrt{x})^3 = x^{3/2}, \quad \text{for } x \geq 0, \\ g(g(x)) &= (g(x))^3 = (x^3)^3 = x^9, \quad \text{for all } x, \\ f(f(x)) &= \sqrt{f(x)} = \sqrt{\sqrt{x}} = (x^{1/2})^{1/2} = x^{1/4}, \quad \text{for } x \geq 0. \end{aligned}$$

♠

Inverse Function.

Next, let's look at the curves shown in **Figure 1.29**.

The curves define two functions f and g because they pass the vertical line test. But they are different. For the function f, if you start with a y value y_1 and draw a horizontal line, then the line will cross the curve at exactly

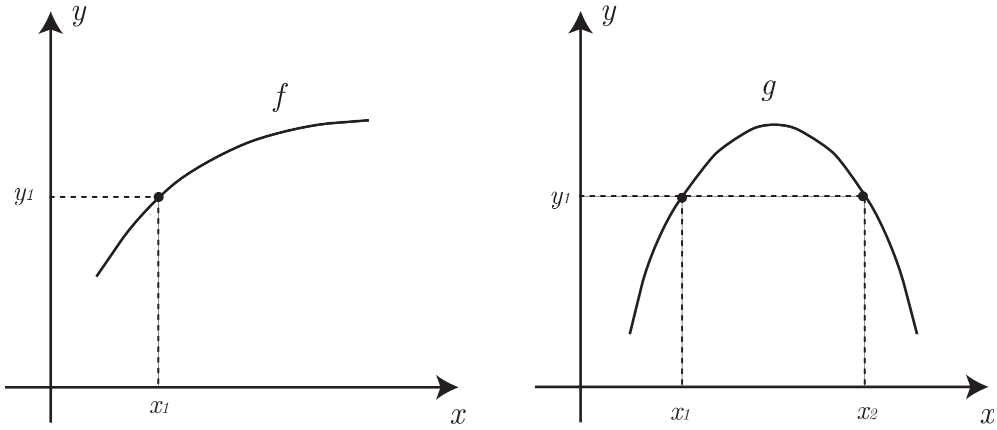

Figure 1.29: The difference between the two curves

one point to determine a unique x value x_1 (which we call the **horizontal line test**). In this sense, the curve of f defines another function, called the **inverse function** of f, denoted by f^{-1}, when we let y be the independent variable and let x be the dependent variable.

The function g has no inverse function because the horizontal line test fails for g. That is, for the y value y_1, there are two x values x_1 and x_2 corresponding to y_1, so that we cannot uniquely determine a corresponding x value.

Note that we are so used to use x for independent variable and y for dependent variable and draw a function in the xy-plane, so that after we find the rule for an inverse function, we switch x and y so as to get a function with independent variable x and dependent variable y. We explain this using the following example.

Example 1.2.10 Find the inverse function of $f(x) = 2x$, if it exists.

Solution. The curve of the function $f(x) = 2x$ is a non-vertical straight line, so it passes the horizontal line test and hence has an inverse. For a given y value, the unique x value is determined by solving x from $y = 2x$, so we obtain

$$x = \frac{1}{2}y. \tag{2.8}$$

This gives the rule defining the inverse function, where an output is to multiply by $\frac{1}{2}$ to an input. To obtain such an inverse function with independent variable x and dependent variable y, we switch x and y in (2.8) to derive

$$y = \frac{1}{2}x, \quad \text{or} \quad f^{-1}(x) = \frac{1}{2}x, \tag{2.9}$$

which satisfies the same rule as in (2.8): take an independent value and multiply by $\frac{1}{2}$ to get the dependent value. ♠

This example indicates that after we determine that a function $y = f(x)$ has an inverse, then we do the following to find its inverse:

1. Solve x in terms of y from $y = f(x)$,

2. Switch x and y to get the inverse function.

Example 1.2.11 Find the inverse function of $f(x) = x^2 + 1$, $x \in [0, 2]$, if it exists.

Solution. Using the knowledge of quadratic functions or a graphing calculator (the subject on *curve sketching* will be covered later, but for our purpose here, you can use a graphing calculator to get graphs), we see that $f(x)$ has inverse on $[0, 2]$ (note that it has no inverse on $[-2, 2]$ or similar intervals). To find the inverse function, we solve x from $y = x^2 + 1$ to obtain

$$x = \sqrt{y-1}.$$

Then we switch x and y to derive the inverse function as

$$y = \sqrt{x-1}, \quad \text{or} \quad f^{-1}(x) = \sqrt{x-1} \quad \text{with domain} \quad [1, 5],$$

where the domain, $[1, 5]$, is the same as the range of $f(x) = x^2 + 1$ on $[0, 2]$. ♠

If we compose the functions f and f^{-1} in Example 1.2.10, then we get

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x,$$
$$f(f^{-1}(x)) = f(\frac{1}{2}x) = 2(\frac{1}{2}x) = x,$$

that is, we start with x and get back to x after applying $f^{-1}(f)$ or $f(f^{-1})$. This explains why f^{-1} is called the *inverse* of f: it undoes what the f does. In this sense, we also say that f is the inverse of f^{-1}, or say that f and f^{-1} are inverse of each other.

Note that

$$f^{-1}(f(x)) = x, \tag{2.10}$$
$$f(f^{-1}(x)) = x, \tag{2.11}$$

are true for any function f with an inverse f^{-1}. For example, you can check (see exercises) that (2.10) and (2.11) are true for the functions in Example 1.2.11.

If we draw $f(x) = 2x$ and $f^{-1}(x) = \frac{1}{2}x$ in Example 1.2.10 in one xy-plane, given in **Figure 1.30**, then we find something very interesting. That is, $f(x) = 2x$ and $f^{-1}(x) = \frac{1}{2}x$ are symmetric about the middle line $y = x$. We leave it to you to check (see exercises; you can use a graphing calculator here) that $f(x) = x^2 + 1$ and $f^{-1}(x) = \sqrt{x-1}$ in Example 1.2.11 are also symmetric about the middle line $y = x$. In fact, it is true in general that a function and its inverse are always symmetric about the middle line $y = x$.

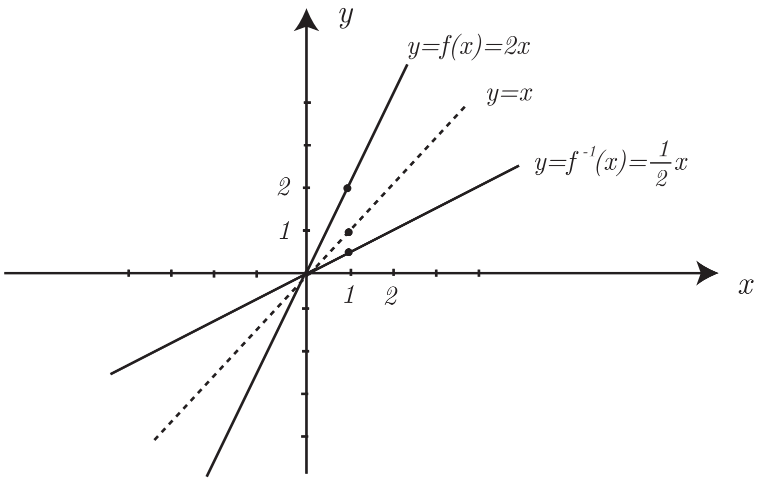

Figure 1.30: The curves of $f(x) = 2x$ and $f^{-1}(x) = \frac{1}{2}x$

Inequalities.

Later on, we will need to find where a given function is positive and where it is negative. So we make some preparation here.

Example 1.2.12 For $f(x) = 2x - 1$, find where it is positive and where it is negative.

Solution. The curve of $f(x) = 2x - 1$ is a straight line, given in **Figure 1.31**, so we call $2x - 1$ is a **linear factor**. We see from Figure 1.31 that the straight line $2x - 1$ crosses the x-axis at $\frac{1}{2}$, so that $2x - 1$ has a fixed sign on the right-hand side of $\frac{1}{2}$, which is positive; and $2x - 1$ has a fixed sign on the left-hand side of $\frac{1}{2}$, which is negative. That is, $f(x)$ is positive on $(\frac{1}{2}, \infty)$, negative on $(-\infty, \frac{1}{2})$. The second picture in Figure 1.31 is called the **sign chart** of $f(x)$. ♠

Example 1.2.13 For $f(x) = x^2 - 1$, find where it is positive and where it is negative.

Solution. By using the identity

$$A^2 - B^2 = (A - B)(A + B),$$

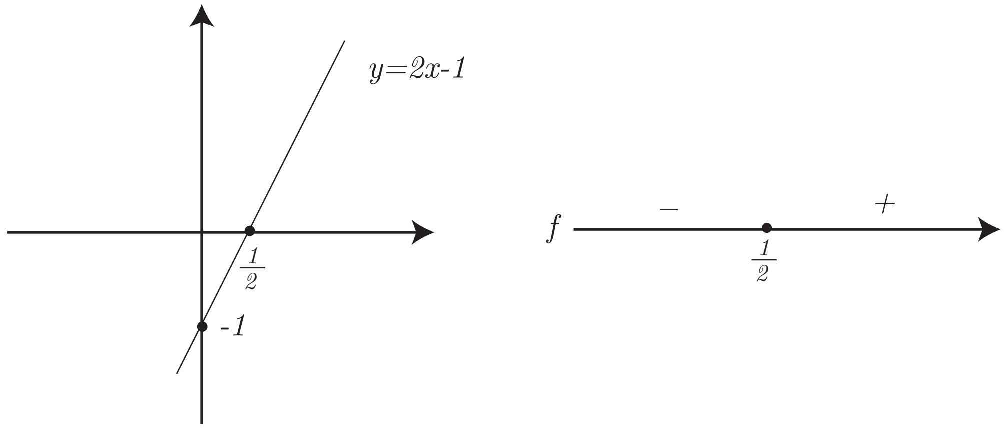

Figure 1.31: The graph of the linear factor $2x - 1$ and its sign chart

we obtain

$$f(x) = (x-1)(x+1).$$

Thus the sign of $f(x)$ is determined by the two linear factors $x - 1$ and $x + 1$, and each can be treated using the idea of Example 1.2.12. Now, $f(x) = 0$ only at $x = -1$ and 1, and -1 and 1 divide the x-axis into three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. For $x \in (-\infty, -1)$, since each linear factor ($x - 1$ or $x + 1$) has a fixed sign, $f(x)$ also has a fixed sign. To find this sign, we can test $f(x)$ at any point in $(-\infty, -1)$, such as at $x = -100$:

$$f(-100) = (-100-1)(-100+1) = (\text{negative})(\text{negative}) = \text{positive},$$

thus we conclude that $f(x)$ must be positive on $(-\infty, -1)$.

Similarly, we can test $f(x)$ at $x = 0$ and $x = 100$ to conclude that $f(x) < 0$ on $(-1, 1)$ and $f(x) > 0$ on $(1, \infty)$. Of course, this matches with the curve of $f(x) = x^2 - 1$ in **Figure 1.32**. (For our purposes here, you can use a graphing calculator or your general knowledge to sketch $f(x) = x^2 - 1$.) ♠

The ideas and details used in the above two examples can also be applied to many other situations. That is, if an expression is given as a product or division using linear factors, then we can determine the signs of the linear factors and then determine the sign of the expression.

Example 1.2.14 Find values of x such that

$$(x-1)(x-3)(x-5) < 0.$$

Solution. Let $f(x) = (x-1)(x-3)(x-5)$, then the sign of $f(x)$ is determined by the three linear factors. Similar to Example 1.2.13, we see that 1, 3, and 5

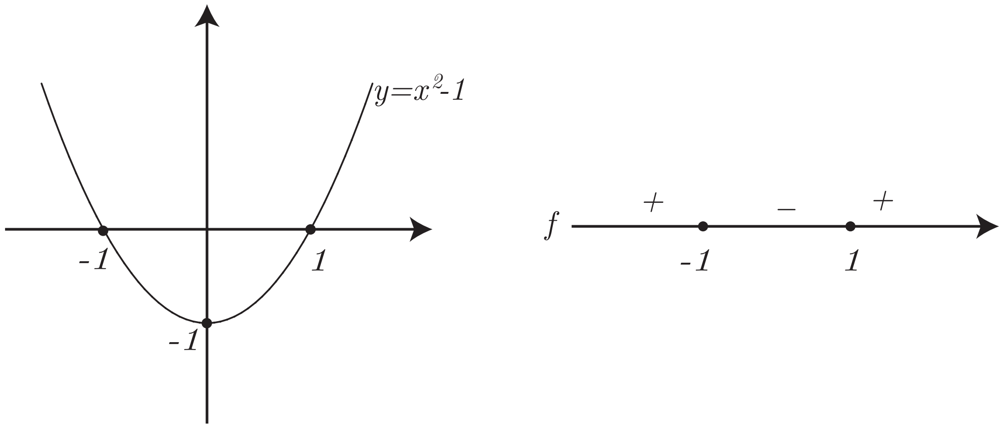

Figure 1.32: The curve of $f(x) = x^2 - 1$ and its sign chart

divide the x-axis into four intervals $(-\infty, 1)$, $(1, 3)$, $(3, 5)$, and $(5, \infty)$. Now, we can test $f(x)$ at $x = 0$, 2, 4, and 6 to derive the sign chart of $f(x)$ in **Figure 1.33**, from which we find that $f(x) < 0$ on $(-\infty, 1)$ and on $(3, 5)$. ♠

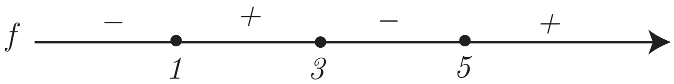

Figure 1.33: The sign chart of $f(x) = (x - 1)(x - 3)(x - 5)$

Example 1.2.15 Find values of x such that

$$\frac{1}{(x-2)^3} < 0.$$

Solution. We should not multiply $(x - 2)^3$ on both sides because we don't know the sign of $(x - 2)^3$, so we don't know whether we need to keep or change the inequality. Now, the numerator is positive, so the sign of $f(x)$ is determined by the denominator $(x - 2)^3$. Moreover, the power of $x - 2$ is 3, an odd number, so the sign of $(x - 2)^3$ is determined by $x - 2$. That is, $\frac{1}{(x-2)^3} < 0$ when $x - 2 < 0$, or $x < 2$. ♠

Example 1.2.16 Find values of x such that

$$\frac{x^2 + x - 6}{x^2 - 3x - 4} < 0.$$

Solution. Similar to the previous example, we should not multiply $x^2 - 3x - 4$ on both sides. Instead, we rewrite the inequality as

$$\frac{(x-2)(x+3)}{(x-4)(x+1)} < 0.$$

Then the sign of $f(x) = \frac{(x-2)(x+3)}{(x-4)(x+1)}$ is determined by the four linear factors. Similar to Example 1.2.13, we see that -3, -1, 2, and 4 divide the x-axis into five intervals $(-\infty, -3)$, $(-3, -1)$, $(-1, 2)$, $(2, 4)$, and $(4, \infty)$. Now, we can test $f(x)$ at $x = -100$, -2, 0, 3, and 100 to derive the sign chart of $f(x)$ in **Figure 1.34**, from which we find that $f(x) < 0$ on $(-3, -1)$ and on $(2, 4)$. ♠

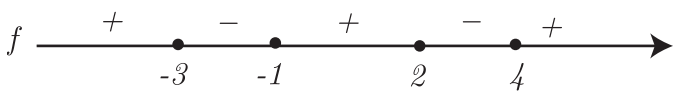

Figure 1.34: The sign chart of $f(x) = \frac{(x-2)(x+3)}{(x-4)(x+1)}$

Example 1.2.17 Find values of x such that

$$\frac{2x^2 - 2x - 10}{x^2 - 3x - 4} < 1.$$

Solution. In this case, we change $\frac{2x^2-2x-10}{x^2-3x-4} < 1$ to

$$\frac{2x^2 - 2x - 10}{x^2 - 3x - 4} - 1 < 0 \quad \text{or} \quad \frac{x^2 + x - 6}{x^2 - 3x - 4} < 0,$$

which is the same as in Example 1.2.16, from which we know that $f(x) < 0$ on $(-3, -1)$ and on $(2, 4)$. ♠

Note in Example 1.2.17 that if we keep 1 on the right-hand side, then the sign chart would not help. Thus we move 1 to the left such that the right-hand side is 0 and then the sign chart will help.

Exercises 1.2

1. Find the domain for

(a) $f(x) = 2$.

(b) $f(x) = x - 2$.

(c) $f(x) = x^3 + 2$.

(d) $f(x) = \sqrt{2x}$.

(e) $f(x) = \sqrt{2-x}$.

(f) $f(x) = \sqrt[3]{x} - x$.

(g) $f(x) = \frac{-2}{\sqrt{2x}}$.

(h) $f(x) = \frac{x-2}{\sqrt[3]{2x}}$.

2. Find the domain and range for the function in Example 1.2.1.

3. Draw the curves for the following functions

$$f(x) = \begin{cases} -2x+5, & x \le 2, \\ 3x-4, & x > 2. \end{cases} \qquad g(x) = \begin{cases} -2x+5, & x \le -1, \\ 3x-4, & -1 < x < 6, \\ -5x-3, & x \ge 6. \end{cases}$$

4. Draw the curve (use a calculator if needed) for $f(x) = \begin{cases} x^2+6, & \text{if } x \neq 0, \\ 2, & \text{if } x = 0. \end{cases}$

5. Find $|3|$, $|-9|$, $|8-1|$, $|1-8|$, $|x-1|$, and $|1-x|$ (that is, get rid of the absolute value sign $|\cdot|$).

6. Find values of x such that $|3x-5| < 8$.

7. Find values of x such that $|3x-5| \ge 8$.

8. Draw the curves for the following functions

$$f(x) = \begin{cases} |-2x+5|, & x \le 2, \\ |3x-4|, & x > 2. \end{cases} \qquad g(x) = \begin{cases} |-2x+5|, & x \le -1, \\ |3x-4|, & -1 < x < 6, \\ |-5x-3|, & x \ge 6. \end{cases}$$

9. Suppose you sell certain T-shirts on campus for \$12 each. Determine the revenue function for the money received by selling x T-shirts. If it costs you \$1.1 to sell each T-shirt (such as postage) and you paid \$90 for advertizement, then determine the cost function for the cost of selling x T-shirts. Finally determine the profit function for the net income of selling x T-shirts, and when you will break even.

10. Solve the following equations by factoring the functions into linear factors.

(a) $x^2 - 9 = 0$.

(b) $x^2 - 3 = 0$.

(c) $x^2 + x - 2 = 0$.
(d) $x^3 - 7x^2 + 10x = 0$.

11. Evaluate $9^{1/2}$, $9^{3/2}$, $81^{3/2}$, $8^{1/3}$, $8^{2/3}$, $9^{-3/2}$, and $64^{-2/3}$.

12. Find values of x such that $2^{3x+2} = 16$.

13. Find values of x such that $8^{3x+2} = 16$.

14. Find values of x such that $2^{2x} - 5 \cdot 2^x + 4 = 0$.

15. Find $f(1)$, $f(w)$, $f(x+h)$, $\frac{f(w)-f(x)}{w-x}$ $(w \neq x)$ and $\frac{f(x+h)-f(x)}{h}$ $(h \neq 0)$ for

(a) $f(x) = -9$.
(b) $f(x) = 2x^2 - x$.
(c) $f(x) = \sqrt{x}$.
(d) $f(x) = 4 - \sqrt{3x}$.
(e) $f(x) = \frac{1}{x}$.
(f) $f(x) = 3 - \frac{2}{x}$.

16. For $f(x) = x^5$, $g(x) = \sqrt{1-x}$, find $f(g(x))$, $g(f(x))$, $f(f(x))$, and $g(g(x))$.

17. Find the inverse function of $f(x) = 3x - 1$, if it exists.

18. Find the inverse function of $f(x) = 2x^3 + 2$, if it exists.

19. Find the inverse function of $f(x) = x^4 - 3$, if it exists.

20. Check that (2.10) and (2.11) are true for the functions in Example 1.2.11.

21. Check (you can use a graphing calculator here) that $f(x) = x^2 + 1$ and $f^{-1}(x) = \sqrt{x-1}$ in Example 1.2.11 are symmetric about the middle line $y = x$.

22. For $f(x) = x^2 - 4$, find where it is positive and where it is negative.

23. Find values of x such that

(a) $(x-1)(x-2)(x-3) < 0$.
(b) $\frac{-2}{x+4} > 0$.
(c) $\frac{x^2-x-30}{x^2-2x-63} < 0$.
(d) $\frac{3x^2-5x-156}{x^2-2x-63} > 2$.
(e) $e^{-x} - \frac{1}{2} > 0$.
(f) $e^{-x} - e^{-\ln 2} < 0$.

1.3 Exponential Functions

Question: *If you deposit \$1 in a bank with an annual interest rate of* 100%, *then how much can you get after one year if the bank can compound the interest as many times as you like? In particular, can you make a fortune this way or will your return always be limited?*

In order to understand and solve this and other related problems, let's start with a simple example.

Example 1.3.1 (Deposit and interest) You deposit \$1 in a bank with an annual interest rate of 3% which is compounded annually (apply interest after every year). Find the amount in the account at the end of one year, two years, three years, and x years.

Solution. Let's use $A(x)$ to denote the amount after x years. Then $A(1)$ equals the principal, \$1, plus the principal multiplied by the interest rate, 0.03. That is,

$$A(1) = 1 + 1(0.03) = 1 + 0.03 = 1.03.$$

To find $A(2)$, one way is to regard $A(1) = 1.03$ as a new principal and deposit $A(1) = 1.03$ for one year. Thus,

$$A(2) = 1.03 + 1.03(0.03) = 1.03(1 + 0.03) = (1.03)(1.03) = 1.03^2.$$

Similarly,

$$A(3) = A(2) + [A(2)](0.03) = 1.03^2 + 1.03^2(0.03) = 1.03^2(1.03) = 1.03^3.$$

Based on these, we infer that the amount after x years should be given by

$$A(x) = 1.03^x. \tag{3.1}$$

♠

$A(x) = 1.03^x$ in (3.1) defines a function, and this function is such that the independent variable x appears as the exponent, and the base is a constant 1.03. Accordingly, we make the following definition.

Definition 1.3.2 *Let $b > 0$ and $b \neq 1$. Then the function*

$$f(x) = b^x \text{ on the domain } (-\infty, \infty)$$

is called the **exponential function with base** *b.*

Remark 1.3.3 To present a rigorous treatment for exponential functions, such as understanding things like $1.32^{0.53}$, we have to wait until finishing some subjects of *integral calculus*. However, for our purposes, the simple treatment given here is still valid for this elementary level calculus course, and it provides us with more functions as we learn calculus and its applications. See Remark 6.2.11 in Chapter 6. ♠

We require $b > 0$ because otherwise if $b = -1$ and $x = \frac{1}{2}$, then $b^x = \sqrt{-1} = i$ is the complex unit, not a real number. We also require $b \neq 1$ because otherwise if $b = 1$ then $b^x = 1^x = 1$ is a constant, which is not interesting to us.

Example 1.3.1 is a case where the interest is applied only once per year (compounded annually). Other cases may happen.

Example 1.3.4 (Deposit and interest) You deposit \$5 in a bank with an annual interest rate of 3% which is compounded semi-annually (apply interest after every six months). Find the amount in the account at the end of one year, two years, three years, and x years.

Solution. Let's use $A(x)$ to denote the amount after x years. Since the interest is compounded twice per year, the bank will pay interest after every six months with a rate of $\frac{0.03}{2}$.

For $A(1)$, let's first find the amount after the first six months, which is given by

$$5 + 5(\frac{0.03}{2}) = 5(1 + \frac{0.03}{2}).$$

Now, to find $A(1)$, we regard $5(1 + \frac{0.03}{2})$ as a new principal and deposit it for another six months with the rate of $\frac{0.03}{2}$. Thus,

$$A(1) = 5(1 + \frac{0.03}{2}) + 5(1 + \frac{0.03}{2})(\frac{0.03}{2}) = 5(1 + \frac{0.03}{2})^2.$$

To find $A(2)$, we follow the same idea and regard $A(1) = 5(1 + \frac{0.03}{2})^2$ as a new principal and deposit it for six months and then another six months (all with the rate of $\frac{0.03}{2}$). Thus,

$$\begin{aligned} A(2) &= [5(1 + \frac{0.03}{2})^2 + 5(1 + \frac{0.03}{2})^2(\frac{0.03}{2})] \\ &\quad +[5(1 + \frac{0.03}{2})^2 + 5(1 + \frac{0.03}{2})^2(\frac{0.03}{2})](\frac{0.03}{2}) \\ &= 5(1 + \frac{0.03}{2})^3 + 5(1 + \frac{0.03}{2})^3(\frac{0.03}{2}) \\ &= 5(1 + \frac{0.03}{2})^4 = 5(1 + \frac{0.03}{2})^{2\times 2}. \end{aligned}$$

Similarly,

$$\begin{aligned} A(3) &= [A(2) + A(2)(\frac{0.03}{2})] + [A(2) + A(2)(\frac{0.03}{2})](\frac{0.03}{2}) \\ &= 5(1 + \frac{0.03}{2})^5 + 5(1 + \frac{0.03}{2})^5(\frac{0.03}{2}) \\ &= 5(1 + \frac{0.03}{2})^6 = 5(1 + \frac{0.03}{2})^{2\times 3}. \end{aligned}$$

Based on these, we infer that the amount after x years should be given by

$$A(x) = 5(1 + \frac{0.03}{2})^{2x}. \tag{3.2}$$

♠

In general, we have the following result (see exercises): If you deposit a principal P in a bank at an annual interest rate r which is compounded m times per year (the year is divided into m equal periods and the interest rate during each period is $\frac{r}{m}$), then at the end of the xth year, the amount is given by

$$A_m(x) = P\Big(1 + \frac{r}{m}\Big)^{mx}. \tag{3.3}$$

Example 1.3.5 (Comparing banks) Assume that Bank A offers an annual interest rate of 3.8% which is compounded ten times per year, and Bank B offers an annual interest rate of 3.85% which is compounded two times per year. To which bank should you deposit your money?

Solution. We will use (3.3) to compare, and we only need to consider the case $x = 1$ (deposit for one year) since other cases are the same. Let P be your money. If you deposit to Bank A, then $r = 0.038$ and $m = 10$, thus, your return after one year is

$$P\Big(1 + \frac{0.038}{10}\Big)^{10} = 1.0386559P.$$

If you deposit to Bank B, then $r = 0.0385$ and $m = 2$, thus, your return after one year is

$$P\Big(1 + \frac{0.0385}{2}\Big)^{2} = 1.0388705P.$$

Accordingly, you should deposit your money to Bank B.

However, note that if Bank B compounds only once per year, then the return after one year becomes $P\Big(1 + \frac{0.0385}{1}\Big)^{1} = 1.0385P$, which is less than $1.0386559P$ from Bank A, so you should deposit your money to Bank A in that case. ♠

Now, we are ready to answer the question raised at the beginning of this section. That is, we can look at a special case of (3.3) where we let $P = 1$, $r = 1$ (that is 100% ! If you can find such a bank, tell us first! We use these kinds of examples in order to look at some *extreme cases*), $x = 1$, and let m be arbitrary. That is, you deposit \$1 in a bank for one year with an annual interest rate of 100% compounded m times per year. Then at the end of one year, your return is given by

$$A_m = \Big(1 + \frac{1}{m}\Big)^{m}. \tag{3.4}$$

For example, if $m = 1$ (compounded annually), then $A_1 = 1 + 1 = 2$. If $m = 2$ (compounded semi-annually), then $A_2 = (1 + \frac{1}{2})^2 = 2.25$, which is bigger than A_1, as expected. That is, when m is increased, A_m will also be

m	A_m
1	2
2	2.25
10	2.59374
1,000	2.71692
1,000,000	2.71828

Figure 1.35: The table for the number e

increased. Will your return be increased without bound (so you will make a big fortune) if the bank compounds the interest millions of millions of times in one year? The following table in **Figure 1.35** tells you that you are not going to get rich this way.

That is, when m gets very big, the amount A_m gets very close to $2.71828\cdots$, where the decimals never end. Accordingly, we define this number (which is an irrational number) as

$$e = 2.71828\cdots,$$

which is similar to how we understand $\pi = 3.14\cdots$, where the decimals never end. The letter e was chosen for this number to honor the great mathematician Leonhard Euler (1707-1783) who discovered many properties and their applications of this number.

The irrational number e is a very important constant in mathematics and can also be derived from other studies. What we did above provides a very easy way to understand it, that is, we can regard e as the amount in the account if \$1 is deposited in a bank for one year with an annual interest rate of 100% compounded millions of millions of times in one year.

The number e is positive and different from 1, thus we have the following definition.

Definition 1.3.6 *The function*

$$f(x) = e^x \text{ on the domain } (-\infty, \infty)$$

is called the **exponential function with base** e*.*

The function e^x is also called the **natural exponential function** because, as we will see later, e^x appears *naturally* in many applications; and the function e^x is built into scientific calculators for you to evaluate its values.

Example 1.3.7 Draw the curves for $f(x) = 3^x$, $g(x) = (\frac{1}{3})^x = 3^{-x}$, $h(x) = e^x$, and $q(x) = (\frac{1}{e})^x = e^{-x}$.

Solution. The subject of *curve sketching* will be studied in detail later on in the course. However, for our purposes here, we can just evaluate the functions at several x values and then use curves to link the points.

For example, for $f(x) = 3^x$, we have $f(0) = 1$, $f(1) = 3$, $f(2) = 9$, $f(-1) = \frac{1}{3}$, and $f(-2) = \frac{1}{9}$, so roughly, the curve of $f(x) = 3^x$ is given in **Figure 1.36**.

For $g(x) = (\frac{1}{3})^x = 3^{-x}$, we have $g(0) = 1$, $g(1) = \frac{1}{3}$, $g(2) = \frac{1}{9}$, $g(-1) = 3$, and $g(-2) = 9$, so a rough sketch of $g(x) = (\frac{1}{3})^x = 3^{-x}$ is given as the second picture in Figure 1.36.

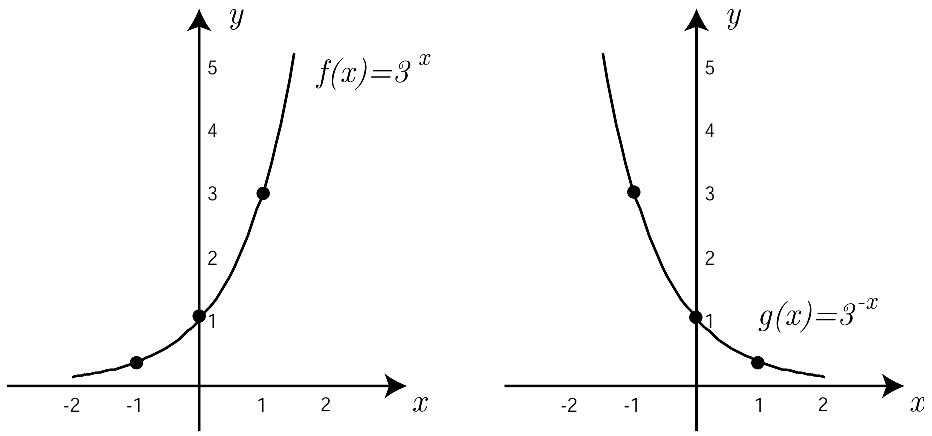

Figure 1.36: The curves of $f(x) = 3^x$ and $g(x) = 3^{-x}$

As $e \approx 2.71828 \approx 3$, the curves of $h(x) = e^x$ and $q(x) = e^{-x}$ should be very similar to those of $f(x) = 3^x$ and $g(x) = 3^{-x}$ respectively. So we obtain the curves for $h(x) = e^x$ and $q(x) = e^{-x}$ in **Figure 1.37**. One may also use a calculator to evaluate a few points and then draw these curves, but for now, Figure 1.37 is good enough for the curves of $h(x) = e^x$ and $q(x) = e^{-x}$. ♠

Example 1.3.7 indicates that we have the following property.

Property 1.3.8 *For the exponential function b^x $(b > 0, b \neq 1)$, if $b > 1$, then as the x value is increased, the function value is increased. If $b < 1$, then as the x value is increased, the function value is decreased.*

We can apply this property to the following type of exponential functions

$$y(x) = e^{kx} = (e^k)^x,$$

where k is a nonzero constant. We write $y(x) = e^{kx} = (e^k)^x = b^x$ with $b = e^k$. If $k > 0$, then $b = e^k > 1$, so $y(x) = e^{kx} = (e^k)^x$ will increase; if $k < 0$, then $b = e^k < 1$, so $y(x) = e^{kx} = (e^k)^x$ will decrease. Accordingly, we say that

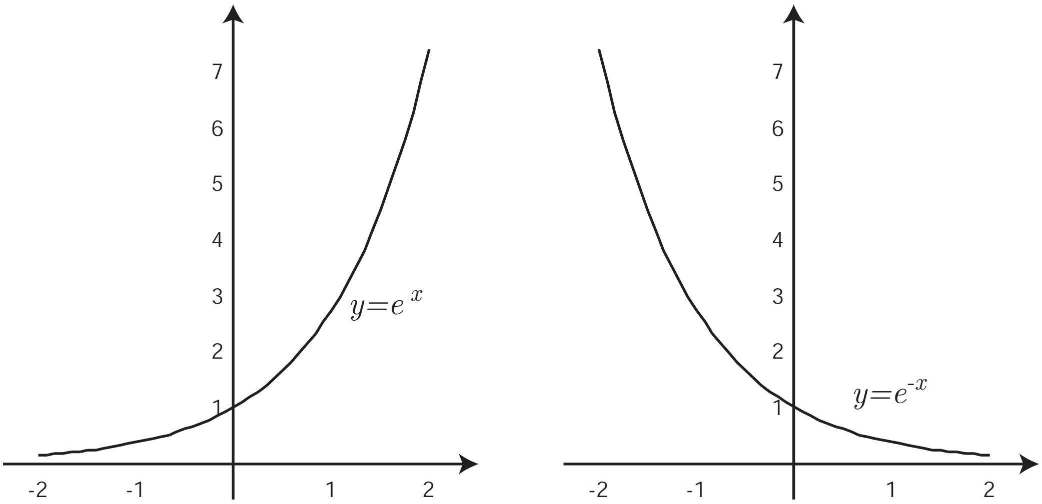

Figure 1.37: The curves of $h(x) = e^x$ and $q(x) = e^{-x}$

1. if $k > 0$, then e^{kx} **grows exponentially**,
2. if $k < 0$, then e^{kx} **decays exponentially**.

The curves of e^{kx} $(k \neq 0)$ are similar to those of $h(x) = e^x$ and $q(x) = e^{-x}$ in Figure 1.37.

Exponential growth and exponential decay are widely used in mathematical modeling in biology, chemistry, finance, physics, and population dynamics. Here we will look at some simple applications. Further study of this topic will be given later.

Example 1.3.9 (Population dynamics) The population of a certain city was $40,000$ in the year 2002. Assume that the population can be modeled as $p(t) = 40,000e^{0.03t}$, where t is the number of years after 2002. Find the population of this city in the year 2005 using this model.

Solution. Note first that we used t in the function $p(t)$ because in this case the independent variable is "time". It takes 3 years from 2002 to 2005, so $t = 3$. Thus, using the model $p(t) = 40,000e^{0.03t}$ and using a calculator, we obtain the population of this city in the year 2005 to be

$$p(3) = 40,000e^{(0.03)3} = 40,000e^{0.09} \approx 43766.97 \approx 43767.$$

That is, if a calculation using a mathematical model results in a decimal, then we should round it to an integer. ♠

Exercises 1.3

1. Find values of x such that $e^{4x+5} = e^{7x}$.

2. Find values of x such that $\frac{1}{e^{4x+5}} = e^{7x}$.

3. Find values of x such that $1 = e^{7x}$.

4. Find values of x such that $1 = \frac{1}{e^{7x}}$.

5. Find values of x such that $1 > e^{7x}$.

6. Find values of x such that $1 < e^{7x}$.

7. Find values of x such that $e^{4x+5} < e^{7x}$.

8. Find values of x such that $e^{4x+5} > e^{7x}$.

9. You deposit \$100 in a bank with an annual interest rate of 3% which is compounded quarterly (apply interest after every three months). Find (derive formulas) the amount in the account at the end of one year, two years, three years, and x years.

10. Derive (3.3).

11. Evaluate $A_m = \left(1 + \frac{1}{m}\right)^m$ for $m = 10$ and 100.

12. Assume that Bank A offers an annual interest rate of 4% which is compounded ten times per year, and Bank B offers an annual interest rate of 4.02% which is compounded three times per year. To which bank should you deposit your money?

13. The population of a certain college was 10,000 in the year 1999. Assume that the population can be modeled as $p(t) = 10,000e^{0.02t}$, where t is the number of years after 1999. Find the population of this college in the year 2003 using this model.

14. Draw the curves for $f(x) = 2^x$, $g(x) = (\frac{1}{2})^x = 2^{-x}$, $h(x) = e^{2x}$, $q(x) = e^{-2x}$, $k(x) = e^{x/2}$, and $w(x) = e^{-x/2}$.

1.4 Logarithmic Functions

Question: *You have learned how to solve x from $3^{x-1} = 27$ because $27 = 3^3$ so that the exponents can be equaled. Now, do you know how to solve x from $3^{x-1} = 4$? A similar question is to follow Example 1.3.9 (where the population of a city was modeled using $p(t) = 40,000e^{0.03t}$) and ask in what year the population reaches 50,000, which means to find t such that $40,000e^{0.03t} = 50,000$, or $e^{0.03t} = 1.25$.*

These questions indicate that sometimes we are given a and b, and we need to solve c such that

$$b^c = a.$$

To learn how to solve this and other related problems, let's start with some simple examples. We know that the c in $10^c = 100$ must be $c = 2$. Similarly, we have

$$10^3 = 1000,\ 10^{-1} = \frac{1}{10},\ 2^2 = 4,\ 2^4 = 16,\ 2^{-2} = \frac{1}{4}.$$

For $10^2 = 100$, it means that the base 10 raised to the exponent of 2 gives 100. Accordingly, we define

$$\text{logarithm} \ = \ \text{exponent},$$

and write

$$\log_{10} 100 = 2,$$

which reads: for the base 10, in order to get 100, the exponent (logarithm) must be 2.

Similarly, $10^{-1} = \frac{1}{10}$ means that the base 10 raised to the exponent of -1 gives $\frac{1}{10}$, and we write

$$\log_{10} \frac{1}{10} = -1,$$

which reads: for the base 10, in order to get $\frac{1}{10}$, the exponent (logarithm) must be -1.

In the same way, we have

$$\begin{aligned} 10^3 = 1000 &\iff \log_{10} 1000 = 3, & 2^2 = 4 &\iff \log_2 4 = 2, \\ 2^4 = 16 &\iff \log_2 16 = 4, & 2^{-2} = \tfrac{1}{4} &\iff \log_2 \tfrac{1}{4} = -2, \end{aligned}$$

where "$\Longleftrightarrow$" means "if and only if" or "the same as saying".

From these examples, we see that for a base b ($b > 0$, $b \neq 1$), we have

$$b^y = x \quad \Longleftrightarrow \quad y = \log_b x, \tag{4.1}$$

from which we see that $y = \log_b x$ defines y as a function of x. Moreover, for all real y values, we always have $x = b^y > 0$. Thus, this function $y = \log_b x$ is only defined for $x \in (0, \infty)$.

Accordingly, we have the following definition.

Definition 1.4.1 *Let $b > 0$ and $b \neq 1$. Then the function defined by*

$$f(x) = \log_b x \ \text{ on the domain } \ (0, \infty)$$

is called the **logarithmic function with base** *b.*

Remark 1.4.2 If you have some knowledge about inverse functions, then you see that $y = b^x$ and $y = \log_b x$ are just inverse functions of each other. For example, for $y = 10^x$ and $y = \log_{10} x$, $10^2 = 100$ sends 2 to 100 and $\log_{10} 100 = 2$ sends 100 back to 2. Here, we will present a treatment for logarithmic functions without assuming the knowledge of inverse functions, in case the subject on inverse functions is not included in this course. If the knowledge of inverse functions is assumed, the presentation here is still appropriate. ♠

If $b = 10$, then we write $\log x$ for $\log_{10} x$, and call

$$f(x) = \log x, \quad x > 0$$

the **common logarithmic function**.

If $b = e$, then we write $\ln x$ for $\log_e x$, and call

$$f(x) = \ln x, \quad x > 0$$

the **natural logarithmic function**, because the base e is taken from the natural exponential function e^x.

Common logarithmic and natural logarithmic functions are very important in applications, and are built into scientific calculators for you to evaluate their values.

Example 1.4.3 Find $\log_5 25$, $\log_3 \frac{1}{27}$, $\log 1$, $\log 10$, $\ln 1$, and $\ln e$.

Solution. For $\log_5 25$, we write $\log_5 25 =?$ and then change it to

$$5^? = 25,$$

from which we find that $? = 2$. Thus, $\log_5 25 = 2$.

For $\log_3 \frac{1}{27} =?$, we change it to $3^? = \frac{1}{27}$, from which we find that $? = -3$. Thus, $\log_3 \frac{1}{27} = -3$.

For $\log 1 = \log_{10} 1 =?$, we change it to $10^? = 1$, from which we find that $? = 0$. Thus, $\log 1 = 0$.

Similarly, we obtain $\log 10 = 1$, $\ln 1 = 0$, and $\ln e = 1$. ♠

From these examples, we find that $(0, 1)$ is on the curve of $y = e^x$ (that is, when $x = 0$, $y = e^0 = 1$), and $(1, 0)$ is on the curve of $y = \ln x$ (when $x = 1$, $y = \ln 1 = 0$). Also, we see that $(1, e)$ is on $y = e^x$ and $(e, 1)$ is on $y = \ln x$. That is, switching the first and second components of these points will make them on the graphs of $y = e^x$ and $y = \ln x$ respectively.

In fact, from (4.1), we find that this is true in general. That is, we have

$$\begin{aligned}
&(A, B) \text{ is on the curve of } y = e^x \\
&\Longleftrightarrow B = e^A \\
&\Longleftrightarrow A = \log_e B = \ln B \\
&\Longleftrightarrow (B, A) \text{ is on the curve of } y = \ln x.
\end{aligned}$$

In geometry, the points (A, B) and (B, A) are the mirror images of each other with respect to the straight line $y = x$, see **Figure 1.38**. Therefore, to obtain the curve of $y = \ln x$, we can just draw the mirror image of $y = e^x$ with respect to the straight line $y = x$, given in **Figure 1.39**. This indicates that if you fold along the straight line $y = x$, then the curves of $y = e^x$ and $y = \ln x$ must match up.

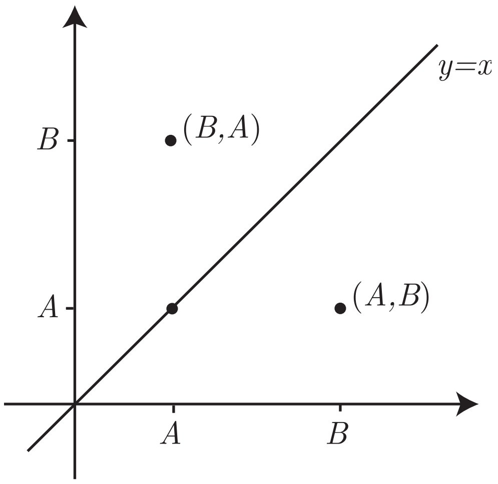

Figure 1.38: The mirror images of (A, B) and (B, A) with respect to $y = x$

We summarize these as a property.

Property 1.4.4 *For the functions $y = e^x$ and $y = \ln x$,*

1. *$y = \ln x$ is only defined for $x > 0$.*
2. $\ln 1 = 0, \ \ln e = 1.$
3. *A point (A, B) is on the curve of $y = e^x \iff$ the point (B, A) is on the curve of $y = \ln x$.*
4. *The curves of $y = e^x$ and $y = \ln x$ are symmetric about the straight line $y = x$.* ♠

Based on Property 1.4.4, we can derive further properties. For example, let (A, B) be on $y = e^x$, that is, $B = e^A$. From Property 1.4.4, (B, A) is on $y = \ln x$, that is, $A = \ln B$. Now, if we plug $A = \ln B$ into the A of $B = e^A$, we obtain

$$B = e^A = e^{(\ln B)}.$$

If we plug $B = e^A$ into the B of $A = \ln B$, we obtain

$$A = \ln B = \ln(e^A).$$

Therefore, we derive

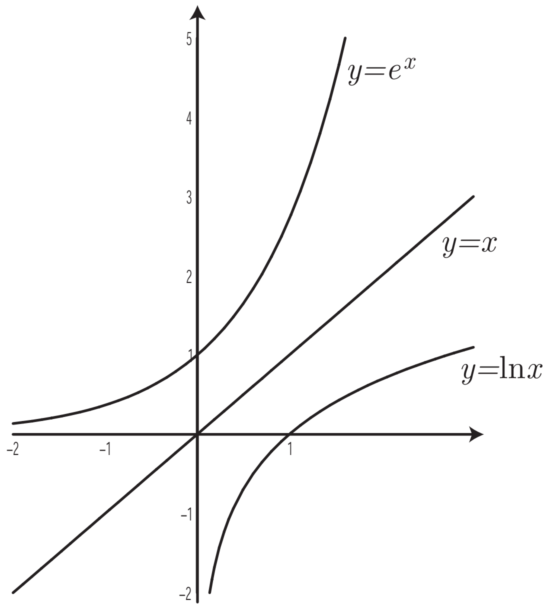

Figure 1.39: The curves for $y = e^x$ and $y = \ln x$

Property 1.4.5 *For the functions $y = e^x$ and $y = \ln x$,*

1. $x = e^{\ln x}, \quad x > 0.$
2. $x = \ln e^x, \quad -\infty < x < \infty.$ ♠

You can use a calculator to check Property 1.4.5 for a few values to convince yourself that Property 1.4.5 is valid. Property 1.4.5 indicates that the functions $y = e^x$ and $y = \ln x$ "cancel" each other (a property of *inverse functions*), or the compositions of the functions $y = e^x$ and $y = \ln x$ result in what you started with. For example, you start with x and apply the natural logarithmic function and get $\ln x$, then, with $\ln x$ as a value, you apply the exponential function with base e and get $e^{(\ln x)}$. But $e^{(\ln x)} = x$, so you are back to x. The same can be done for $\ln e^x = x$.

The formula $x = e^{\ln x}\,(x > 0)$ in Property 1.4.5 also indicates that any positive number can be rewritten as a number with base e.

Sometimes, we need to use logarithmic functions to simplify products, quotients, and functions with exponents. For this purpose, we present the following property based on Property 1.4.5.

Property 1.4.6 *For $A > 0$, $B > 0$, and $r \neq 0$, we have*

1. $\ln(AB) = \ln A + \ln B.$
2. $\ln \frac{A}{B} = \ln A - \ln B.$

3. $\ln(A^r) = r \ln A$.

Verification. (1). Treating AB as one value and applying Property 1.4.5, we get $e^{\ln(AB)} = AB$. We also know that $A = e^{\ln A}$ and $B = e^{\ln B}$, thus, we obtain

$$e^{\ln(AB)} = AB = (e^{\ln A})(e^{\ln B}) = e^{\ln A + \ln B}.$$

We can then apply laws of exponents to derive $\ln(AB) = \ln A + \ln B$.

(2). Similar to how (1) was derived, we have

$$e^{\ln \frac{A}{B}} = \frac{A}{B} = \frac{e^{\ln A}}{e^{\ln B}} = e^{\ln A - \ln B},$$

so we obtain $\ln \frac{A}{B} = \ln A - \ln B$.

(3). Similarly, we have

$$e^{\ln(A^r)} = A^r = (e^{\ln A})^r = e^{r \ln A},$$

so we obtain $\ln(A^r) = r \ln A$. ♠

Property 1.4.6 indicates that numbers cannot be taken out of a logarithmic function. For example, we know that $\ln(e \times 1) = \ln e = 1$. But if we take the constant e out, we get $e \ln 1 = e(0) = 0$, different from $\ln e = 1$.

Example 1.4.7 Find $\ln(\ln e^e)$.

Solution. We start from the inside, that is, using Property 1.4.5, we get $\ln e^e = e$, thus,

$$\ln(\ln e^e) = \ln(e) = 1.$$

♠

Example 1.4.8 If $\ln A = 3$ and $\ln B = 4$, find $\ln(AB)$, $\ln \frac{A}{B}$, and $\ln A^2$.

Solution. Using Property 1.4.6, we get

$$\begin{aligned} &\ln(AB) = \ln A + \ln B = 3 + 4 = 7, \\ &\ln \frac{A}{B} = \ln A - \ln B = 3 - 4 = -1, \\ &\ln A^2 = 2 \ln A = 2(3) = 6. \end{aligned}$$

♠

Recall that we solved $3^{x-1} = 27$ where we wrote $3^{x-1} = 3^3$ so we had a common base and then we could equal the exponents. The following are some examples where Property 1.4.5 and/or Property 1.4.6 must be used in order to solve.

Example 1.4.9 Find values of x such that $e^{2x-1} = 6$.

Solution 1. Using Property 1.4.5, we can rewrite $6 = e^{\ln 6}$, so we obtain

$$e^{2x-1} = e^{\ln 6}.$$

Now, with the same base e, we can equal the exponents to get

$$2x - 1 = \ln 6, \quad \text{or} \quad x = \frac{1 + \ln 6}{2}.$$

Solution 2. We can apply the natural logarithmic function on both sides of $e^{2x-1} = 6$ directly and use Property 1.4.6 to obtain

$$\ln[e^{2x-1}] = \ln 6, \quad \text{or} \quad 2x - 1 = \ln 6,$$

which gives the same answer as in Solution 1. ♠

Example 1.4.10 Find values of x such that $5e^{2x-1} = 6$.

Solution 1. We first simplify it to a form that is similar to that in Example 1.4.9 and then solve. That is, we rewrite it as $e^{2x-1} = \frac{6}{5}$, and then apply the natural logarithmic function on both sides and obtain

$$\ln e^{2x-1} = \ln \frac{6}{5}, \quad \text{or} \quad 2x - 1 = \ln \frac{6}{5}, \quad \text{or} \quad x = \frac{1 + \ln(6/5)}{2}.$$

Solution 2. We can apply the natural logarithmic function on both sides of $5e^{2x-1} = 6$ directly and use Property 1.4.6 to obtain

$$\ln[5e^{2x-1}] = \ln 6, \quad \text{or} \quad \ln 5 + \ln e^{2x-1} = \ln 6,$$

which gives $\ln 5 + 2x - 1 = \ln 6$, the same as in Solution 1.

Note here that $\ln[5e^{2x-1}] \neq 5\ln[e^{2x-1}]$, that is, numbers cannot be taken out of a logarithmic function. ♠

Example 1.4.11 Find values of x such that $5e^{2x-1} = e^{6x}$.

Solution 1. We first simplify it to a form that is similar to that in Example 1.4.9 and then solve. That is, we rewrite it as $5 = e^{6x-2x+1}$, and then apply the natural logarithmic function on both sides and obtain

$$\ln 5 = \ln e^{6x-2x+1} = 4x + 1, \quad \text{or} \quad x = \frac{\ln 5 - 1}{4}.$$

Solution 2. We apply the natural logarithmic function on both sides and use Property 1.4.6 to obtain

$$\ln[5e^{2x-1}] = \ln 5 + \ln e^{2x-1} = \ln e^{6x}, \quad \text{or} \quad \ln 5 + 2x - 1 = 6x,$$

so we get $x = \frac{\ln 5 - 1}{4}$, the same as in Solution 1. ♠

We can now answer the question raised at the beginning of this section.

Example 1.4.12 Find values of x such that $3^{x-1} = 4$.

Solution. We apply the natural logarithmic function on both sides and use Property 1.4.6 to obtain

$$\ln 3^{x-1} = \ln 4, \quad \text{or} \quad (x-1)\ln 3 = \ln 4,$$

thus, $x - 1 = \frac{\ln 4}{\ln 3}$ and then

$$x = 1 + \frac{\ln 4}{\ln 3}.$$

♠

Example 1.4.13 (Population dynamics) Following Example 1.3.9 and assume that the population of a certain city was 40,000 in the year 2002 and that the population can be modeled as $p(t) = 40,000e^{0.03t}$, where t is the number of years after 2002. Find, using the model, in what year the population reaches 50,000.

Solution. We need to find t such that $p(t) = 50,000$. Using the model, we have $p(t) = 40,000e^{0.03t}$, so we obtain

$$40,000e^{0.03t} = 50,000, \quad \text{or} \quad e^{0.03t} = \frac{5}{4} = 1.25,$$

from which we obtain $\ln e^{0.03t} = \ln 1.25$, or $0.03t = \ln 1.25$, so that

$$t = \frac{\ln 1.25}{0.03} \approx 7.$$

Therefore, according to the model, the population of the city reaches 50,000 in the year 2009. ♠

The following examples indicate how the logarithmic functions are used to simplify products, quotients, and functions with exponents.

Example 1.4.14 Rewrite

$$\ln \frac{(x+1)^{10}(x+2)^{20}}{(x+3)^{30}(x+4)^{40}}$$

using terms without exponents.

Solution. We use Property 1.4.6 to break the original one into pieces and get

$$\begin{aligned}
&\ln \frac{(x+1)^{10}(x+2)^{20}}{(x+3)^{30}(x+4)^{40}} \\
&= \ln[(x+1)^{10}(x+2)^{20}] - \ln[(x+3)^{30}(x+4)^{40}] \\
&= [\ln(x+1)^{10} + \ln(x+2)^{20}] - [\ln(x+3)^{30} + \ln(x+4)^{40}] \\
&= 10\ln(x+1) + 20\ln(x+2) - 30\ln(x+3) - 40\ln(x+4).
\end{aligned}$$

♠

Going in the reverse direction of Example 1.4.14, we can solve the following.

Example 1.4.15 Rewrite

$$2 \ln 3x - 5 \ln(x + 3) + 7 \ln(x - 2)^2$$

as one term of the form $\ln(\cdots)$.

Solution. First, we need to clear the coefficients. Next, when we have more terms, we just look at two terms at a time and apply Property 1.4.6 to reduce them into one term. Thus we have

$$2 \ln 3x - 5 \ln(x + 3) + 7 \ln(x - 2)^2 = \ln(3x)^2 - \ln(x + 3)^5 + \ln[(x - 2)^2]^7$$
$$= \ln \frac{9x^2}{(x + 3)^5} + \ln(x - 2)^{14} = \ln \frac{9x^2(x - 2)^{14}}{(x + 3)^5}.$$

♠

Exercises 1.4

1. Find $\log_5 125$, $\log_3 \frac{1}{9}$, and $\log 1000$.
2. Find values of x such that $3e^{4x+5} = 7$.
3. Find values of x such that $3e^{4x+5} = e^7$.
4. Find values of x such that $3e^{4x+5} = e^{7x}$.
5. Find values of x such that $3^{4x+5} = 7$.
6. Find values of x such that $6^{4x+5} = e^{7x}$.
7. Find values of x such that $3^{4x+5} = 5^{7x}$.
8. Find $\ln(\ln e^{e^e})$.
9. If $\ln A = -2$ and $\ln B = 3$, find $\ln(AB)$, $\ln \frac{A}{B}$, and $\ln A^2$.
10. For $A > 0, B > 0$, and $C > 0$, verify that $\ln(ABC) = \ln A + \ln B + \ln C$. Then come up with a statement concerning n positive numbers, and verify it.
11. Rewrite
$$\ln \frac{(3x + 1)^{110}(5x + 2)^{220}}{(7x + 3)^{330}(9x + 4)^{440}}$$
using terms without exponents.

12. Rewrite

$$x \ln 2x + 12 \ln(7x + 3) - 5 \ln(3x - 2)^2$$

as one term of the form $\ln(\cdots)$.

13. Assume that the population of a certain city was $36,000$ in the year 1998 and that the population can be modeled as $p(t) = 36,000e^{0.02t}$, where t is the number of years after 1998. Find, using the model, in what year the population reaches $40,000$.

14. If you deposit \$100 to a bank with an annual interest rate of 7% which is compounded annually, how long will it take for the money to double?

15. The population of a college was $20,000$ in the year 2002 and was 20,600 in the year 2004. Assume that the population can be modeled as $p(t) = 20,000e^{kt}$ for a constant k, where t is the number of years after 2002. Find the population of this college in the year 2008 using this model.

Chapter 2

Limits, Continuity, and the Derivative

We start by looking at how to approximate a function locally using a tangent line in geometry, then we look at how to find average velocity in elementary physics. They demonstrate the need to study the notion of limits.

Then we introduce the notions of limit and continuity, and the operations of limits, which lead to the notion of derivatives.

After this, we derive derivatives of some simple functions, based on which we are able to present the so-called "power rule" for taking derivatives.

Then we will show how to approximate a function locally using a tangent line.

This will set the stage for the differential calculus.

2.1 Why do We Need to Study Limits?

James' story: As a graduate student I would often take my daughter to the city park to expel some energy. She loved to ride the Merry-Go-Round, see **Figure 2.1**, and no matter how fast I would spin her she would yell, "faster daddy faster". She would get off the ride and stumble around like a drunk. After her head stopped spinning, she would shout, "do it again". Her enthusiasm for dizziness worried me when I thought of her later going to college. On one of these outings I grabbed a handle on the Merry Go Round and took off in a dead sprint running as fast as my legs would go and gave a final two armed thrust sending the Merry Go Round into a full speed whirl. Gasping for breath I looked back just in time to see the centrifugal force pry my terrified daughter's fingers from the side bar sending the poor kid sailing and then skidding...... The worst part of the incident was my intrigue in the fact that my daughter's flight trajectory actually followed the **tangent line** to the circle representing the base of the merry-go-round at the point where her little fingers slipped off, see **Figure 2.2**. In a sick "geeky" kind of way I found it neat that this is exactly what physics would predict for the situation. NASA actually used this same idea in the Apollo missions. After lift off they would allow the spacecraft to significantly gain speed by taking several orbits of the earth and then at the correct instant would use its rockets to slip free of earth's gravity, see **Figure 2.3**. This created a slingshot that shot the spacecraft along the tangent line toward lunar orbit. This quite literally saved the astronaut's days in space. Tangent lines and their applications are vital to all disciplines that rely on mathematics, and we will see some of them in this chapter. **End of the story**.

Figure 2.1: A Playground Merry-Go-Round

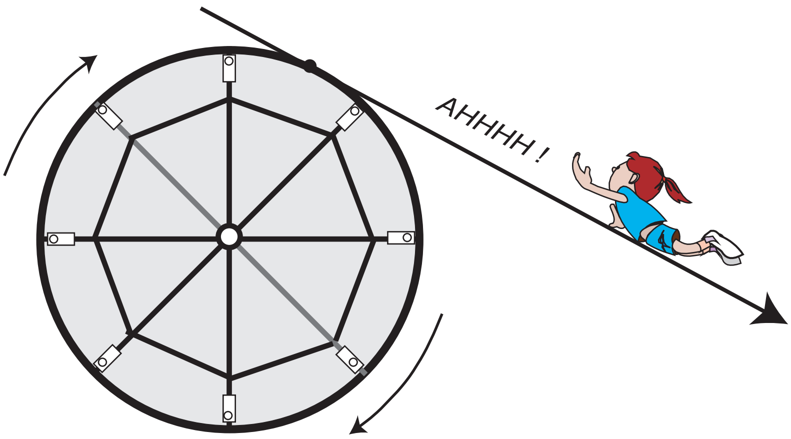

Figure 2.2: Daughter Thrown Along Tangent Line

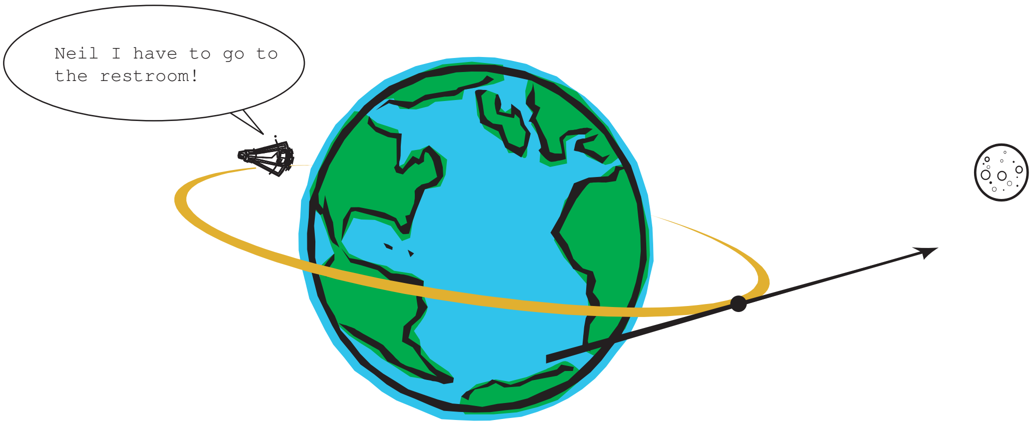

Figure 2.3: Spacecraft to be "Slung" Along Tangent Line to the Moon

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Now, let's start with the following questions.

Question 1: *Do you know how to approximate* $\sqrt{81.34}$ *without using a calculator? Next, if you do use a calculator to approximate, then do you know what happens after you press some buttons?*

Question 2: *When you are driving a car, do you understand what the speedometer is telling you?*

To understand these and other related questions, let's first begin with some simple cases.

Example 2.1.1 Let $f(x) = x^2$. For $h = 1, 0.5, 0.25, 0.1, 0.01$, draw the secant lines passing through $(0, f(0))$ and $(h, f(h))$, and find their slopes.

Solution. The secant lines for those h values are given in **Figure 2.4**.

Next, let $(x_1, y_1) = (0, f(0)) = (0, 0)$ and $(x_2, y_2) = (h, f(h)) = (h, h^2)$, then the slope is given by

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(h) - f(0)}{h - 0} = \frac{f(h)}{h} = \frac{h^2}{h} = h.$$

So that for $h = 1, 0.5, 0.25, 0.1, 0.01$, the slopes are $1, 0.5, 0.25, 0.1, 0.01$.

♠

Question: *What will happen to the slope* $\frac{f(h)-f(0)}{h-0}$ *and the corresponding secant line when* h *approaches zero?*

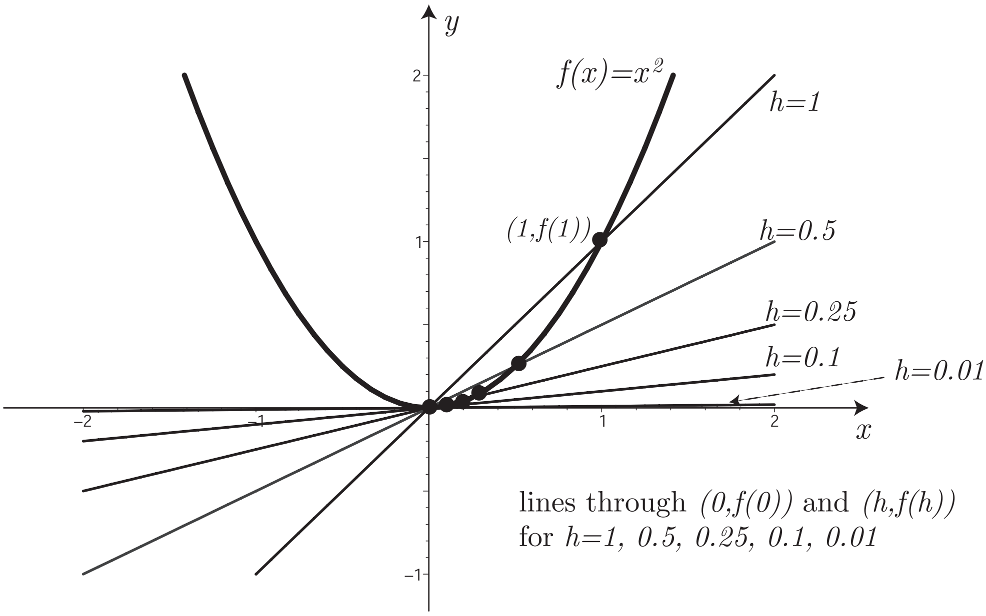

Figure 2.4: The tangent line to $f(x) = x^2$ at $x = 0$: $y = 0$

From the calculation, we see that when h gets close to zero, the slope $\frac{f(h)-f(0)}{h-0}$ also gets close to zero. In this sense, we say that "as h approaches zero, $\frac{f(h)-f(0)}{h-0}$ has a **limit** which is zero", indicating the tendency that $\frac{f(h)-f(0)}{h-0}$ will get as close to zero as we want when h approaches zero.

From the geometry in Figure 2.4, we see that these secant lines get more and more flat and eventually approach the x-axis, or the straight line $y = 0$. This straight line $y = 0$ is so special that it doesn't just pass the curve of $f(x) = x^2$ at $x = 0$ in an arbitrary way, instead, it is **tangent** to the curve at $x = 0$ in the sense that it is the **closest** to the curve of $f(x) = x^2$ at $x = 0$ **locally**. That is, the curves of $y = x^2$ and $y = 0$ are almost the same near $x = 0$, and the straight line $y = 0$ is the only straight line to have this property. Therefore, $y = 0$ is called the **tangent line** to the function $f(x) = x^2$ at $x = 0$; and the slopes of these secant lines approach the slope of the tangent line at $x = 0$.

Based on the above description of tangent lines, we see that for any non-vertical straight line, the tangent line at any point of the straight line is this straight line itself.

Next, let's look at a general case shown in **Figure 2.5**. At the point $(c, g(c))$, we see from Figure 2.5 that there is a unique straight line L that is *tangent* to the curve of the function g at the point $(c, g(c))$. This straight line L is called the *tangent line* to the function g at the point $(c, g(c))$. If we can find this tangent line L, then we can use L (which is easy to deal with

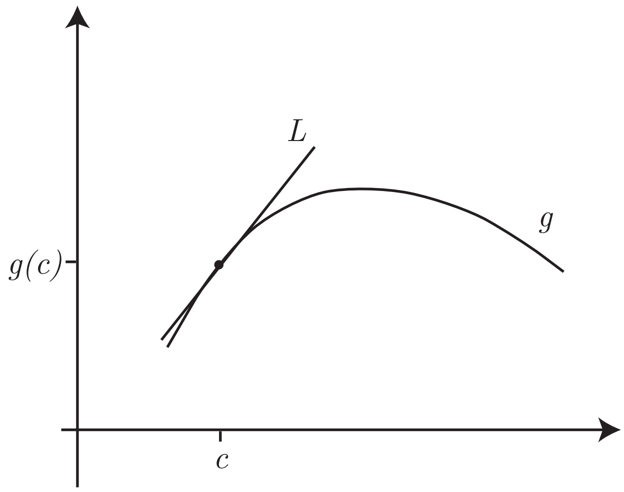

Figure 2.5: The function g and its tangent line

since it is a straight line) to **approximate** the function g **locally** near the point of tangency $(c, g(c))$ because near that point the curves of g and L are almost the same, see Figure 2.5. This will be very useful especially in the cases when the function g is difficult to evaluate, such as evaluating $\sqrt{81.34}$ when you don't have a calculator.

Question: *How do we find this tangent line L?*

Note that the point $(c, g(c))$ is the only point we know on the straight line L, so from the study in Chapter 1, we see that to find the straight line L we now must find its slope. But we don't have another point on L (L is unknown yet), so the slope of L cannot be found directly. Therefore, we consider the following procedure.

Look at **Figure 2.6**, where we choose another point $(w, g(w))$ on the curve of the function g (this is allowed because the function g is given).

Now, with the two points $(c, g(c))$ and $(w, g(w))$, we can calculate the slope for the corresponding secant line to be

$$\frac{g(w) - g(c)}{w - c} \quad or \quad \frac{g(c+h) - g(c)}{h} \text{ if letting } w = c + h. \tag{1.1}$$

If we are lucky enough, then $\frac{g(w)-g(c)}{w-c}$ gives the slope of the tangent line L. However, in general we do not expect this to happen. Then what do we do?

Now, we explore the idea shown in **Figure 2.7**. That is, we regard c as a fixed value and let w approach c, so that we get many secant lines passing through $(c, g(c))$ and $(w, g(w))$ with slopes $\frac{g(w)-g(c)}{w-c}$ for various values of w.

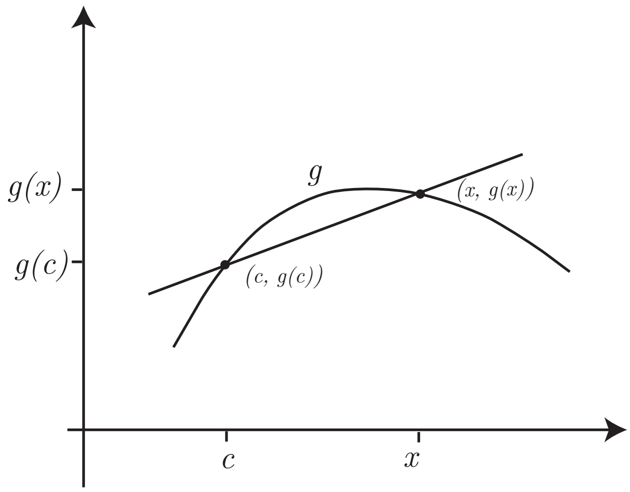

Figure 2.6: The straight line passing through $(c, g(c))$ and $(w, g(w))$

Question: *What will happen to the slope* $\frac{g(w)-g(c)}{w-c}$ *when* w *approaches* c*?*

Now, the point $(c, g(c))$ is fixed and other points $(w, g(w))$ (for various w values) are getting very close to the point $(c, g(c))$ as w gets close to c. Thus, from geometry, we see that these secant lines should approach the tangent line L. Consequently, we infer that as w approaches c, the slope $\frac{g(w)-g(c)}{w-c}$ of the corresponding secant line should approach a limit (a value), which should be the slope of the tangent line L. (For example, in Example 2.1.1, the secant lines approach the tangent line $y = 0$, and the slopes of those secant lines approach zero, which is the slope of the tangent line $y = 0$.)

Therefore, if we treat c as a fixed value and treat w as the independent variable and then treat $\frac{g(w)-g(c)}{w-c}$ as a function of w, such as writing

$$f(w) = \frac{g(w) - g(c)}{w - c},$$

then, we intuitively imply that

> " as the independent variable w approaches c, the function $f(w) = \frac{g(w)-g(c)}{x-c}$ approaches a limit."

Next, let's look at some other cases.

Example 2.1.2 Assume that the position of a moving object (such as a car you are driving) is given by $p(t) = t^2 + 1$, where t denotes how many minutes after the initial time, 0. Then find the average velocity of the object on the time interval $[1,\, 1 + h]$ for $h = 1,\, 0.1,\, 0.01,\, 0.001$.

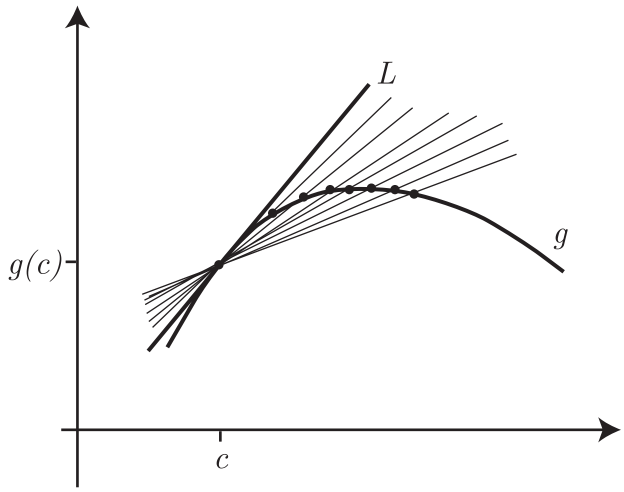

Figure 2.7: The secant lines approaching the tangent line

Solution. The average velocity on the time interval $[1,\ 1+h]$ is given by

$$\text{average velocity} = \frac{\text{distance}}{\text{time}} = \frac{p(1+h)-p(1)}{(1+h)-1} = \frac{p(1+h)-p(1)}{h}.$$

For $h = 1$, or on the time interval $[1, 2]$, the average velocity is

$$\frac{p(1+h)-p(1)}{h} = \frac{p(2)-p(1)}{1} = \frac{5-2}{1} = 3.$$

Similarly, for $h = 0.1$, the average velocity on the interval $[1, 1+h] = [1, 1.1]$ is

$$\frac{p(1+h)-p(1)}{h} = \frac{p(1.1)-p(1)}{0.1} = \frac{2.21-2}{0.1} = \frac{0.21}{0.1} = 2.1.$$

The rest of the calculations are given in the table in **Figure 2.8**. ♠

Question: *What will happen to the average velocity* $\frac{p(1+h)-p(1)}{h}$ *on interval* $[1, 1+h]$ *when* h *approaches zero?*

From the table, we see that as h approaches zero, the average velocity $\frac{p(1+h)-p(1)}{h}$ approaches 2. Accordingly, we say that "as h approaches zero, $\frac{p(1+h)-p(1)}{h}$ has a **limit** which is 2", indicating the tendency that $\frac{p(1+h)-p(1)}{h}$ will get as close to 2 as we want when h approaches zero. Next, note that as h gets close to zero, the interval $[1, 1+h]$ shrinks to just one point at $t = 1$, so that the interval $[1, 1+h]$ can be regarded as if it is "instantaneous" at $t = 1$. Thus, the value 2 can be regarded as if the velocity is obtained

h	$\frac{p(1+h)-p(1)}{h}$
1	3
0.1	2.1
0.01	2.01
0.001	2.001

Figure 2.8: The average velocity $\frac{p(1+h)-p(1)}{h}$

"instantaneous" at $t = 1$. Therefore, in physics, the limit, 2, is called the **instantaneous velocity** at $t = 1$.

Next, let's use the curve in **Figure 2.9** to denote the change in your car's mileage as you drive from city A to city B, where $m(t)$ gives the mileage at time t (note that as time goes on, the mileage will *increase* or at least stay the same during traffic jam; can you find a case where such a curve decreases?)

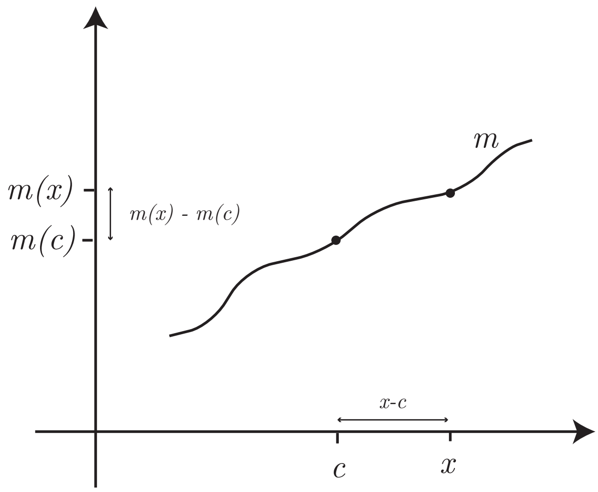

Figure 2.9: A typical curve for your trip

Now, to find the average velocity on the time interval $[c, w]$ in Figure 2.9, we use the total miles traveled, $m(w) - m(c)$, to divide by the total time, $w - c$, to obtain

$$\frac{m(w) - m(c)}{w - c} \quad or \quad \frac{m(c+h) - m(c)}{h} \text{ if letting } w = c + h. \tag{1.2}$$

Question: *What will happen to the average velocity* $\frac{m(w)-m(c)}{w-c}$ *on interval*

$[c, w]$ *when* w *approaches* c*?*

Now, we can argue that when w approaches c, the interval $[c, w]$ can be regarded as if it is "instantaneous" at time $t = c$. Thus, the average velocity $\frac{m(w)-m(c)}{w-c}$ should approach a *limit* (a value), which gives the *instantaneous velocity* at time $t = c$. The instantaneous velocity is what you see on your car's speedometer as you drive, which may change with time.

Therefore, if we treat c as a fixed value and treat w as the independent variable and then treat $\frac{m(w)-m(c)}{w-c}$ as a function of w, such as writing

$$f(w) = \frac{m(w) - m(c)}{w - c},$$

then, we intuitively imply that

> "as the independent variable w approaches c, the function $f(w) = \frac{m(w)-m(c)}{w-c}$ approaches a limit."

From the above, we see that the notion of limits is related to the slope of tangent line in geometry and the instantaneous velocity in physics. Therefore, we need to study limits in order to derive tangent line and instantaneous velocity and apply them in geometry and in physics. Of course, we point out that the notion of limits has wide applications also in many other important areas of studies.

Accordingly, we have enough reason to study the following issue, given in the next section (note that we are so used to denote the independent variable by x):

> *For a function* $f(x)$*, as its independent variable* x *approaches a value, whether or not the function value* $f(x)$ *approaches a limit.*

We end this section by observing that (1.1) and (1.2) are of the same form, which is important for the study in Section 2.3.

Exercises 2.1

1. For $f(x) = x^3$, find the slope of the secant line passing through $(0, f(0))$ and $(h, f(h))$ for $h = 1$, 0.1, 0.01, 0.001. Then approximate the slope of the tangent line of $f(x)$ at $x = 0$ and find this tangent line.

2. For $f(x) = x^2$, find the slope of the secant line passing through $(4, f(4))$ and $(w, f(w))$ for $w = 5$, 4.1, 4.01, 4.001. Then approximate the slope of the tangent line of $f(x)$ at $x = 4$ and find this tangent line.

3. If the position of a moving object is given by $p(t) = t^2 + 5$, where t denotes how many minutes after the initial time, 0. Then find the average velocity on the interval $[0, h]$ for $h = 1$, 0.1, 0.01, 0.001. Then approximate the instantaneous velocity of the object at $t = 0$.

4. If the position of a moving object is given by $p(t) = t^3 + 4$, where t denotes how many minutes after the initial time, 0. Then find the average velocity on the interval $[1, 1+h]$ for $h = 1, 0.1, 0.01, 0.001$. Then approximate the instantaneous velocity of the object at $t = 1$.

5. Find and simplify $\frac{f(w)-f(x)}{w-x}$ $(w \neq x)$ and $\frac{f(x+h)-f(x)}{h}$ $(h \neq 0)$ for

(a) $f(x) = 1$.

(b) $f(x) = x$.

(c) $f(x) = x + 5$.

(d) $f(x) = x^2$.

(e) $f(x) = x^2 - 6$.

(f) $f(x) = x^3 - 5x + 4$.

(g) $f(x) = \sqrt{2x-3}$.

(h) $f(x) = \frac{1}{x}$.

(i) $f(x) = \frac{1}{2-x}$.

2.2 Limits and Continuity

Question: [Payment at a gas station] *Following Example 1.2.1 where the payment*

$$f(x) = \begin{cases} 0, & x = 0, \\ 2.5x + 1.25, & x > 0, \end{cases} \tag{2.1}$$

is shown in **Figure 2.10** *(copied from Figure 1.25). What will happen to your payment if you add very very little gas?*

In this case, as long as you add gas ($x > 0$), your payment will be $f(x) = 2.5x + 1.25$. Thus, if you add very very little gas, say only a few drops, then your money for the gas is almost zero, so your payment will be almost \$1.25.

Using the geometry in Figure 2.10, the above result indicates that when x approaches zero from the right-hand side ($x > 0$ and x approaches zero), the corresponding function value $f(x)$ approaches 1.25. In this sense, we say that "the **right-hand limit** of $f(x)$ is 1.25 as x approaches 0 from the right-hand side", and we denote it as

$$\lim_{x \to 0^+} f(x) = 1.25, \tag{2.2}$$

where $x \to 0^+$ means that "$x > 0$ and $x \to 0$".

Based on these, we learn the following two important things.

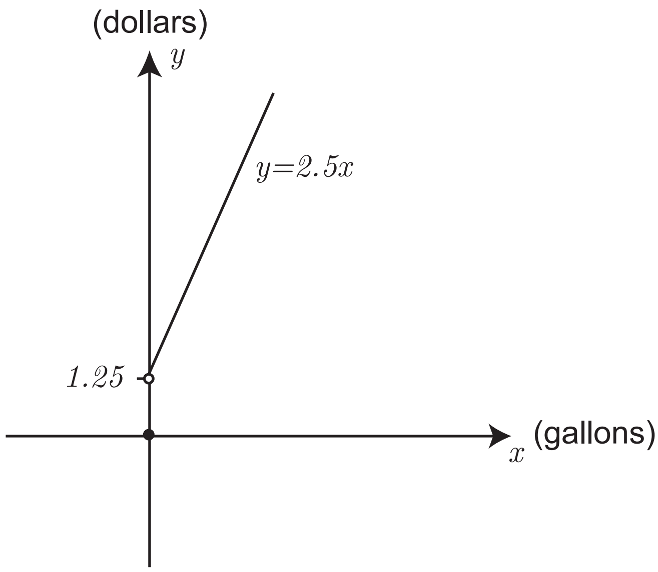

Figure 2.10: Payment at a gas station

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1. In taking this limit as x approaches 0, we cannot plug in $x = 0$ (otherwise, for $x = 0$, the payment $f(x) = f(0) = 0$, a wrong answer). Also for example, in the previous section, when we let w go to c in $\frac{g(w)-g(c)}{w-c}$, we cannot plug in $w = c$ because otherwise the denominator becomes zero. We will see later that sometimes we can do "plug-in".

2. The limit 1.25 shows the **tendency to a fixed destination** of the function value $f(x)$ as x approaches 0 from the right-hand side. That is, in this case, the limit is 1.25, but $f(x)$ will never achieve 1.25. In geometry, it says that the hole in Figure 2.10 will not affect the limit, because we only look at the tendency.

The $f(x)$ from the above question is not defined for $x < 0$. In other cases that a function $f(x)$ is defined on both sides of a point c, we use

$$\lim_{x \to c^-} f(x)$$

to denote the **left-hand side limit**, where $x \to c^-$ means that "$x < c$ and $x \to c$", that is, x approaches c from the left-hand side. And we use

$$\lim_{x \to c} f(x)$$

to denote the **limit**, where $x \to c$ means that "$x < c$ or $x > c$, and $x \to c$", that is, x approaches c from both sides. For simplicity, we sometimes use "x approaches c" to mean "x approaches c from both sides".

If we have a table of data, then to find a limit is just to make a "good guess".

Example 2.2.1 Find $\lim_{x\to 3} f(x)$ by using the table

$$\begin{array}{rcccccc} x: & 2.9, & 2.99, & 2.999, & \cdots & 3.001, & 3.01, & 3.1, \\ f(x): & 8.410, & 8.940, & 8.994, & \cdots & 9.006, & 9.060, & 9.610. \end{array}$$

Solution. From the table, we see that as x approaches 3 (from both sides), the value of $f(x)$ is getting very close to 9. Accordingly, we conclude that

$$\lim_{x\to 3} f(x) = 9.$$

That is, as x approaches 3, the limit of the function $f(x)$ is 9. ♠

Example 2.2.2 Find $\lim_{h\to 0} \frac{e^h-1}{h}$ by using the table in **Figure 2.11**.

h	$\frac{e^h-1}{h}$
0.1	1.0517
0.01	1.0050
0.001	1.0005
-0.1	0.9516
-0.01	0.9950
-0.001	0.9995

Figure 2.11: A table for $\frac{e^h-1}{h}$

Solution. From the table, we see that when h approaches 0 (from both sides), the value of $\frac{e^h-1}{h}$ is getting very close to 1. Therefore, we conclude that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

♠

In general, we can use the following **Figure 2.12** to understand the limit $\lim_{x\to c} f(x) = L$: For any small y-axis interval I_L containing L, you can always find an x-axis interval I_c containing c, such that if x is "thrown" to c in I_c, the corresponding function value $f(x)$ will be "caught" in I_L.

Next, let's continue to explore the concept of limits by using geometry. For the function shown in **Figure 2.13**, we see that as x approaches 3 (from both sides), the function value $f(x)$ approaches 2, thus,

$$\lim_{x\to 3} f(x) = 2.$$

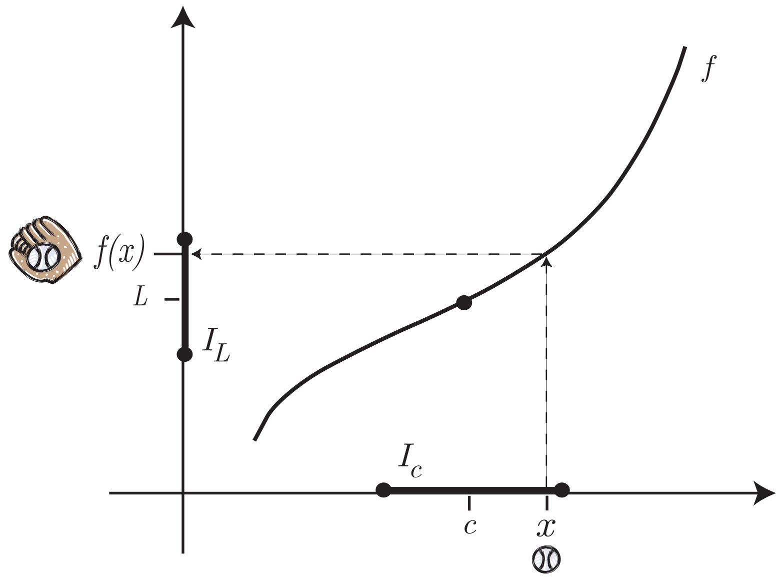

Figure 2.12: $\lim_{x\to c^+} f(x) = L$

The functions shown in **Figure 2.14** and **Figure 2.15** are similar to that in Figure 2.13, but the difference is that in Figure 2.14, $f(3) = 4$, not 2; in Figure 2.15, the function $f(x)$ is not defined at $x = 3$. Recall that when finding limit at $x = 3$, we only look at the tendency of the function value when x is close to 3 and do not consider the case of $x = 3$, so that whether $f(3)$ is defined or how it is defined (the holes in the figures) will not affect the limits. Therefore, similar to the above cases, we see that for the functions in Figure 2.14 and Figure 2.15, we still have $\lim_{x\to 3} f(x) = 2$, that is, in each case, as x approaches 3 from both sides, the function value approaches 2.

Next, let's look at the function shown in **Figure 2.16**. We see that as x approaches 3 from the right-hand side, the right-hand side limit is 4; similarly, the left-hand side limit is 2. Accordingly, as x approaches 3 (from both sides), we have no idea whether x approaches 3 from the right or from the left, therefore, we don't have a tendency or destination for the function value to approach. Thus, in this case, we say that "the limit of $f(x)$ does not exist as x approaches 3", denoted as

$$\lim_{x\to 3} f(x) \quad \text{DNE},$$

where "DNE" means "does not exist".

Now, we can summarize the ideas used in the above cases and make the following definition.

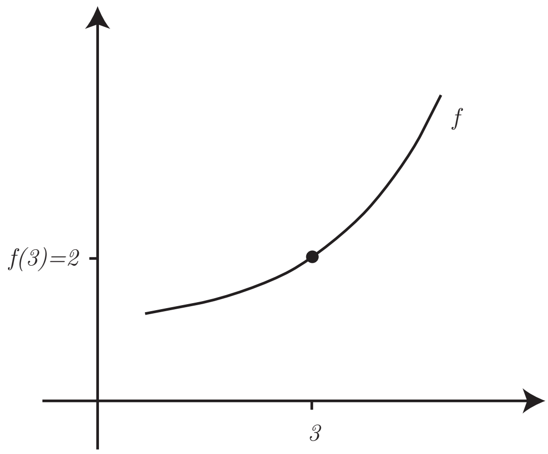

Figure 2.13: $\lim_{x\to 3} f(x) = 2$

Definition 2.2.3 *Let L and c be finite real numbers and let f be defined on both sides of c. If $f(x)$ approaches L as x approaches but is always different from c, then we say that "the function $f(x)$ has a* **limit** *L as x approaches c". We denote this limit by*

$$\lim_{x\to c} f(x) = L.$$

One sided limits can also be defined in similar ways. And from the discussions of the functions in Figures 2.13 – 2.16, we can generalize the idea and derive the following result.

Theorem 2.2.4 $\lim_{x\to c} f(x) = L$ *if and only if* $\lim_{x\to c^+} f(x) = L$ *and* $\lim_{x\to c^-} f(x) = L$.

Example 2.2.5 Find $\lim_{x\to 1} f(x)$, $\lim_{x\to 2} f(x)$, and $\lim_{x\to 3} f(x)$, where the curve of $f(x)$ is given in **Figure 2.17**.

Solution. Applying Theorem 2.2.4, we conclude that

$$\lim_{x\to 1} f(x) = 5, \qquad \lim_{x\to 2} f(x) = 4, \qquad \lim_{x\to 3} f(x) \text{ DNE}.$$

♠

Now, we know that all functions in figures 2.13 – 2.15 have limits as x approaches 3. But these functions are different. In Figure 2.13, we have

$$\lim_{x\to 3} f(x) = 2 = f(3), \tag{2.3}$$

which means that we just need to **plug in** $x = 3$ or **evaluate** at $x = 3$.

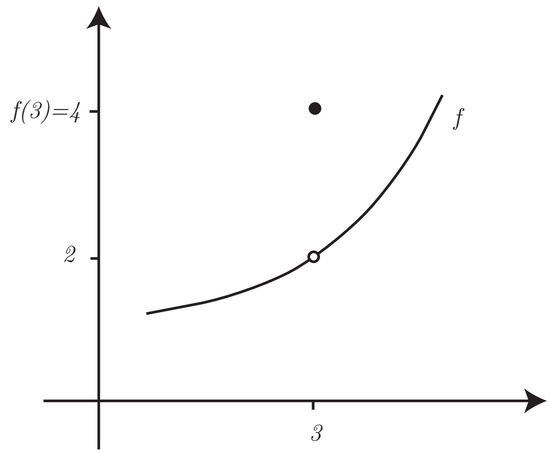

Figure 2.14: $\lim_{x\to 3} f(x) = 2$

In Figure 2.14, we have

$$\lim_{x\to 3} f(x) = 2 \neq f(3), \tag{2.4}$$

since $f(3) = 4$.

In Figure 2.15, we have

$$\lim_{x\to 3} f(x) = 2, \tag{2.5}$$

but the function $f(x)$ is not defined at $x = 3$.

The geometrical explanation is that in Figure 2.13, you can draw the curve to pass through the point $(3, f(3)) = (3, 2)$ **continuously without lifting your pen**. For figures 2.14 – 2.16, you have to lift your pen in order to draw the curves.

The "plug in" for the function in Figure 2.13 made the finding of limit so easy. Accordingly, we need to find conditions under which the curve of Figures 2.13 is *good*, or *continuous*; and the curves in figures 2.14 – 2.16 are *bad*, or *discontinuous*, as they have holes or breaks so we don't want to call them continuous.

To exclude the curve in Figure 2.16 from being continuous, we require the function to have a limit there; to exclude the curve in Figure 2.15, we require the function to be defined there; to exclude the curve in Figure 2.14, we require the limit to be the same as the function value there. Accordingly, we have the following definition based on these discussions.

Definition 2.2.6 *The function $f(x)$ is said to be* **continuous** *at $x = c$ if the following three conditions are satisfied,*

1. $\lim_{x\to c} f(x)$ *exists,*

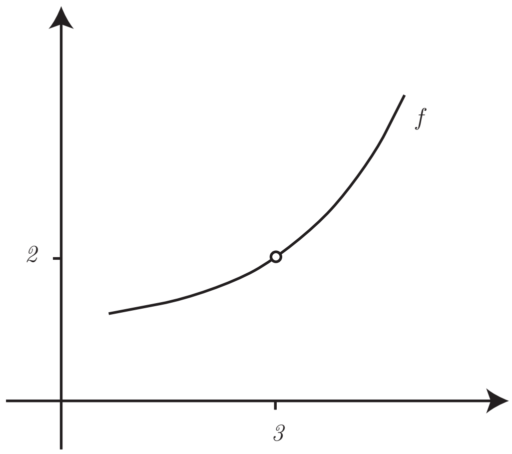

Figure 2.15: $\lim_{x\to 3} f(x) = 2$

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2. $f(c)$ is defined,

3. $\lim_{x\to c} f(x) = f(c)$.

The function $f(x)$ is continuous on an interval I if $f(x)$ is continuous at every point of the interval I.

Theorem 2.2.7 *If $f(c)$ is defined, then the three conditions in Definition 2.2.6 can be reduced to the last one:* $\lim_{x\to c} f(x) = f(c)$, *as it implies the three conditions.* ♠

Accordingly, in geometry, if the curve of a function has no holes or breaks, then the function is continuous; otherwise it is discontinuous. The curves in figures 2.13 – 2.16 are such examples.

For our purposes here in this elementary calculus course, we state without proof that most functions we encounter in this book are continuous. For example, functions of the forms

$$\sqrt{6x^8 - x^3 + 7x - 3},\ \ (4x^3 - x)^3,\ \ \frac{4x^7 - 3x^3}{x^3 - 2},\ \ xe^x + \ln x$$

are all continuous in their respective domains.

The good thing about continuous functions we have seen from the above is that

> **for continuous functions, finding limits is as simple as "plugging in" or "evaluating". Of course, if you see trouble, such as getting zero in the denominator, then simplify and then plug in**.

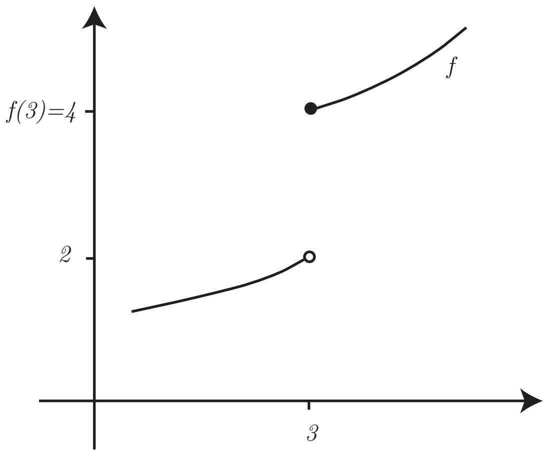

Figure 2.16: A function that has no limit as x approaches 3

Accordingly, we have the following result.

Property 2.2.8 (Properties of limits for continuous functions) *If $f(x)$ and $g(x)$ are continuous at $x = c$, then*

1. $\lim_{x\to c}[f(x) \pm g(x)] = f(c) \pm g(c)$,
2. $\lim_{x\to c}[kf(x)] = kf(c)$ *for any constant* k,
3. $\lim_{x\to c}[f(x)g(x)] = f(c)g(c)$,
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$ *if* $g(c) \neq 0$,
5. $\lim_{x\to c}[f(x)]^r = [f(c)]^r$ *for any constant* r *that results in a real number.*
♠

We now give some examples utilizing this idea of "plugging in" or "evaluating".

Example 2.2.9 Find the following limits using algebra.

1. $\lim_{x\to 3}(4x^2 - 1)$.
2. $\lim_{x\to 3} x^2\sqrt{x-2}$.
3. $\lim_{x\to 3} \frac{x^3+\sqrt{x-2}}{x^2-4}$.
4. $\lim_{x\to 1} \frac{(\ln x)^2-1}{\ln x-1}$.
5. $\lim_{x\to 0} \frac{3x^5+7x^2}{4x^6+3x^2}$.

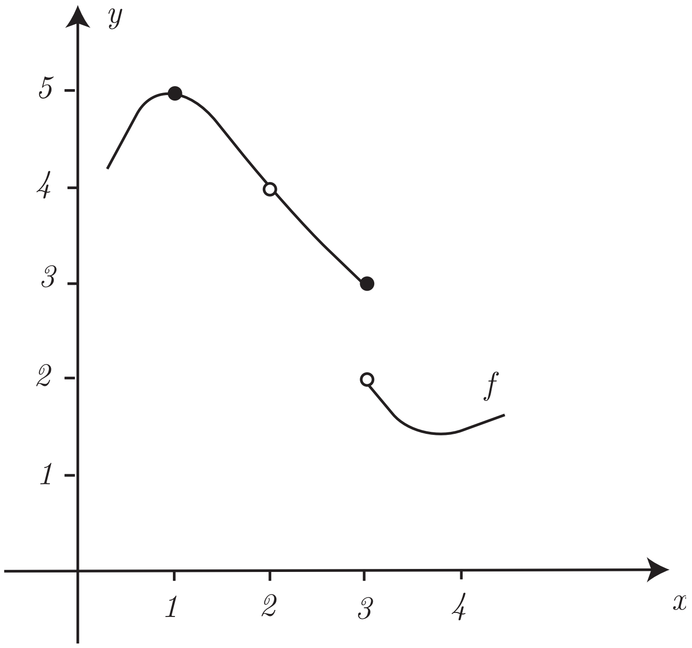

Figure 2.17: The curve of $f(x)$

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6. $\lim_{x\to 1} \frac{x^2+4x-5}{x^2-3x+2}$.
7. $\lim_{x\to 3} \frac{x^2-9}{x-3}$.
8. $\lim_{x\to e} \frac{(\ln x)^2-1}{\ln x-1}$.
9. $\lim_{x\to 0} \frac{e^{2x}-1}{e^x-1}$.
10. $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$.
11. $\lim_{x\to 0} \frac{x}{\sqrt{x+4}-2}$.

Solution. They are all continuous functions, so we obtain

1. $\lim_{x\to 3}(4x^2-1) = 4\cdot 9 - 1 = 35$.
2. $\lim_{x\to 3} x^2\sqrt{x-2} = 9\cdot\sqrt{1} = 9$.
3. $\lim_{x\to 3} \frac{x^3+\sqrt{x-2}}{x^2-4} = \frac{27+\sqrt{1}}{9-4} = \frac{28}{5}$.
4. Since $\ln 1 = 0$, we get

$$\lim_{x\to 1} \frac{(\ln x)^2-1}{\ln x-1} = \frac{(\ln 1)^2-1}{\ln 1-1} = \frac{-1}{-1} = 1.$$

5. In this case, if we directly plug in $x = 0$, then we get $\frac{0}{0}$, which is undefined. Thus, we need to simplify first and then plug in. Note that as $x \to 0$, x^5 and x^6 are much smaller than x^2, so they can be ignored. One way to ignore them is to divide by x^2 (the smallest power), so that they will be gone after taking the limit. That is, we have

$$\lim_{x\to 0} \frac{3x^5 + 7x^2}{4x^6 + 3x^2} = \lim_{x\to 0} \frac{x^2(3x^3 + 7)}{x^2(4x^4 + 3)} = \lim_{x\to 0} \frac{3x^3 + 7}{4x^4 + 3} = \frac{7}{3}.$$

Note that the cancellation of x^2 (or dividing by x^2) is allowed because $x \neq 0$.

6. Same as above, we cannot plug in directly. Now, we factor $x^2 - 3x + 2 = (x-1)(x-2)$ and find that $x-1$ gives trouble (becomes zero if plugging in $x = 1$). Thus we need to come up with $x - 1$ also in the numerator so they cancel. Indeed, we have $x^2 + 4x - 5 = (x-1)(x+5)$, so we get

$$\lim_{x\to 1} \frac{x^2 + 4x - 5}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{(x-1)(x+5)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x+5}{x-2} = \frac{6}{-1} = -6.$$

Note that the cancellation of $x - 1$ is allowed because $x \neq 1$.

7. We cannot plug in directly, and we need to come up with $x - 3$ also in the numerator. To simplify, we use the identity

$$A^2 - B^2 = (A - B)(A + B) \tag{2.6}$$

to factor $x^2 - 9$ as $x^2 - 9 = x^2 - 3^2 = (x-3)(x+3)$, and then obtain

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

Note that the cancellation of $x - 3$ is allowed because $x \neq 3$.

8. Since $\ln e = 1$, we cannot plug in directly, and we need to come up with $\ln x - 1$ also in the numerator. As $1 = 1^2$, we simplify using $A^2 - B^2 = (A - B)(A + B)$ and obtain

$$\lim_{x\to e} \frac{(\ln x)^2 - 1}{\ln x - 1} = \lim_{x\to e} \frac{(\ln x - 1)(\ln x + 1)}{\ln x - 1} = \lim_{x\to e}(\ln x + 1) = \ln e + 1 = 1 + 1 = 2.$$

Note that the cancellation of $\ln x - 1$ is allowed because $x \neq e$.

9. We cannot plug in directly, and we need to come up with $e^x - 1$ also in the numerator. To do so, we note that $e^{2x} = (e^x)^2$. Thus, same as above, we have

$$\lim_{x\to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x\to 0} \frac{(e^x - 1)(e^x + 1)}{e^x - 1} = \lim_{x\to 0}(e^x + 1) = 1 + 1 = 2.$$

Note that the cancellation of $e^x - 1$ is allowed because $x \neq 0$.

10. We cannot plug in directly, and we need to come up with x also in the numerator. Now, the difficulty is dealing with the square root $\sqrt{x+4}$. Thus, we use (2.6) in the reverse direction to create squares in order to get rid of the square root. That is, we treat $\sqrt{x+4}$ as A and 2 as B, and multiply $(A+B)$ top and bottom and then use $(A-B)(A+B) = A^2 - B^2$ to obtain

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{(\sqrt{x+4}\,)^2-2^2}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{x+4-4}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}\\ &= \frac{1}{2+2} = \frac{1}{4}.\end{aligned}$$

Note that the cancellation of x is allowed because $x \neq 0$.

11. We cannot plug in directly. In this case, it is not a good idea to come up with $\sqrt{x+4}-2$ also in the numerator (try and see why). Instead, we will use $(A-B)(A+B) = A^2 - B^2$ to simplify the denominator and then cancel the things causing trouble. Therefore, we have

$$\begin{aligned}\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2} &= \lim_{x\to 0}\frac{x(\sqrt{x+4}+2)}{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}\\ &= \lim_{x\to 0}\frac{x(\sqrt{x+4}+2)}{(\sqrt{x+4}\,)^2-2^2}\\ &= \lim_{x\to 0}\frac{x(\sqrt{x+4}+2)}{x+4-4}\\ &= \lim_{x\to 0}\frac{x(\sqrt{x+4}+2)}{x}\\ &= \lim_{x\to 0}(\sqrt{x+4}+2) = 4.\end{aligned}$$

Note that the cancellation of x is allowed because $x \neq 0$. ♠

The solutions given above are typical: if you can plug in with no trouble, just do it. Otherwise, try to use algebra skills to simplify and cancel the things causing trouble, then plug in.

Next, let's look at some piecewise defined functions for which the left and right limits should be used, so that Theorem 2.2.4 is especially useful.

Example 2.2.10 Find $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2+6, & \text{if } x \neq 0, \\ 2, & \text{if } x = 0. \end{cases}$

Solution 1. The function is defined differently around $x = 0$, so it causes some uncertainty around $x = 0$. When this is the case, we need to proceed with caution, which means we need to use the left and right limits. As $x \to 0^-$, we know that $x < 0$, so that $x \neq 0$. Thus, we replace $f(x)$ by x^2+6 and obtain

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+6) = 0+6 = 6,$$

where we used the fact that x^2+6 is a continuous function. Similarly, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+6) = 0+6 = 6.$$

Therefore, by using Theorem 2.2.4, we get

$$\lim_{x\to 0} f(x) = 6.$$

Solution 2. When $x \to 0$, we know that $x \neq 0$, thus we replace $f(x)$ by x^2+6 and obtain

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^2+6) = 6.$$

You can also sketch (use a calculator if needed) $f(x)$ and then find limit. If you do, you will see that the idea here is similar to that in Figure 2.14 where a hole will not affect the limit. ♠

Example 2.2.11 Find $\lim_{x\to 1} f(x)$ where

$$f(x) = \begin{cases} 2x+1, & x \leq 1, \\ -3x+2, & x > 1. \end{cases} \tag{2.7}$$

Solution. In this case, the uncertainty is around $x = 1$ and we should use the left and right limits because when x approaches 1, x can be on both sides of 1, so that in $\lim_{x\to 1} f(x)$, we don't know whether $f(x)$ takes the form of $2x+1$ or $-3x+2$. Accordingly, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x+1) = 2+1 = 3,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-3x+2) = -3+2 = -1.$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, the limit of $f(x)$ does not exist as x approaches 1, by using Theorem 2.2.4. ♠

Example 2.2.12 Find a constant a such that $f(x)$ has a limit at $x = 1$, where

$$f(x) = \begin{cases} 2x+1, & x \leq 1, \\ x+a, & x > 1. \end{cases} \tag{2.8}$$

Solution. Similar to the above example, we use the left and right limits and obtain

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+1) = 3,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+a) = 1+a.$$

For $f(x)$ to have a limit at $x=1$, we need $3 = 1+a$. Thus, if we solve and choose $a=2$, then $f(x)$ has a limit at $x=1$. ♠

Example 2.2.13 Find constants a and b such that $f(x)$ is continuous at $x=1$, where

$$f(x) = \begin{cases} 2x+a, & x<1, \\ b, & x=1, \\ x^2+2a, & x>1. \end{cases} \tag{2.9}$$

Solution. We have $f(1)=b$, and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+a) = 2+a,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+2a) = 1+2a.$$

For $f(x)$ to be continuous at $x=1$, $f(x)$ must have a limit at $x=1$ and the limit must be the same as $f(1)$. Thus, we have

$$2+a = 1+2a = b,$$

from which we solve $a=1$ and $b=3$. With these choices, $f(x)$ is continuous at $x=1$. ♠

Limits involving infinity ($\pm\infty$).

So far, the L and c in $\lim_{x\to c} f(x) = L$ are both finite numbers. However, sometimes it is useful to consider the cases where L and/or c are $\pm\infty$.

For example, in **Figure 2.18**, we see that as x is increased without bound, the function value $f(x)$ gets very close to 3. Thus we say that "as x approaches positive infinity, $f(x)$ has a limit 3", denoted by

$$\lim_{x\to\infty} f(x) = 3.$$

In **Figure 2.19**, we see that as x approaches 0, the function value $f(x)$ will increase without bound. Thus we say that "as x approaches 0, the function $f(x)$ approaches positive infinity (or has a limit ∞)", denoted by

$$\lim_{x\to 0} f(x) = \infty.$$

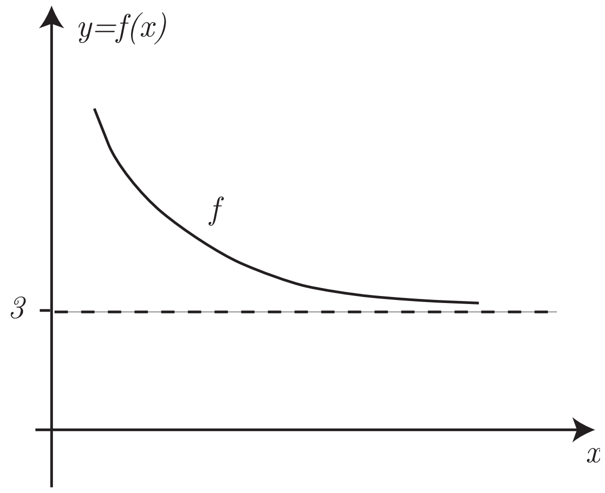


Figure 2.18: $\lim_{x\to\infty} f(x) = 3$

Similarly, for the straight line in **Figure 2.20**, we have

$$\lim_{x\to\infty} f(x) = -\infty; \quad \lim_{x\to-\infty} f(x) = \infty.$$

Accordingly, if you know the graph of a function, then you can look at the tendency and make a conclusion about the limit. If a function is given by a formula, then you can use a graphing calculator to find limit, or evaluate the function at some appropriate x values and make a conclusion, as we do next.

Example 2.2.14 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to\pm\infty} \frac{C}{x^k}$, where k is a positive constant and C is any constant (and x^k is defined for $x < 0$).

Solution. For $\lim_{x\to\infty} \frac{1}{x}$, we choose some big x values, such as $x = 10, 100, 1000, \cdots$, and obtain $\frac{1}{10} = 0.1, \frac{1}{100} = 0.01, \frac{1}{1000} = 0.001, \cdots$, from which we conclude that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

This is similar to sharing one pizza (the numerator is 1) by many many people (the denominator x increases without bound), so that everyone gets very very little ($\frac{1}{x}$ goes to zero).

For $\lim_{x\to\pm\infty} \frac{C}{x^k}$, the numerator C is a constant, and as $x \to \infty$, the denominator x^k increases without bound (as $x \to -\infty$, x^k may be negative, but the absolute value increases without bound), thus, similar to the case of $\frac{1}{x}$, the quotient also goes to zero. Therefore,

$$\lim_{x\to\pm\infty} \frac{C}{x^k} = 0.$$

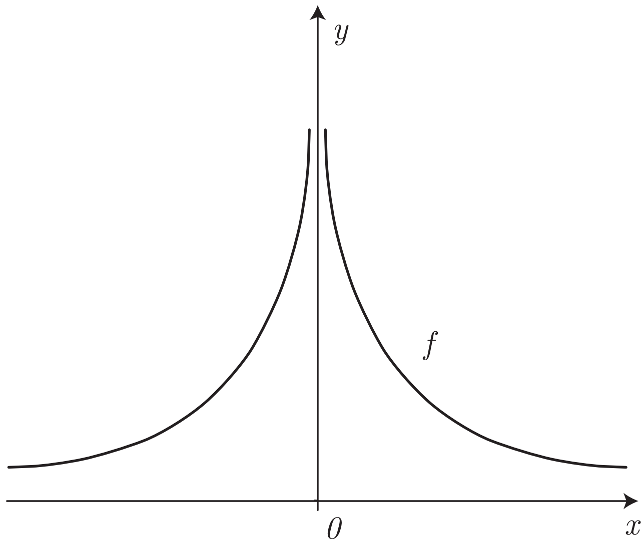


Figure 2.19: $\lim_{x\to 0} f(x) = \infty$

♠

The result in Example 2.2.14 can be represented in words as a

Rule:

$$\frac{\text{the numerator is or approaches a constant}}{\text{the denominator approaches } \pm\infty} \longrightarrow 0.$$

Example 2.2.15 Find $\lim_{x\to\infty} \frac{3x^9+7x^4+x}{8x^9-3x^6+4}$.

Solution. You may still plug in big x values and make a conclusion, but it will get a little messy in this case. The following is a better approach. As $x \to \infty$, x^9 will become the *dominating term* (if $x = \$10$, then what is x^9? if $x = \$100$, then what is x^9?), and others with smaller powers can be ignored. One way to ignore them is to divide by x^9 (the biggest power), so that they will be gone after taking the limit. That is, we have

$$\lim_{x\to\infty} \frac{3x^9+7x^4+x}{8x^9-3x^6+4} = \lim_{x\to\infty} \frac{(3x^9+7x^4+x)/x^9}{(8x^9-3x^6+4)/x^9} = \lim_{x\to\infty} \frac{3+\frac{7}{x^5}+\frac{1}{x^8}}{8-\frac{3}{x^3}+\frac{4}{x^9}} = \frac{3}{8},$$

where $\frac{7}{x^5}$ and other similar terms go to zero by using the above rule.

Note that you should not plug in $x \to \infty$ directly: if you do, you will get $\frac{\infty}{\infty}$, which is undefined. ♠

Rules: *To find limits of rational functions as $x \to \pm\infty$, divide with the biggest power (see Example 2.2.15); to find limits of rational functions as $x \to 0$, divide with the smallest power (see Example 2.2.9 (5)).*

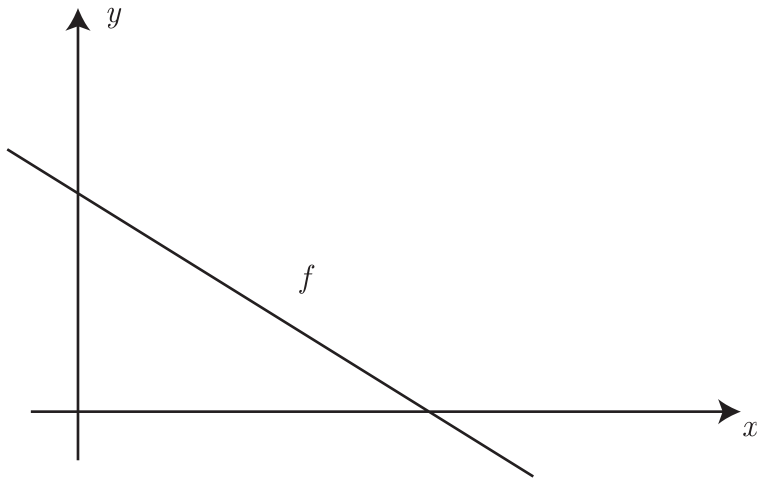


Figure 2.20: $\lim_{x\to\infty} f(x) = -\infty$; $\lim_{x\to-\infty} f(x) = \infty$

Example 2.2.16 Find $\lim_{x\to-\infty} \frac{6x^3+7x}{7x^5-x^2+4}$.

Solution. Similar to the above example, we have

$$\lim_{x\to-\infty} \frac{6x^3+7x}{7x^5-x^2+4} = \lim_{x\to-\infty} \frac{(6x^3+7x)/x^5}{(7x^5-x^2+4)/x^5} = \lim_{x\to-\infty} \frac{\frac{6}{x^2}+\frac{7}{x^4}}{7-\frac{1}{x^3}+\frac{4}{x^5}} = \frac{0}{7} = 0.$$

♠

Example 2.2.17 Find $\lim_{x\to\infty}(2x^3 - 9x^2 + 12x + 16)$.

Solution. As $x \to \infty$, x^3 will become the dominating term and others with smaller powers can be ignored, so that the limit should be ∞. One way to verify this is to go through the limit

$$\begin{aligned}\lim_{x\to\infty}(2x^3 - 9x^2 + 12x + 16) &= \lim_{x\to\infty} x^3\Big(\frac{2x^3 - 9x^2 + 12x + 16}{x^3}\Big) \\ &= \lim_{x\to\infty} x^3\Big(2 - \frac{9}{x} + \frac{12}{x^2} + \frac{16}{x^3}\Big) = \infty,\end{aligned}$$

because $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to\infty}(2 - \frac{9}{x} + \frac{12}{x^2} + \frac{16}{x^3}) = 2$. ♠

Next, we look at the cases where the numerators approach constants and the denominators approach zero.

Example 2.2.18 Find $\lim_{x\to0^+} \frac{1}{x}$; $\lim_{x\to0^-} \frac{1}{x}$.

Solution. For $\lim_{x\to0^+} \frac{1}{x}$, we choose some small and positive x values, such as $x = 0.1, 0.01, 0.001, \cdots$, and obtain $\frac{1}{0.1} = 10, \frac{1}{0.01} = 100, \frac{1}{0.001} = 1000, \cdots$, from which we conclude that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Similarly, we obtain

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

A consequence is that $\lim_{x \to 0} \frac{1}{x}$ does not exist. ♠

Example 2.2.19 Find $\lim_{x \to 1^+} \frac{x+1}{x-1}$; $\lim_{x \to 1^-} \frac{x+1}{x-1}$.

Solution. This case is similar to Example 2.2.18. For example, as $x \to 1^+$, the numerator approaches 2 and the denominator is positive and approaches 0. Thus, we obtain

$$\lim_{x \to 1^+} \frac{x+1}{x-1} = \infty; \quad \lim_{x \to 1^-} \frac{x+1}{x-1} = -\infty.$$

A consequence is that $\lim_{x \to 1} \frac{x+1}{x-1}$ does not exist. This example is important when we study curve sketching for rational functions, such as $\frac{x+1}{x-1}$. ♠

Example 2.2.20 Find $\lim_{x \to 0} \frac{1}{x^2}$.

Solution. When x approaches 0, x may be positive or negative, but x^2 is always positive and will also approach 0. Therefore, similar to Example 2.2.18, we get

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

♠

From these examples, we derive the following

Rule:

$$\frac{\text{the numerator is or approaches a constant}}{\text{the denominator approaches 0}} \longrightarrow \pm\infty,$$

where the sign is determined by from which side the denominator approaches zero.

Having learned the notion of limits, we remark that the number e introduced in Section 1.3 is actually defined using a limit:

$$e = \lim_{m \to \infty} \Big(1 + \frac{1}{m}\Big)^m, \tag{2.10}$$

which we understand as the amount in the account if \$1 is deposited in a bank for one year with an annual interest rate of 100% compounded *momentarily* or *continuously*, as the number of compounds m goes to infinity. Note that the m in (2.10) can be replaced by any other letter, so that a consequence is that if we take a limit on $A_m(x) = P\left(1+\frac{r}{m}\right)^{mx}$ given in (3.3) of Section 1.3 (the amount after depositing a principal P in a bank for x years at an annual interest rate r which is compounded m times per year), then we obtain

$$\begin{aligned} \lim_{m\to\infty} A_m(x) &= P\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{mx} = P\lim_{m\to\infty}\left[\left(1+\frac{1}{(m/r)}\right)^{m/r}\right]^{rx} \\ &= P\lim_{m^*\to\infty}\left[\left(1+\frac{1}{m^*}\right)^{m^*}\right]^{rx} = Pe^{rx}, \end{aligned} \tag{2.11}$$

where we denoted $m^* = \frac{m}{r}$ and used (2.10).

(2.11) is called **continuous compounding**, and is similar to the population growth models studied in Section 1.3. Note that this type of compounding grows exponentially fast, and it gives the maximum return after depositing a principal P for x years at an annual interest rate r, which is used as an extreme case when doing some comparisons.

Example 2.2.21 (Continuous compounding) Assume that \$1500 is deposited to a bank with an annual interest rate of 6% compounded continuously. Find the amount after two years. How long will it take for the initial deposit to triple?

Solution. Using (2.11), the amount after x years is given by $A(x) = Pe^{0.06x}$, so that the amount after two years is given by

$$A(2) = 1500e^{0.06\cdot 2} = 1500e^{0.12} \approx 1691.25$$

Next, to find when it will triple, we need to solve x such that

$$Pe^{0.06x} = 3P,$$

so that we get $0.06x = \ln 3$, or $x = \frac{\ln 3}{0.06} \approx 18$. That is, it takes about 18 years for the initial deposit to triple. ♠

Guided Practice 2.2

For the following find the limits using algebra.

1. $\lim_{x\to 3} 4x$.

2. $\lim_{x\to 9} \sqrt{x}$.

3. $\lim_{x\to -3} \frac{x}{x+6}$.

4. $\lim_{x\to 2} \frac{x^2-1}{x-1}$.

5. $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

6. $\lim_{x\to -2} \frac{x^2-x-6}{x+2}$.

7. $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+3}$.

8. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-2}$.

9. $\lim_{x\to\infty} \frac{2x^2}{x^2+1}$.

10. $\lim_{x\to\infty} \frac{2x^3}{x^2+1}$.

11. $\lim_{x\to\infty} \frac{2x}{x^2+1}$.

Exercises 2.2

1. Use a calculator to verify the numbers in Table 2.11.

2. For the function f in Figure 2.21, find $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x), f(a)$; $\lim_{x\to b^-} f(x)$, $\lim_{x\to b^+} f(x)$, $\lim_{x\to b} f(x), f(b)$; $\lim_{x\to c^-} f(x)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c} f(x), f(c)$; $\lim_{x\to d^-} f(x)$, $\lim_{x\to d^+} f(x)$, $\lim_{x\to d} f(x), f(d)$; $\lim_{x\to e^-} f(x)$, $\lim_{x\to e^+} f(x)$, $\lim_{x\to e} f(x), f(e)$; $\lim_{x\to -\infty} f(x)$, $\lim_{x\to\infty} f(x)$.

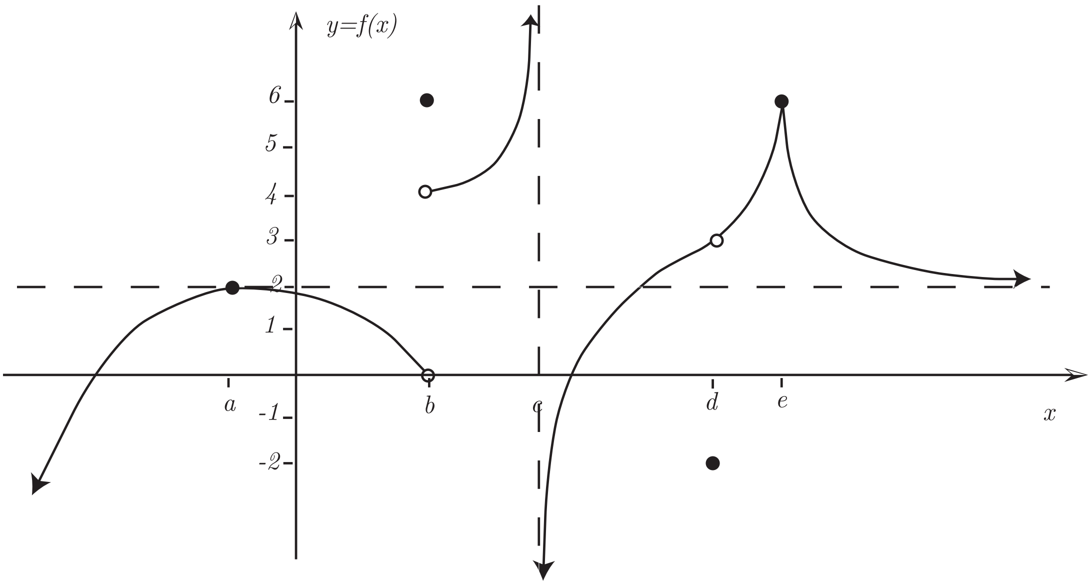


Figure 2.21: The graph of f

3. For each point of $\{a, b, c, d, e\}$ in Figure 2.21, is $f(x)$ continuous at the point? why or why not?

4. For the following, find the limits using graphs or tables of data, then find the limits using algebra.

(a) $\lim_{x\to 3}(x^3 - 1)$.

(b) $\lim_{x\to 3} x^3\sqrt{x^2 - 2x}$.

(c) $\lim_{x\to 3} \frac{x^2+\sqrt{x+2}}{x^2+5}$.

(d) $\lim_{x\to 1} \frac{(\ln x)^5-1}{\ln x-1}$.

(e) $\lim_{x\to 0} \frac{5x^9+3x^3}{2x^7+5x^3}$.

(f) $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

(g) $\lim_{x\to e} \frac{(\ln x)^3-1}{\ln x-1}$.

(h) $\lim_{x\to 0} \frac{e^{4x}-1}{e^x-1}$.

(i) $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$.

(j) $\lim_{x\to 1} \frac{2x^2-8x+8}{x^2+2x-8}$.

(k) $\lim_{x\to 2} \frac{2x^2-8x+8}{x^2+2x-8}$.

(l) $\lim_{x\to 2} \frac{3x^2-7x+2}{x^2+x-6}$.

(m) $\lim_{x\to\infty} \frac{5x^7+7x^4+x}{9x^7-3x^6+4}$.

(n) $\lim_{x\to-\infty} \frac{9x^7+7x}{7x^9-8x^2+4}$.

(o) $\lim_{x\to-1^+} \frac{x-1}{x+1}$; $\lim_{x\to-1^-} \frac{x-1}{x+1}$.

(p) $\lim_{x\to 0} \frac{|x|}{x}$.

(q) $\lim_{x\to 2} \frac{|x^2-4|}{2-x}$.

5. Find $\lim_{x\to 1} f(x)$ (using table, graph, or algebra), where

$$f(x) = \begin{cases} 5x+2, & x \le 1, \\ -3x+10, & x > 1. \end{cases}$$

6. Find a constant a such that $f(x)$ has a limit at $x = 1$, where

$$f(x) = \begin{cases} 6x-9, & x \le 1, \\ x+2a, & x > 1. \end{cases}$$

7. Find constants a and b such that $f(x)$ is continuous at $x = 1$, where

$$f(x) = \begin{cases} ax+b, & x < 1, \\ 2, & x = 1, \\ x^2+a, & x > 1. \end{cases}$$

8. Use Definition 2.2.6 to verify that $f(x) = |x|$ is a continuous function.

9. Assume that \$1000 is deposited to a bank with an annual interest rate of 6%. Find the amount after three years if (a). the bank compounds 20 times per year; (b). the bank compounds continuously.

10. Assume that \$1000 is deposited to a bank with an annual interest rate of 5% compounded continuously. Find the amount after two years. How long will it take for the initial deposit to triple? If the bank compounds only 10 times per year, then how long will it take for the initial deposit to triple?

2.3 The Derivative and the Power Rule

Having learned how to understand and perform operations on limits, we now go back to Section 2.1 where we see in geometry that as $w \to c$,

$$\frac{g(w) - g(c)}{w - c} \tag{3.1}$$

can be used to approximate the slope of the tangent line of a function g at $(c, g(c))$. Also, in the example of traveling from city A to city B in Section 2.1, we know that as $w \to c$,

$$\frac{m(w) - m(c)}{w - c} \tag{3.2}$$

can be used to approximate the instantaneous velocity of your car at time c.

Note that (3.1) and (3.2) are of the same format. Accordingly, due to their importance in geometry and physics applications, we generalize these ideas and make the following definition. Note also that after taking limits as $w \to c$, the w in (3.1) and (3.2) will be gone and we are left with some expressions in c, where the c in (3.1) and (3.2) can be an arbitrary place in geometry and an arbitrary time as you drive. In this regard we now treat c as the independent variable and change c to x (because we are so used to use x for the independent variable) in the following definition, so that the resulting limit will be a function in x.

Definition 2.3.1 *Let $f(x)$ be a function defined on an interval (a, b). If the limit*

$$\lim_{w \to x} \frac{f(w) - f(x)}{w - x} \tag{3.3}$$

exists and is finite, then we say that "f has a **derivative** *at x" or "f is* **differentiable** *at x". We denote this derivative at x as*

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}, \quad \text{or} \quad \frac{d}{dx} f(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}, \tag{3.4}$$

where "$'$" is called "prime".

Next, we say that "f has derivatives in (a, b)" or "f is differentiable in (a, b)" if f has a derivative at every point of (a, b). Since $f'(x)$ is a function, we call it a **derivative function**.

Note that in (3.3), if we let $w = x+h$, then $w-x = h$, thus the derivative at x can also be written as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.5}$$

Also note that if $g(t)$ is a function of the independent variable t, then its derivative is written as $g'(t)$ or $\frac{d}{dt}g(t)$. The same remark also applies to functions with other independent variables.

Recall that in the example of traveling from city A to city B in Section 2.1, $\frac{m(w)-m(c)}{w-c}$ gives the average velocity of your car on the interval $[c, w]$. In general, for a quantity $q(x)$, $\frac{q(w)-q(c)}{w-c}$ gives the **average rate of change** of the quantity on the interval $[c, w]$. Upon taking a limit as $w \to c$, this average rate can be regarded as if it is obtained **instantaneously** at c, so we call the limit (derivative) the **instantaneous rate of change**, or just the **rate of change**. If we also look at the connection between tangent lines and derivatives, then we have the following ways to understand derivatives.

1. In geometry, the derivative means the **slope** of a tangent line.

2. For a quantity, the derivative means the (instantaneous) **rate of change**, such as the *instantaneous velocity* of a moving car.

We require the limit in the definition of a derivative to be finite, because, for example, the instantaneous velocity cannot be infinite; and the slope of a tangent line cannot be infinite because otherwise the tangent line becomes a vertical straight line for which we do not define slope.

Next, let's find derivative functions for some simple functions. According to Definition 2.3.1, we need to complete the following steps:

1. Find $f(w)$ (or $f(x + h)$ if using (3.5)).

2. Simplify $\frac{f(w)-f(x)}{w-x}$ (or $\frac{f(x+h)-f(x)}{h}$ if using (3.5)).

3. Take the limit of $\frac{f(w)-f(x)}{w-x}$ as $w \to x$ (or of $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$).

Example 2.3.2 Use definition (i.e., limit) to find (the derivative function) $f'(x)$ for $f(x) = 1$.

Solution 1. Use our knowledge of functions from Chapter 1, we know that this function (machine) is such that the output is always 1 for all inputs. Thus

$$f(w) = 1.$$

Then

$$\begin{aligned} f'(x) &= \lim_{w \to x} \frac{f(w) - f(x)}{w - x} \\ &= \lim_{w \to x} \frac{1 - 1}{w - x} = \lim_{w \to x} \frac{0}{w - x} = 0. \end{aligned}$$

This matches with the geometry that the slope of the horizontal straight line $f(x) = 1$ (or $y = 1$) is zero.

Solution 2. Here we use (3.5). From the definition of the function, we get $f(x + h) = 1$, so that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{1 - 1}{h} = \lim_{h \to 0} \frac{0}{h} = 0. \end{aligned}$$

♠

Example 2.3.3 Use definition (i.e., limit) to find $f'(x)$ for $f(x) = x$.

Solution 1. For this function, the output is the same as the input. Thus,

$$f(w) = w.$$

Then

$$\begin{aligned} f'(x) &= \lim_{w \to x} \frac{f(w) - f(x)}{w - x} \\ &= \lim_{w \to x} \frac{w - x}{w - x} \\ &= \lim_{w \to x} \frac{1}{1} = 1. \end{aligned}$$

Now, we can still use geometry to check the result because the slope of the straight line $f(x) = x$ (or $y = x$) is 1.

Solution 2. Here we use (3.5). From the definition of the function, we get $f(x + h) = x + h$, so that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x + h) - x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} = 1. \end{aligned}$$

♠

Example 2.3.4 Use definition (i.e., limit) to find $f'(x)$ for $f(x) = x^2$.

Solution 1. For this function, the output is to square the input. Thus,

$$f(w) = w^2.$$

Then, we try to come up with $w - x$ also in the numerator by using $A^2 - B^2 = (A - B)(A + B)$ and obtain

$$\begin{aligned} f'(x) &= \lim_{w \to x} \frac{f(w - f(x)}{w - x} \\ &= \lim_{w \to x} \frac{w^2 - x^2}{w - x} \\ &= \lim_{w \to x} \frac{(w - x)(w + x)}{w - x} \\ &= \lim_{w \to x} (w + x) = 2x, \end{aligned}$$

where x is regarded as a constant in taking the limit as $w \to x$ since x has nothing to do with w.

Solution 2. Here we use (3.5). From the definition of the function, we get $f(x + h) = (x + h)^2$, so that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h)}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

♠

Remark 2.3.5 A common mistake in solving Example 2.3.4 (and similar ones) is to write $f(x+h) = x^2+h$, because then we would get $\frac{f(x+h)-f(x)}{h} = \frac{x^2+h-x^2}{h} = \frac{h}{h} = 1$, so the function would look like a straight line $y = x + C$ (for a constant C) in order for the slope to be 1. But $f(x) = x^2$ is not a straight line. ♠

Example 2.3.6 Use definition (i.e., limit) to find $f'(x)$ for $f(x) = \sqrt{x}$, $(x \geq 0)$.

Solution 1. For this function, the output is to take a square root of the input. Thus,

$$f(w) = \sqrt{w}.$$

Then,

$$f'(x) \quad = \quad \lim_{w \to x} \frac{\sqrt{w} - \sqrt{x}}{w - x},$$

and we need to come up with $w - x$ also in the numerator. Now, similar to the examples of finding limits in the previous section, we multiply by $\sqrt{w} + \sqrt{x}$ top and bottom and then use $(A - B)(A + B) = A^2 - B^2$ to get rid of the square roots. Therefore,

$$\begin{aligned}
f'(x) &= \lim_{w \to x} \frac{\sqrt{w} - \sqrt{x}}{w - x} \\
&= \lim_{w \to x} \frac{(\sqrt{w} - \sqrt{x})(\sqrt{w} + \sqrt{x})}{(w - x)(\sqrt{w} + \sqrt{x})} \\
&= \lim_{w \to x} \frac{(\sqrt{w})^2 - (\sqrt{x})^2}{(w - x)(\sqrt{w} + \sqrt{x})} \\
&= \lim_{w \to x} \frac{w - x}{(w - x)(\sqrt{w} + \sqrt{x})} \\
&= \lim_{w \to x} \frac{1}{\sqrt{w} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}, \quad \text{if } x > 0.
\end{aligned}$$

Note that in this case, the function $\sqrt{x}$ is defined for $x \geq 0$, but $\sqrt{x}$ has derivatives only for $x > 0$.

Solution 2. Here we use (3.5). From the definition of the function, we get $f(x + h) = \sqrt{x + h}$, so that

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{x + h} - \sqrt{x})(\sqrt{x + h} + \sqrt{x})}{h(\sqrt{x + h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{(\sqrt{x + h})^2 - (\sqrt{x})^2}{h(\sqrt{x + h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}, \quad \text{if } x > 0.
\end{aligned}$$

♠

Example 2.3.7 Use definition (i.e., limit) to find $f'(x)$ for $f(x) = \frac{1}{x}$, $(x \neq 0)$.

Solution 1. For this function, the output is to use 1 to divide by the input. Thus,

$$f(w) = \frac{1}{w}.$$

Then

$$\begin{aligned}
f'(x) &= \lim_{w \to x} \frac{f(w) - f(x)}{w - x} \\
&= \lim_{w \to x} \frac{\frac{1}{w} - \frac{1}{x}}{w - x} = \lim_{w \to x} [\frac{1}{w} - \frac{1}{x}] \frac{1}{w - x} \\
&= \lim_{w \to x} [\frac{x}{wx} - \frac{w}{wx}] \frac{1}{w - x} \\
&= \lim_{w \to x} [\frac{x - w}{wx}] \frac{1}{w - x} \\
&= \lim_{w \to x} [\frac{-(w - x)}{wx}] \frac{1}{w - x} \\
&= \lim_{w \to x} \frac{-1}{wx} \\
&= \frac{-1}{x^2} = -x^{-2}.
\end{aligned}$$

Solution 2. Here we use (3.5). From the definition of the function, we get $f(x + h) = \frac{1}{x+h}$, so that

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} [\frac{1}{x + h} - \frac{1}{x}] \frac{1}{h} \\
&= \lim_{h \to 0} [\frac{x}{x(x + h)} - \frac{x + h}{x(x + h)}] \frac{1}{h} \\
&= \lim_{h \to 0} [\frac{x - (x + h)}{x(x + h)}] \frac{1}{h} \\
&= \lim_{h \to 0} [\frac{-h}{x(x + h)}] \frac{1}{h} \\
&= \lim_{h \to 0} \frac{-1}{x(x + h)} \\
&= \frac{-1}{x^2} = -x^{-2}.
\end{aligned}$$

♠

Remark 2.3.8 After seeing these examples, you should practice a few similar examples and then choose a method you would like to work with. If dealing with $x+h$ causes troubles, such as in Example 2.3.7 where a common mistake is to write $x - x + h$ for $x - (x+h)$, then $\frac{f(w)-f(x)}{w-x}$ should be used to avoid these algebra troubles. ♠

If you put the results of examples 2.3.2 – 2.3.7 together in the following table,

$$\begin{array}{rlllll} f(x): & 1, & x, & x^2, & \sqrt{x} = x^{\frac{1}{2}}, & \frac{1}{x} = x^{-1}, \\ f'(x): & 0, & 1, & 2x, & \frac{1}{2}x^{-\frac{1}{2}}, & -x^{-2}, \end{array}$$

you will be able to find a very interesting phenomenon. That is, to find the derivative of a *power function* x^α where α is a constant, it seems that all you need is to put the power α in front as the coefficient, and then subtract 1 from the power α to obtain $\alpha x^{\alpha-1}$. In fact, we have the following result, which we can use from now on.

Theorem 2.3.9 (Power Rule) *For any fixed real number α, the derivative of the power function x^α is given by*

$$\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1} \tag{3.6}$$

if $x^{\alpha-1}$ results in a real value. ♠

For example, for $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ to be valid, we need $x > 0$.

Example 2.3.10 Find $\frac{d}{dx}x^{99/100}$.

Solution. Using the power rule , we obtain

$$\frac{d}{dx}x^{99/100} = \frac{99}{100}x^{-1/100}, \quad \text{if } x > 0.$$

♠

For the function in Example 2.3.10, it is very difficult to derive the derivative using Definition 2.3.1 (try and see why). Then just imagine how difficult it is to *prove* the power rule using definition for *any* real number α, which could be π or e. To avoid using Definition 2.3.1 in a proof, we will wait and prove the power rule after learning the derivatives of exponential and logarithmic functions.

The following example indicates that the power rule can be used to obtain certain limits.

Example 2.3.11 Find $\lim_{h\to 0} \frac{(1+h)^{1000}-1}{h}$.

Solution. We rewrite $\lim_{h\to 0} \frac{(1+h)^{1000}-1}{h}$ as

$$\lim_{h\to 0} \frac{(1+h)^{1000}-1^{1000}}{h},$$

and then compare it with the definition of derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \quad \text{and} \quad f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}.$$

Thus we conclude that

$$\lim_{h\to 0} \frac{(1+h)^{1000}-1^{1000}}{h} = f'(1), \quad \text{where } f(x) = x^{1000}.$$

Therefore, as $f'(x) = 1000x^{999}$, we obtain

$$\lim_{h\to 0} \frac{(1+h)^{1000}-1^{1000}}{h} = f'(1) = 1000.$$

♠

To take derivatives of some combinations of functions, we need the following simple derivative rules.

Property 2.3.12 *Let k be a constant. If $f(x)$ and $g(x)$ have derivatives, then*

1. $\frac{d}{dx}[kf(x)] = k\frac{d}{dx}f(x)$.
2. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$.

Verification. (1). We treat $kf(x)$ as a new function and apply definition (i.e., limit) to obtain

$$\begin{aligned} \frac{d}{dx}[kf(x)] &= \lim_{h\to 0} \frac{kf(x+h)-kf(x)}{h} \\ &= k\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= k\frac{d}{dx}f(x). \end{aligned}$$

(2). Similarly, we have

$$\begin{aligned} \frac{d}{dx}[f(x) \pm g(x)] &= \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\ &= \lim_{h\to 0} [\frac{f(x+h)-f(x)}{h} \pm \frac{g(x+h)-g(x)}{h}] \\ &= \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x). \end{aligned}$$

♠

Example 2.3.13 Find $\frac{d}{dx}(3x^{2/3} - 5x^4 + 7x^2 + 9)$.

Solution. Property 2.3.12(2) is only stated for two functions, but you can verify that it is valid for more functions (by repeating the rule). Thus, we take derivative one by one and obtain

$$\begin{aligned} \frac{d}{dx}(3x^{2/3} - 5x^4 + 7x^2 + 9) &= 3\frac{2}{3}x^{-1/3} - 5 \cdot 4x^3 + 7 \cdot 2x + 0 \\ &= 2x^{-1/3} - 20x^3 + 14x. \end{aligned}$$

♠

Note that $2x^{-1/3} - 20x^3 + 14x$ from Example 2.3.13 is also a function (for $x \neq 0$), so we can take its derivative and obtain

$$\frac{d}{dx}(2x^{-1/3} - 20x^3 + 14x) = -\frac{2}{3}x^{-4/3} - 60x^2 + 14.$$

Now, if we write $f(x) = 3x^{2/3} - 5x^4 + 7x^2 + 9$ for the function in Example 2.3.13, then we have taken the derivative of f twice (called the **second derivative**), for which we denote

$$f''(x) = \frac{d^2}{dx^2}f(x) = -\frac{2}{3}x^{-4/3} - 60x^2 + 14.$$

We can continue to take more derivatives of this function $f(x)$. But if we take derivative of f 1000 times, then should we write the prime "′" 1000 times? Of course not. So we introduce the following notation.

Notation.

We use

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n}{dx^n}f(x)$$

to denote the n-th derivative of $f(x)$.

Typically, we write $f''(x)$ and $f'''(x)$ for the second and third derivatives. Beyond the third derivative, we use $f^{(n)}(x)$ or $\frac{d^n}{dx^n}f(x)$.

In physics, if $p(t)$ denotes the position of a moving object at time t, then we have seen that $p'(t)$ gives the instantaneous velocity. Now, if we take the second derivative, then $p''(t)$ gives the rate of change of velocity, which is called the **acceleration**.

In particular, the general position function for a free-falling object, neglecting air resistance, is determined to be

$$p(t) = -\frac{1}{2}gt^2 + v_0t + p_0,$$

where the time t is measured in seconds, the height $p(t)$ is measured in feet, v_0 is the initial velocity, p_0 is the initial height, and $g = 32$ (ft/sec^2) is the downward acceleration due to gravity. Note that if $v_0 > 0$, then the initial motion is upward; if $v_0 < 0$, then the initial motion is downward; if $v_0 = 0$, then no initial motion.

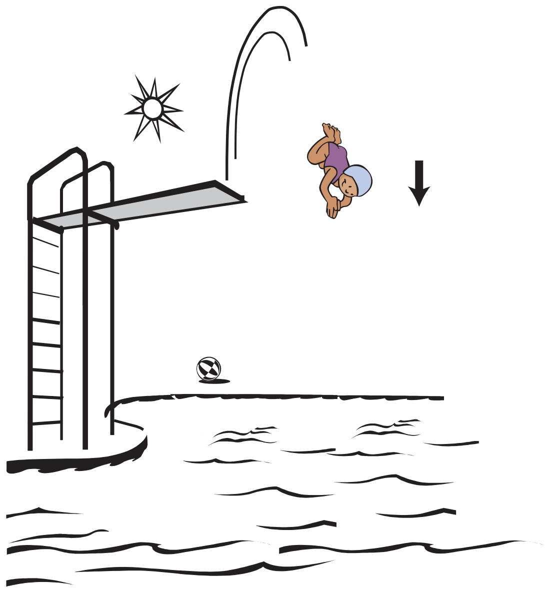


Figure 2.22: Jumping from a diving board

Example 2.3.14 If you jump with an initial velocity of 16 ft/sec from a diving board that is 96 feet high, then find

1. the height, the velocity, and the acceleration after $t = 2$ seconds.
2. when do you hit the water?
3. what is your velocity and acceleration at impact?

Solution. 1. See **Figure 2.22**. Since $v_0 = 16$ and $p_0 = 96$, the position function is given by

$$p(t) = -16t^2 + 16t + 96.$$

Taking derivatives, we get

$$\begin{aligned} p'(t) &= -32t + 16, \\ p''(t) &= -32. \end{aligned}$$

When $t = 2$, the height is $p(2) = -16(2)^2 + 16(2) + 96 = 64$ feet; the velocity is $p'(2) = -32(2) + 16 = -48$ ft/sec; and the acceleration is $p''(2) = -32$ ft/sec^2.

2. To find when do you hit the water, we let $p(t) = 0$ and solve for t:

$$p(t) = -16t^2+16t+96 = -16(t^2-t-6) = -16(t-3)(t+2) = 0 \quad \text{or} \quad t = -2, 3.$$

Since we need $t > 0$, we get $t = 3$. That is, you hit the water when $t = 3$ seconds.

3. To find your velocity and acceleration at impact, we plug $t = 3$ into $p'(t)$ and $p''(t)$. So we obtain the velocity to be $p'(3) = -32(3) + 16 = -80$ ft/sec and the acceleration to be $p''(3) = -32$ ft/sec^2. ♠

The following is an example of a curve with a vertical tangent line.

Example 2.3.15 Find $f'(0)$ for $f(x) = \sqrt{x}$, $x \geq 0$.

Solution. The power rule $(x^{1/2})' = \frac{1}{2\sqrt{x}}$ is not applicable because it is valid only for $x > 0$. Using the knowledge of functions or using a graphing calculator, we see that the curve of $f(x) = \sqrt{x}$ is given in **Figure 2.23**, where $f(x) = \sqrt{x}$ has a vertical tangent line at $x = 0$, which indicates that $f(x)$ has no derivative at $x = 0$.

Now, let's use the definition of derivative to check. We have, at $x = 0$,

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sqrt{h} - 0}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty,$$

thus $f'(0)$ does not exist. ♠

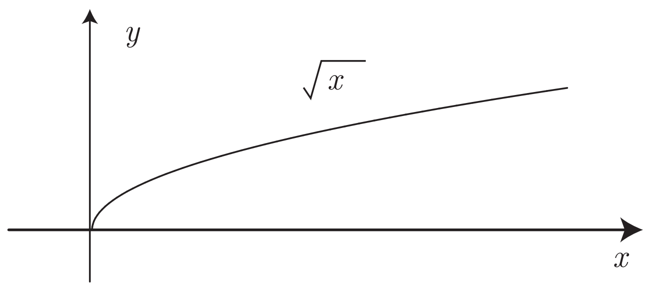


Figure 2.23: A curve with a vertical tangent line

The following is an example of a curve with a point at which there is no tangent line.

Example 2.3.16 Find $f'(0)$ for $f(x) = |x|$.

Solution. The graph of $f(x) = |x|$ is given in **Figure 2.24**. At $x = 0$, the curve has a "sharp corner" in the sense that there is no unique way to put a tangent line there. So, it indicates that $f(x)$ has no derivative at $x = 0$.

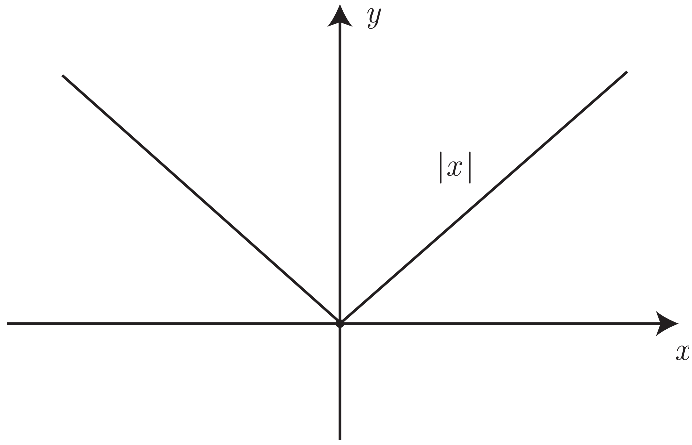


Figure 2.24: The curve of $f(x) = |x|$

This can also be seen as follows: if we use secant lines to approximate the tangent line at $x = 0$, then any secant line on the right-hand side of $x = 0$ has slope 1, and any secant line on the left-hand side of $x = 0$ has slope -1. Since $1 \neq -1$, the slope at $x = 0$ cannot be defined.

Now let's use the definition of derivative to check. Since $f(x)$ is piecewise defined, we should use the right and left limits for $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$. Thus we have

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1,$$

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,$$

therefore, $f'(0)$ does not exist. ♠

Finally, let's look at the relationship between the notions of *continuity* and *differentiability*. We can use the absolute value function $f(x) = |x|$ to make a point. From Example 2.3.16, we know that $f(x) = |x|$ has no derivative at $x = 0$. But the curve of $f(x) = |x|$ has no holes or breaks, so $f(x) = |x|$ is continuous (can be verified in details). Thus we conclude that

Continuity does not imply differentiability.

Question: *Does differentiability imply continuity?*

The answer is *yes* because of the following result.

Theorem 2.3.17 *If a function $f(x)$ has a derivative at $x = c$, then $f(x)$ is continuous at $x = c$.*

Verification. Using the assumption that $f(x)$ has a derivative at $x = c$, we know that the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and is finite, which also indicates that $f(c)$ is defined. To show that $f(x)$ is continuous at $x = c$, we now need to show that

$$\lim_{x \to c} f(x) = f(c),$$

which, after letting $x = c + h$, is equivalent to

$$\lim_{h \to 0} f(c+h) = f(c), \quad \text{or} \quad \lim_{h \to 0} [f(c+h) - f(c)] = 0.$$

To this end, we write

$$f(c+h) - f(c) = [\frac{f(c+h) - f(c)}{h}] \cdot h,$$

so we obtain

$$\begin{aligned} \lim_{h \to 0} [f(c+h) - f(c)] &= [\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}][\lim_{h \to 0} h] \\ &= [f'(c)][0] = 0, \end{aligned}$$

since $f'(c)$ is a finite number. This completes the verification. ♠

A consequence is that if $f(x)$ is not continuous at $x = c$, then $f(x)$ has no derivative at $x = c$.

Applications in Business.

Note that the notions of revenue functions, cost functions, and profit functions were briefly mentioned in an example in Chapter 1. Here, we provide further details.

In running a business (such as producing and selling certain tables), it is a common practice to give discount for larger orders. That is, the unit price is determined by how many units are ordered. If we use x to denote the number of units in an order, and use $u(x)$ to denote the corresponding unit price (price per unit if x units are ordered), then $u(x)$ is called a **price function** (also called a **demand equation**). Typically, $u(x)$ decreases as x increases. For example, $u(x)$ may be given by

$$u(x) = 800 - x,$$

so that for an order of $x = 10$ units, the price is $800 - 10 = 790$ per unit; for an order of $x = 15$ units, the price is $800 - 15 = 785$ per unit.

If we multiply x by $u(x)$, then $xu(x)$ gives the revenue of producing and selling x units with unit price $u(x)$. We denote this revenue by $R(x)$. That is,

$$R(x) = xu(x)$$

gives the revenue of producing and selling x units of the product, and we call $R(x)$ a **revenue function**.

In most cases, there is a cost to produce certain products, such as the cost of materials and/or labor. So we use $C(x)$ to denote the cost of producing and selling x units of the product, and call it a *cost function*. Now, if we subtract the cost from the revenue, then we get the profit. That is,

$$P(x) = R(x) - C(x)$$

gives the profit of producing and selling x units of the product, and we call $P(x)$ a **profit function**.

In real-life situations, the x values should take positive integers only. However, in order to use calculus (a very powerful tool as we will see) to help us analyze problems, we assume that these functions, $u(x)$, $R(x)$, $C(x)$, and $P(x)$, are defined for *all* x values of their corresponding domains. Also note here that to make situations simple, we assume that when producing x units of the product, they can all be sold.

Consider the cost function $C(x)$. In business, the *cost of producing one additional unit*, $C(x+1) - C(x)$, is called the **marginal cost**. In many business applications, people are dealing with large quantities, so that 1 is regarded as a very small number, such as one candy in a candy factory. Thus we can treat $1 \approx h \approx 0$ and obtain

$$\begin{aligned} C(x+1) - C(x) &= \frac{C(x+1) - C(x)}{1} \approx \frac{C(x+h) - C(x)}{h} \\ &\approx \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} = C'(x). \end{aligned} \tag{3.7}$$

Therefore, in business applications, $C'(x)$ is called the **marginal cost function**, which gives the rate of change of the cost function. Note that the marginal cost function gives only an approximation of the marginal cost, because of the steps involved in (3.7) where 1 is treated as close to zero.

We also call $\overline{C}(x) = \frac{C(x)}{x}$ the **average cost function** because it gives the cost per unit, and then call $\overline{C}'(x)$ the **marginal average cost function**. Similarly, $R'(x)$ is called the **marginal revenue function**, and $P'(x)$ is called the **marginal profit function**.

Example 2.3.18 If the cost of producing x units of a certain product is

$$C(x) = x^3 - 30x^2 + 500x + 100 \text{ (\$)},$$

and its revenue is

$$R(x) = 900x - 3x^2,$$

then find

1. The marginal cost function $C'(x)$.
2. $C'(10)$ and $C(10+1) - C(10)$, and then compare.
3. The average cost function.
4. The marginal average cost function.
5. The marginal revenue function.
6. The marginal profit function.

Solution. 1. The marginal cost function is given by

$$C'(x) = 3x^2 - 60x + 500.$$

2. $C'(10) = 300 - 600 + 500 = 200\,(\$)$, and

$$\begin{aligned}&C(10+1) - C(10)\\&= [11^3 - 30(11)^2 + 500(11) + 100] - [10^3 - 30(10)^2 + 500(10) + 100]\\&= 201\ (\$),\end{aligned}$$

which is so close to $C'(10) = 200\,(\$)$. This means that calculation from a marginal cost function gives a good approximation of the marginal cost.

3. The average cost function is given by

$$\overline{C}(x) = \frac{C(x)}{x} = x^2 - 30x + 500 + \frac{100}{x}.$$

4. The marginal average cost function is given by

$$\overline{C}'(x) = 2x - 30 - \frac{100}{x^2}.$$

5. The marginal revenue function is given by

$$R'(x) = 900 - 6x.$$

6. The profit function is given by $P(x) = R(x) - C(x)$, thus the marginal profit function is given by

$$P'(x) = R'(x) - C'(x) = 900 - 6x - (3x^2 - 60x + 500) = -3x^2 + 54x + 400.$$

♠

Example 2.3.19 If the cost and revenue functions for a company are given in **Figure 2.25**, then should the company produce the 12th unit?

Solution. Marginal functions are derivative functions, which in geometry are slopes. From Figure 2.25, the slope of the revenue function at $x = 11$ is bigger than that of the cost function. This means the marginal revenue function is bigger than the marginal cost function at $x = 11$, or the company will make more in extra revenue than it will spend in extra cost if it produces one additional unit. Therefore, the company should produce the 12th unit. ♠

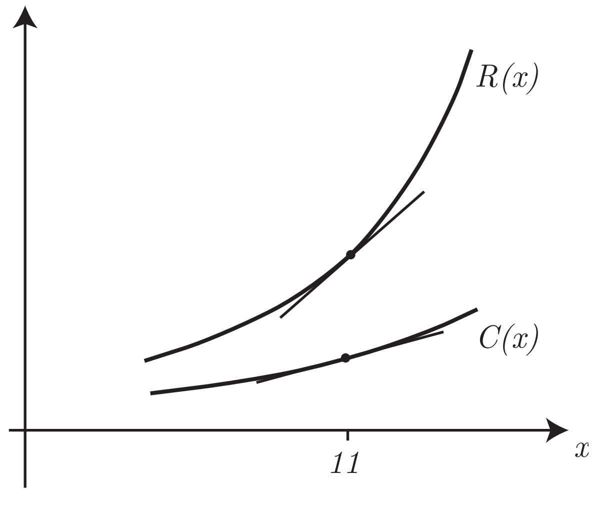


Figure 2.25: Cost and revenue functions

More applications in business using these functions will be given later when we study optimizations.

Guided Practice 2.3

1. Use definition (i.e., limit) to find $f'(x)$ for

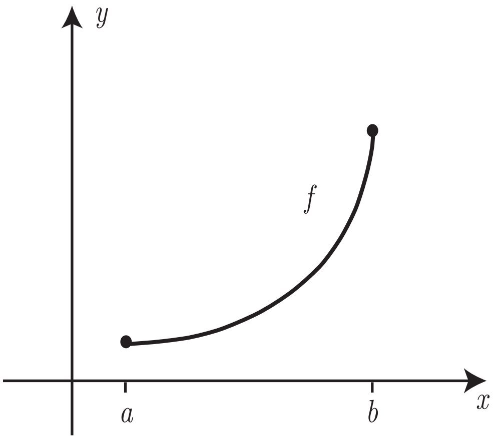
(a) $f(x) = 2x$.
(b) $f(x) = 3x^2 + 1$.
(c) $f(x) = \frac{1}{x+1}$.

2. Find $f'(x)$ for the following functions $f(x)$ using *Power Rule*.

(a) $f(x) = x^2 + 5x + 8$.
(b) $f(x) = \frac{2}{x^5}$.
(c) $f(x) = \sqrt{x} + 2x$.
(d) $f(x) = \frac{1}{6}x^6 + 6x^{1/6}$.

Exercises 2.3

1. For the two functions in **Figure 2.26**, describe the differences in the changes of their derivatives. If the functions represent the changes in mileage of two cars (with the variable x denoting the time), then describe the differences in driving behavior of the two drivers.



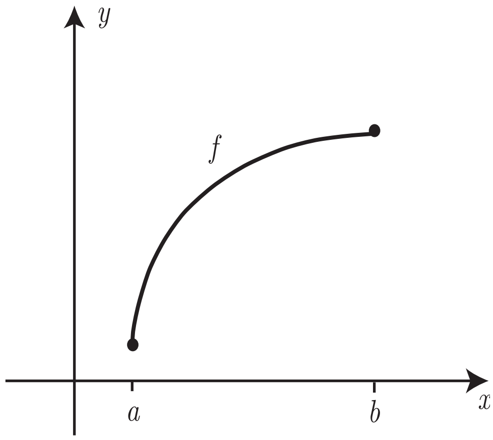


Figure 2.26: Two different functions

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2. Use definition (i.e., limit) to find $f'(x)$ for

(a) $f(x) = 3$.
(b) $f(x) = 2x + 1$.
(c) $f(x) = (x - 4)^2$.
(d) $f(x) = \sqrt{3x - 1} + 5$.
(e) $f(x) = \frac{2}{3x} + 2$.
(f) $f(x) = \frac{2}{3x-1}$.
(g) $f(x) = 4x - \frac{3}{2x}$.

3. Find $f'(x)$ for

(a) $f(x) = 3$.
(b) $f(x) = 2x + 1$.
(c) $f(x) = x^2 - x^{2/3} + 8$.
(d) $f(x) = \sqrt{x} + 5x^5 - 6x^{1/6}$.
(e) $f(x) = \frac{2}{3x} + 2$.
(f) $f(x) = 4x - \frac{3}{2x}$.

4. Find $f'(x)$ for

(a) $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{x}$.
(b) $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{\sqrt{x}}$.

5. Find a curve of a function that is continuous but has no derivatives at two or more places.

6. If you jump with an initial velocity of 16 ft/sec from a diving board that is 32 feet high, then find

(a) the height, the velocity, and the acceleration when $t = 1$ seconds.

(b) when do you hit the water?

(c) what is your velocity and acceleration at impact?

7. If you throw a ball up with an initial velocity of 16 ft/sec from a cliff that is 192 feet high, then find

(a) the height, the velocity, and the acceleration when $t = 2$ seconds.

(b) when does the ball hit the ground?

(c) what is ball's velocity and acceleration at impact?

8. If the cost of producing x units of certain product is

$$C(x) = 2x^3 - 25x^2 + 450x + 200 \text{ (\$)},$$

and its revenue is

$$R(x) = 1000x - 4x^2,$$

then find

(a) The marginal cost function $C'(x)$.

(b) $C'(15)$ and $C(15+1) - C(15)$, and then compare.

(c) The average cost function.

(d) The marginal average cost function.

(e) The marginal revenue function.

(f) The marginal profit function.

9. If the cost and revenue functions for two companies are given in Figure 2.27, then should each company produce the 15th unit?

10. Find the following limits.

(a) $\lim_{h\to 0} \frac{(1+h)^{1234}-1}{h}$.

(b) $\lim_{w\to 1} \frac{w^{4321}-1}{w-1}$.

(c) $\lim_{h\to 0} \frac{(2+h)^5-32}{h}$.

(d) $\lim_{w\to 2} \frac{w^5-32}{w-2}$.

(e) $\lim_{h\to 0} \frac{(3+h)^4-81}{h}$.

(f) $\lim_{w\to 3} \frac{w^4-81}{w-3}$.

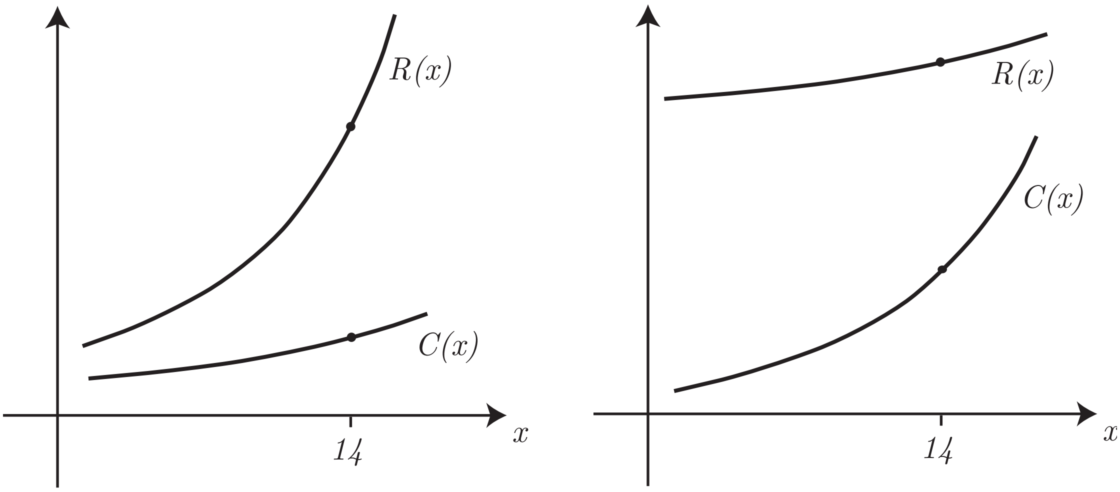

Figure 2.27: Cost and revenue functions

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11. Let f be differentiable at x_0. Define

$$g(h) = \begin{cases} \frac{f(x_0+h)-f(x_0)}{h}, & h \neq 0, \\ c, & h = 0. \end{cases} \tag{3.8}$$

Find $\lim_{h\to 0} g(h)$. Then find c such that $g(h)$ is continuous at $h = 0$.

12. Let f be differentiable at 2. Define

$$g(x) = \begin{cases} \frac{f(x)-f(2)}{x-2}, & x \neq 2, \\ c, & x = 2. \end{cases} \tag{3.9}$$

Find c such that $g(x)$ is continuous at $x = 2$.

2.4 Tangent Lines and Linear Approximations

Question: *Can you approximate* $\sqrt{81.34}$ *without using a calculator? Next, if you do use a calculator to approximate, then do you know what happens after you press some buttons?*

To answer these questions, let's recall that at the beginning of Chapter 2, we asked the question of how to find the tangent line of a function at a point, from which the notion of *derivative* was derived so that the *differential calculus* was born.

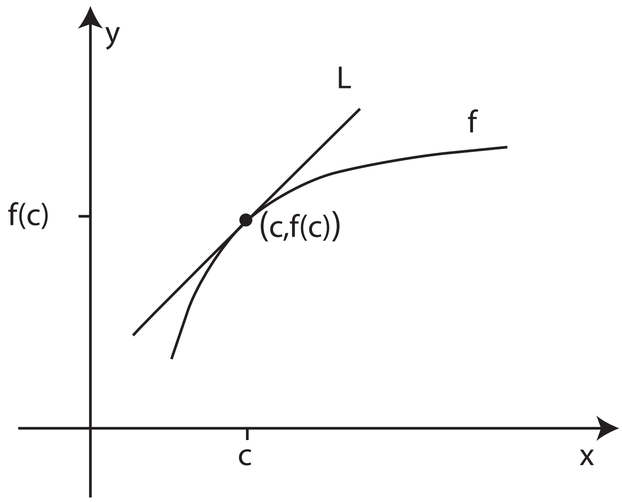

Figure 2.28: The tangent line at the point $(c, f(c))$

Now, we know that derivatives give slopes of tangent lines, so we are able to use them to find tangent lines. Let $f(x)$ be a function and consider the tangent line at the point $(c, f(c))$, see **Figure 2.28**.

Since the slope m of the tangent line at the point $(c, f(c))$ is the derivative evaluated at $x = c$, we get

$$m = f'(c),$$

therefore, with the point $(c, f(c))$ and the slope $m = f'(c)$, the **tangent line** at the point $(c, f(c))$ is given by

$$y = f(c) + f'(c)(x - c). \tag{4.1}$$

From the geometry in Figure 2.28, we see that near the point $(c, f(c))$ (the point of tangency), the original function $f(x)$ and the tangent line are almost the same, that is,

$$f(x) \approx f(c) + f'(c)(x - c), \quad \text{when} \quad x \approx c, \tag{4.2}$$

which provides an approximation of $f(x)$ near $x = c$ by using a straight line.

Note that (4.2) can also be obtained as follows: From

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \approx \frac{f(c+h) - f(c)}{h}, \quad \text{when} \;\; h \approx 0,$$

we get

$$f(c+h) \approx f(c) + f'(c)h, \quad \text{when} \;\; h \approx 0. \tag{4.3}$$

Now, let $x = c + h$ so that $h = x - c$. Then $x \approx c$ when $h \approx 0$, so that (4.3) becomes (4.2).

Example 2.4.1 Find the tangent line of $f(x) = x^2$ at $x = 3$.

Solution. We first plug $x = 3$ into the function to get the point $(3, f(3)) = (3, 9)$. Next, since $f'(x) = 2x$, the slope of $f(x) = x^2$ at $x = 3$ is given by $m = f'(3) = 2 \cdot 3 = 6$. Thus the tangent line is

$$y = 9 + 6(x - 3).$$

♠

Next, let's look at how to use tangent lines to approximate some complicated functions. Based on Figure 2.28 and Example 2.4.1, we need to do the following:

1. Analyze the value to be approximated and determine a function, and then choose a point c so that $f(c)$ and $f'(c)$ are easy to evaluate.

2. Find the tangent line at $(c, f(c))$.

3. Use the tangent line to approximate.

Now we answer the question raised at the beginning of this section.

Example 2.4.2 Approximate $\sqrt{81.34}$ using a tangent line.

Solution. According to the value $\sqrt{81.34}$, we change 81.34 to x and keep the same format so we can define a function $f(x) = \sqrt{x}$. Then we choose $c = 81$ because $c = 81$ is close to 81.34 and $f(c) = \sqrt{c} = \sqrt{81} = 9$ is easy to evaluate. Moreover, since $f'(x) = \frac{1}{2\sqrt{x}}$, the slope $m = f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{18}$ is also easy to evaluate. Thus the tangent line at $(c, f(c)) = (81, 9)$ is given by

$$y = 9 + \frac{1}{18}(x - 81).$$

Now, from **Figure 2.29**, we see that when x is close to 81, the function $f = \sqrt{x}$ and its tangent line at $(81, 9)$ are almost the same, that is,

$$\sqrt{x} \approx 9 + \frac{1}{18}(x - 81) \quad \text{when} \quad x \approx 81. \tag{4.4}$$

Since $81.34 \approx 81$, we obtain, from (4.4),

$$\begin{aligned} \sqrt{81.34} &\approx 9 + \frac{1}{18}(81.34 - 81) \\ &\approx 9.018888. \end{aligned} \tag{4.5}$$

Note that the last "≈" can be calculated by hand. ♠

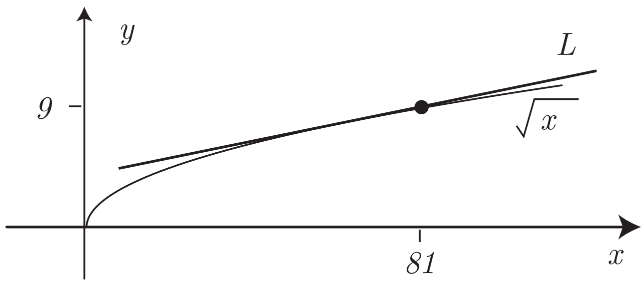

Figure 2.29: The function $f = \sqrt{x}$ and its tangent line at $(81, 9)$

If you evaluate using a calculator, you will get $\sqrt{81.34} \approx 9.018869$, which is very close to 9.018888. This shows that the approximation carried out in Example 2.4.2 is very good. Note that 9.018888 is a little bit bigger than 9.018869 because, from Figure 2.29, the tangent line is slightly above the curve of $\sqrt{x}$ near $x = 81$. We also see from Figure 2.29 that it only makes sense to use that tangent line to approximate $\sqrt{x}$ **locally** near $x = 81$, which is called a **local linear approximation**, because for some other x values, the tangent line and $\sqrt{x}$ are far from each other.

Remark 2.4.3 Speaking of using calculators, we point out that calculators (and computers) can only perform four simple operations: addition, subtraction, multiplication, and division, so that calculators cannot "evaluate" complicated functions (such as $\sqrt{x}$, e^x, $\ln x$) directly, even though you may have the impression that these functions are *evaluated at once* when you press those buttons on your calculators. Instead, calculators are **programed** to **approximate** these functions by using other functions which involve only the four simple operations, such as straight lines, polynomial functions, or rational functions. For example, e^x can be approximated as $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$, so that when the e^x button is pressed, a program is called to use $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$ to approximate.

Here, we present the tangent line approximation so as to help you understand how calculators are working: using programs and approximations, where straight line approximations is the first step to understand approximations. ♠

Remark 2.4.4 In Example 2.4.2, if you use a calculator to directly evaluate $\sqrt{81.34}$, then you are missing the point. Here, *we ourselves are working as a calculator* so as to receive $\sqrt{81.34}$, then process (approximate with a tangent line), and then give 9.018888. Next, if you choose, for example, $c = 81.33$ which is closer to 81.34 than 81 is, then what about $f(c) = \sqrt{81.33}$? If you say you can use a calculator to approximate $f(c) = \sqrt{81.33}$, then why not just use a calculator to directly approximate $\sqrt{81.34}$? These remarks should indicate the importance of selecting a value c so as to make $f(c)$ and $f'(c)$ easy to evaluate without using a calculator. ♠

Guided Practice 2.4

1. Find the tangent line on the graph of f at the point $(x, f(x))$, and then plot both f and the tangent line at the point for

(a) $f(x) = x^2$ at $x = 3$.

(b) $f(x) = x^3$ at $x = 1$.

(c) $f(x) = \frac{1}{x}$ at $x = 2$.

2. Without using a calculator make your best guess at a decimal approximation to $2\sqrt{1.1} + 1$. Now use the tangent line to approximate $2\sqrt{1.1} + 1$, and try to do this without the aid of a calculator. Finally use a calculator to approximate $2\sqrt{1.1} + 1$, and compare this to your guess and the approximation using the tangent line.

Exercises 2.4

1. Find the tangent line for

(a) $f(x) = 2x + 1$ at $x = 1$.

(b) $f(x) = x^5 + 6x^3$ at $x = 1$.

(c) $f(x) = \sqrt{x}$ at $x = 9$.

(d) $f(x) = \sqrt[3]{x}$ at $x = 8$.

2. Use a tangent line to approximate

(a) $\sqrt{4.1}$.

(b) $\sqrt{16.2}$.

(c) $\sqrt[3]{8.03}$.

(d) $\sqrt[3]{27.1}$.

(e) $\sqrt[4]{16.05}$.

(f) $9.05^{-1/2}$.

(g) $9.05^{-3/2} + 4\sqrt{9.05}$.

Chapter 3

The Derivative Rules

The derivative rules are given in this chapter. They allow us to take the derivatives of all the functions we encounter in this course.

As an application, we will learn how to take derivatives implicitly and solve some related rate problems.

3.1 The Product and Quotient Rules

We know by now how to take derivatives of functions of the form

$$7x^5 + 3x^2 - 2x + 1.$$

However, we sometimes need to take derivatives of functions of the form

$$(1 + x + 2x^2 + 3x^3)(4x^4 + 5x^5 + 6x^6),$$

or

$$\frac{1 + x + 2x^2 + 3x^3}{4x^4 + 5x^5 + 6x^6}.$$

For example, to find tangent lines of $(1+x+2x^2+3x^3)(4x^4+5x^5+6x^6)$ or of $\frac{1+x+2x^2+3x^3}{4x^4+5x^5+6x^6}$, we need to take their derivatives. So that in general, we need to find ways to take derivatives of

$$f(x)g(x) \quad \text{and} \quad \frac{f(x)}{g(x)},$$

where f and g are differentiable. For these purposes, let's introduce the following rules.

Theorem 3.1.1 (Product and Quotient Rules) *If $f(x)$ and $g(x)$ have derivatives, then $f(x)g(x)$ and $\frac{f(x)}{g(x)}$ (if $g(x) \neq 0$) also have derivatives, and their derivatives are given by*

1. $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, (product rule).
2. $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$, (quotient rule).

Verification. 1. The idea is to regard $f(x)g(x)$ as a new function and then find its derivative using definition (i.e., limit). That is,

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Then the problem is that $f(x+h)g(x+h)$ and $f(x)g(x)$ don't "recognize" each other. Now, the idea is to bring someone who "knows" both of them. That is, we subtract and then add $f(x)g(x+h)$, which allows us to regroup and obtain

$$\begin{aligned}
[f(x)g(x)]' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \\
&= \lim_{h\to 0} \{\frac{[f(x+h) - f(x)]g(x+h)}{h} + \frac{f(x)[g(x+h) - g(x)]}{h}\} \\
&= \lim_{h\to 0} [\frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x),
\end{aligned}$$

where we have used the result that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x), \quad \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

and the result that

$$\lim_{h\to 0} g(x+h) = g(x)$$

since g has derivatives and hence is continuous, from the discussion in Chapter 2.

2. Similar to above, we now regard $\frac{f(x)}{g(x)}$ as a new function to obtain

$$[\frac{f(x)}{g(x)}]' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$\begin{aligned}
&= \lim_{h\to 0}\frac{1}{h}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{1}{h}\frac{[f(x+h)-f(x)]g(x)-f(x)[g(x+h)-g(x)]}{g(x+h)g(x)}\\
&= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}[\frac{f(x+h)-f(x)}{h}g(x)-f(x)\frac{g(x+h)-g(x)}{h}]\\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.
\end{aligned}$$

♠

We recommend that you write the product rule using the order given here, that is, $f(x)$ first and $g(x)$ second. This way, the numerator of the quotient rule is similar to the product rule: the only difference is to change the "+" to "−".

The way to understand the product and quotient rules is the following:

1. [(first function)(second function)]′ = (first function)′(second function)+ (first function)(second function)′.
2. $[\frac{\text{numerator}}{\text{denominator}}]' = \frac{(\text{numerator})'(\text{denominator}) - (\text{numerator})(\text{denominator})'}{(\text{denominator})^2}$.

We can check the product and quotient rules for simple functions. For example, to check the product rule, we can let $f(x) = x$ and $g(x) = x$, then we have $f(x)g(x) = x \cdot x = x^2$, hence by using the power rule we get

$$[f(x)g(x)]' = (x^2)' = 2x;$$

and we also obtain

$$f'(x)g(x) + f(x)g'(x) = (1)x + x(1) = 2x,$$

therefore they give the same answer. To check the quotient rule, we can let $f(x) = x^2$ and $g(x) = x$, then we have $\frac{f(x)}{g(x)} = \frac{x^2}{x} = x$, hence we get

$$[\frac{f(x)}{g(x)}]' = (x)' = 1;$$

and we also obtain

$$\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} = \frac{2x(x)-x^2(1)}{x^2} = \frac{x^2}{x^2} = 1,$$

therefore they give the same answer.

Note that in general,

$$[f(x)g(x)]' \neq [f'(x)][g'(x)], \quad [\frac{f(x)}{g(x)}]' \neq \frac{f'(x)}{g'(x)}.$$

Now we look at some examples.

Example 3.1.2 Find $\frac{d}{dx}(1+2x^2)(4x^4+5x)$.

Solution. Let $f(x) = 1+2x^2$ and $g(x) = 4x^4+5x$, then we can use the product rule to get

$$\begin{aligned}
&\frac{d}{dx}(1+2x^2)(4x^4+5x) = f'(x)g(x) + f(x)g'(x)\\
&= (1+2x^2)'(4x^4+5x) + (1+2x^2)(4x^4+5x)'\\
&= 4x(4x^4+5x) + (1+2x^2)(16x^3+5).
\end{aligned}$$

In this case, we can check the answer because we can multiply and get

$$(1+2x^2)(4x^4+5x) = 8x^6+4x^4+10x^3+5x, \tag{1.1}$$

and then take its derivative. We leave it to you to take the derivative of (1.1) and see that the answer is the same as above. ♠

Example 3.1.3 Find $\frac{d}{dx}(1+x+2x^2+3x^3)(4x^4+5x^5+6x^6)$.

Solution. Let $f(x) = 1+x+2x^2+3x^3$ and $g(x) = 4x^4+5x^5+6x^6$, then we can use the product rule to get

$$\begin{aligned}
&\frac{d}{dx}(1+x+2x^2+3x^3)(4x^4+5x^5+6x^6)\\
&= (1+x+2x^2+3x^3)'(4x^4+5x^5+6x^6)\\
&+(1+x+2x^2+3x^3)(4x^4+5x^5+6x^6)'\\
&= (1+4x+9x^2)(4x^4+5x^5+6x^6)\\
&+(1+x+2x^2+3x^3)(16x^3+25x^4+36x^5).
\end{aligned}$$

♠

Example 3.1.4 Find $\frac{d}{dx}\frac{x^2+x-2}{x-1}$.

Solution. Let $f(x) = x^2+x-2$ and $g(x) = x-1$, then we can use the quotient rule to get

$$\begin{aligned}
&\frac{d}{dx}\frac{x^2+x-2}{x-1} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}\\
&= \frac{(x^2+x-2)'(x-1) - (x^2+x-2)(x-1)'}{(x-1)^2}\\
&= \frac{(2x+1)(x-1) - (x^2+x-2)}{(x-1)^2}\\
&= \frac{2x^2-x-1-x^2-x+2)}{(x-1)^2}\\
&= \frac{x^2-2x+1}{(x-1)^2}\\
&= \frac{(x-1)^2}{(x-1)^2} = 1.
\end{aligned}$$

In this case, you can check the answer because we can factor

$$\frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1} = x+2,$$

so that the derivative is 1. ♠

Example 3.1.5 Find $\frac{d}{dx}\frac{1+x+2x^2+3x^3}{4x^4+5x^5+6x^6}$.

Solution. Let $f(x) = 1+x+2x^2+3x^3$ and $g(x) = 4x^4+5x^5+6x^6$, then we can use the quotient rule to get

$$\begin{aligned}
&\frac{d}{dx}\frac{1+x+2x^2+3x^3}{4x^4+5x^5+6x^6}\\
&= \frac{1}{(4x^4+5x^5+6x^6)^2}\Big[(1+x+2x^2+3x^3)'(4x^4+5x^5+6x^6)\\
&-(1+x+2x^2+3x^3)(4x^4+5x^5+6x^6)'\Big]\\
&= \frac{1}{(4x^4+5x^5+6x^6)^2}\Big[(1+4x+9x^2)(4x^4+5x^5+6x^6)\\
&-(1+x+2x^2+3x^3)(16x^3+25x^4+36x^5)\Big].
\end{aligned}$$

♠

The next example involves both the product and quotient rules.

Example 3.1.6 Find $\frac{d}{dx}\frac{(2x^2+3x^3)(7x^7-8x)}{5x^5+6x^6}$.

Solution. We use the quotient rule for this example and use the product rule when taking the derivative of the numerator.

Let $f(x) = (2x^2+3x^3)(7x^7-8x)$, $g(x) = 5x^5+6x^6$, then we have

$$\begin{aligned}
f'(x) &= (4x+9x^2)(7x^7-8x) + (2x^2+3x^3)(49x^6-8),\\
g'(x) &= 25x^4+36x^5.
\end{aligned}$$

Now, we just put them into the quotient rule to obtain

$$\begin{aligned}
&\frac{d}{dx}\frac{(2x^2+3x^3)(7x^7-8x)}{5x^5+6x^6} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}\\
&= \frac{1}{(5x^5+6x^6)^2}\Big\{[(4x+9x^2)(7x^7-8x)+(2x^2+3x^3)(49x^6-8)](5x^5+6x^6)\\
&-(2x^2+3x^3)(7x^7-8x)(25x^4+36x^5)\Big\}.
\end{aligned}$$

♠

Example 3.1.7 Approximate $\frac{\sqrt{3.97}-1}{\sqrt{3.97}+1}$ using a tangent line.

Solution. According to the value $\frac{\sqrt{3.97}-1}{\sqrt{3.97}+1}$, we let $f(x) = \frac{\sqrt{x}-1}{\sqrt{x}+1}$ and choose

$a = 4$ so that a is close to 3.97 and $f(4) = \frac{\sqrt{4}-1}{\sqrt{4}+1} = \frac{1}{3}$ is easy to evaluate. Moreover, since

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x}-1)'(\sqrt{x}+1) - (\sqrt{x}-1)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2} \\ &= \frac{\frac{1}{2\sqrt{x}}(\sqrt{x}+1) - (\sqrt{x}-1)\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} \quad \text{(multiply } 2\sqrt{x} \text{ top and bottom)} \\ &= \frac{(\sqrt{x}+1) - (\sqrt{x}-1)}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{2}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}, \end{aligned}$$

we see that the slope is given by

$$m = f'(4) = \frac{1}{\sqrt{4}(\sqrt{4}+1)^2} = \frac{1}{18}.$$

Thus the tangent line at $(4, f(4)) = (4, \frac{1}{3})$ is given by

$$y = \frac{1}{3} + \frac{1}{18}(x-4).$$

Now, when x is close to 4, the function $f = \frac{\sqrt{x}-1}{\sqrt{x}+1}$ and its tangent line at $(4, \frac{1}{3})$ are almost the same, that is,

$$\frac{\sqrt{x}-1}{\sqrt{x}+1} \approx \frac{1}{3} + \frac{1}{18}(x-4) \quad \text{when } x \approx 4. \tag{1.2}$$

Since $3.97 \approx 4$, we obtain, from (1.2),

$$\begin{aligned} \frac{\sqrt{3.97}-1}{\sqrt{3.97}+1} &\approx \frac{1}{3} + \frac{1}{18}(3.97-4) \\ &\approx 0.3316667. \end{aligned}$$

♠

If you use a calculator to find $\frac{\sqrt{3.97}-1}{\sqrt{3.97}+1}$, you will get a value that is very close to 0.3316667. This shows again that the tangent line is a very good local linear approximation.

Guided Practice 3.1

Find $f'(x)$ for the following functions $f(x)$ using *Product Rule* and *Power Rule*, or *Quotient Rule* and *Power Rule*.

1. $f(x) = (x^2+1)(x^3+1)$.

2. $f(x) = (3x^{1/3}+1)(\frac{2}{5}x^{5/2}+x)$.

3. $f(x) = \sqrt{2x}(x+2)$.

4. $f(x) = \frac{3x}{2x+1}$

5. $f(x) = \frac{x^2+1}{\frac{1}{3}x^3-2}$

6. $f(x) = \frac{2x^{1/2}}{2x^2+3}$

Exercises 3.1

1. Find the similarity and the difference between the product rule and the quotient rule.

2. Check the product rule for $2x^3 = (2x)(x^2)$.

3. Check the quotient rule for $2x^3 = \frac{2x^4}{x}$.

4. Find a function $f(x)$ and a function $g(x)$ such that $[f(x)g(x)]' \neq [f'(x)][g'(x)]$.

5. Find a function $f(x)$ and a function $g(x)$ such that $[\frac{f(x)}{g(x)}]' \neq \frac{f'(x)}{g'(x)}$.

6. For $\frac{x^2}{4}$, rewrite it as $\frac{1}{4}x^2$ and take a derivative of it using the power rule to x^2 (i.e., treat $\frac{1}{4}$ as a constant). Then take a derivative of $\frac{x^2}{4}$ using the quotient rule. Which is easier? What does this tell you?

7. For $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{x}$, rewrite it as $f(x) = 5x^4 + 6x^2 - 6x^{-5/6} + 9x^{-1}$ and take a derivative of it using the power rule. Then take a derivative of $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{x}$ using the quotient rule. Which is easier? What does this tell you?

8. Find $f'(1)$ given that $g(1) = 2$ and $g'(1) = 3$ for the following functions f defined in terms of g.

(a) $f(x) = x^2g(x)$.
(b) $f(x) = xg(x) + 5x + 8$.
(c) $f(x) = \frac{g(x)}{x}$.
(d) $f(x) = \frac{x}{g(x)}$.

9. Find $f'(x)$ for

(a) $f(x) = (7-x)(x+2)$.
(b) $f(x) = (x-2)(x^2-x-5)$.
(c) $f(x) = (x+3x^5)(x^5-3x)$.
(d) $f(x) = (2\sqrt{x}-9x^6)(x^3-x^7-2)$.

(e) $f(x) = (1 + 3x^5)(x^{-2/5} - 3x)$.

(f) $f(x) = (2\sqrt{x} - 3x + 5x^4 - 9x^6)(x^{2/3} - x^7 - 2)$.

(g) $f(x) = \frac{Ax}{Bx+C}$ where A, B, and C are constants.

(h) $f(x) = \frac{Ax+B}{Cx}$ where A, B, and C are constants.

(i) $f(x) = \frac{5}{4x^5+3x}$.

(j) $f(x) = \frac{x^7-5x}{4x^5}$.

(k) $f(x) = \frac{x^7-5}{4x^5+3x}$.

(l) $f(x) = \frac{\sqrt{x}-3x}{x^3-x^7-2}$.

(m) $f(x) = \frac{5x^4-9x^6}{x^{-2/3}-x^7-2}$.

(n) $f(x) = \frac{2\sqrt{x}-3x+5x^4-9x^6}{x^{2/3}-x^7-2}$.

(o) $f(x) = \frac{(x^{2/3}-x^7)(3x^4+5\sqrt{x})}{2\sqrt{x}-3x}$.

(p) $f(x) = \frac{2\sqrt{x}-3x}{(x^{2/3}-x^7)(3x^4+5\sqrt{x})}$.

10. Use a tangent line to approximate

(a) $\frac{2-\sqrt{9.1}}{1+\sqrt{9.1}}$.

(b) $\frac{3-\sqrt{16.2}}{1+\sqrt{16.2}}$.

(c) $\frac{3+\sqrt[3]{8.03}}{1-\sqrt[3]{8.03}}$.

(d) $\frac{4-\sqrt[4]{16.05}}{3+\sqrt[4]{16.05}}$.

3.2 The Chain Rule

To take derivatives of functions such as $(4x+1)^2$, one way is to multiply out and then take the derivative.

Question: *How do we take the derivative of* $(4x+1)^{2000}$*? or of* $(4x+1)^{\frac{1}{3}}$*?*

Now, we don't believe anyone wants to multiply out $(4x + 1)^{2000}$ and then take the derivative; as for $(4x+1)^{\frac{1}{3}}$, it can not even be simplified. This indicates that we must find a different approach when taking derivatives of these kinds of functions.

As we have seen in Chapter 1, the function $y(x) = (4x + 1)^{2000}$ can be decomposed as

$$y = f(z) = z^{2000}, \quad z = g(x) = 4x + 1,$$

because the composition of the two functions, $y = [g(x)]^{2000} = (4x+1)^{2000}$ gives back the original function. Thus, we need to find a way to take derivatives of functions of the form

$$y(x) = [g(x)]^{\alpha},$$

where α is a constant.

In general, for two functions

$$y = f(z), \quad z = g(x),$$

after we compose f and g (assume g is in the domain of f), we get a function in x in the form

$$y(x) = f(g(x)),$$

and we need to find a way to take the derivative of $y(x) = f(g(x))$.

Let's approach this in an informal way first. We write $y'(x)$ as $\frac{dy}{dx}$, and regard dy, dx, dz as "numbers" so we can "divide" and "multiply" by dz to obtain

$$\frac{dy}{dx} = \frac{dy}{\star}\frac{\star}{dx} = \frac{dy}{dz}\frac{dz}{dx} = f'(z)g'(x) = f'(g(x))g'(x). \tag{2.1}$$

That is, if we decompose a composition $y(x) = f(g(x))$ as two functions $y = f(z)$ and $z = g(x)$, then we know how to take derivatives of these two functions $\frac{dy}{dz}$ and $\frac{dz}{dx}$. Now, (2.1) suggests that to find the derivative of y with respect to x, we just multiply $\frac{dy}{dz}$ and $\frac{dz}{dx}$. This idea can also be seen in **Figure 3.1**, which shows a *chain reaction* according to (2.1), in the sense that a change in x causes a change in z, which then causes a change in y, so that $\frac{dy}{dx}$ should be the "relay" of $\frac{dy}{dz}$ and $\frac{dz}{dx}$.

Let's check (2.1) for some simple cases.

Example 3.2.1 (Rate of return) Assume that you put your money (x) into a bank A where the return (z) is to double the input (the rate of return is 2), and then you take your money from bank A and put into a bank B where the return (y) is to triple the input (the rate of return is 3). Then, after going through banks A and B (composition), how much will your initial money x become?

Solution. If you put in \$1, then \$1 becomes (3)(2)(\$1) = \$6. So, in general, your initial money x will become $6x$. That is, the rate of return after going through banks A and B (composition) is $(3)(2) = 6$. In formulas, we have

$$z = 2x, \quad y = 3z,$$

so that $y = 3z = 3(2x) = 6x$, and

$$\frac{dz}{dx} = 2, \quad \frac{dy}{dz} = 3, \quad \frac{dy}{dx} = 6,$$

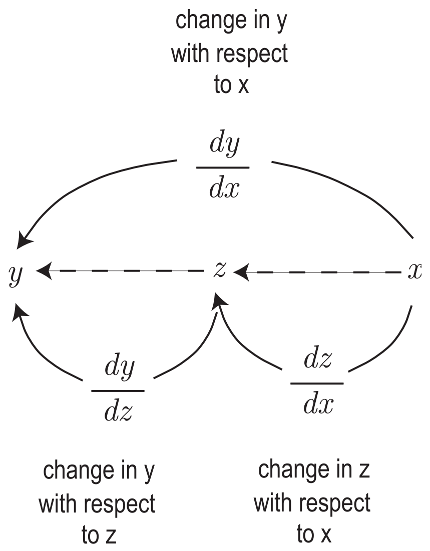

Figure 3.1: A chain reaction

where, in $\frac{dy}{dz} = 3$, z is the independent variable when taking the derivative (this is similar to $(3x)' = 3$ when x is the independent variable), therefore,

$$\frac{dy}{dx} = 6 = 3 \times 2 = \frac{dy}{dz}\frac{dz}{dx},$$

which is the same as (2.1). ♠

Example 3.2.2 Check (2.1) for $y = (4x + 1)^2$.

Solution. First, we simplify and get

$$y = 16x^2 + 8x + 1,$$

so that

$$\frac{dy}{dx} = 32x + 8. \tag{2.2}$$

Next, we decompose $y = (4x + 1)^2$ as

$$y = f(z) = z^2, \quad z = g(x) = 4x + 1.$$

Then, we have

$$\frac{dy}{dz} = f'(z) = 2z, \quad \frac{dz}{dx} = g'(x) = 4,$$

where, in $\frac{dy}{dz} = 2z$, z is the independent variable when taking the derivative (this is similar to $(x^2)' = 2x$ when x is the independent variable). Hence

$$\frac{dy}{dz}\frac{dz}{dx} = f'(z)g'(x) = 2z \cdot 4 = 8z. \tag{2.3}$$

Now, if we ask you whether (2.2) and (2.3) are the same, then you could say "no". But you understand that we have to *compare apple with apple*, which means that to compare (2.2) with (2.3), we need to plug $z = g(x) = 4x + 1$ into (2.3) to get

$$\frac{dy}{dz}\frac{dz}{dx} = 8z = 8(4x + 1) = 32x + 8,$$

which is now the same as (2.2). ♠

Example 3.2.3 Check (2.1) for $y = (4x + 1)^3$.

Solution. We first simplify and get

$$y = (16x^2 + 8x + 1)(4x + 1) = 64x^3 + 48x^2 + 12x + 1,$$

so that

$$\frac{dy}{dx} = 192x^2 + 96x + 12. \tag{2.4}$$

Next, we decompose $y = (4x + 1)^3$ as

$$y = f(z) = z^3, \quad z = g(x) = 4x + 1.$$

Then, we have

$$\frac{dy}{dz} = f'(z) = 3z^2, \quad \frac{dz}{dx} = g'(x) = 4,$$

and

$$\begin{aligned} \frac{dy}{dz}\frac{dz}{dx} &= 3z^2 \cdot 4 = 3(4x + 1)^2 \cdot 4 \\ &= 12(16x^2 + 8x + 1) \\ &= 192x^2 + 96x + 12, \end{aligned}$$

which is the same as (2.4). ♠

More examples can be given to indicate that (2.1) is valid. In fact, (2.1) is true in general, and is called the *chain rule*, in the sense that Figure 3.1 shows a chain reaction according to (2.1). We state this as follows and verify it at the end of this section.

Theorem 3.2.4 (Chain Rule) *For two functions $y = f(z)$ and $z = g(x)$, if $g(x)$ is inside the domain of f, and f and g have derivatives in their respective domains, then for $y(x) = f(g(x))$, we have*

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}, \tag{2.5}$$

or equivalently, as $\frac{dy}{dz} = f'(z)$ and $\frac{dz}{dx} = g'(x)$, we have

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x). \tag{2.6}$$

♠

For $f(g(x))$, we call f the *outside* function and g the *inside* function, so (2.6) means that to find $\frac{d}{dx}f(g(x))$, we first take the derivative of the outside function f and evaluate at the inside function $g(x)$ (for example, in Example 3.2.3, this is $(z^3)' = 3z^2 = 3g^2(x) = 3(4x+1)^2$), and then multiply by the derivative of the inside function $g(x)$ (in Example 3.2.3, this is $g'(x) = 4$).

Based on the results of Example 3.2.2 and Example 3.2.3, that is,

$$\frac{d}{dx}(4x+1)^2 = 2(4x+1)\cdot 4,$$
$$\frac{d}{dx}(4x+1)^3 = 3(4x+1)^2\cdot 4,$$

we see that we are actually doing the *power rule* first, and then multiply the derivative of the inside function. That is, if we apply the chain rule to a power function $f(z) = z^\alpha$ (with α a constant), then, since $f'(z) = \alpha z^{\alpha-1}$, we obtain the following result, which is a special case of the chain rule.

Theorem 3.2.5 (General Power Rule) *If $g(x)$ has derivatives and if $[g(x)]^\alpha$ is well-defined (gives a real value) for the constant α, then*

$$\frac{d}{dx}[g(x)]^\alpha = \alpha[g(x)]^{\alpha-1}\cdot g'(x). \tag{2.7}$$

Example 3.2.6 Find $\frac{d}{dx}(4x+1)^3$.

Solution. Let $g(x) = 4x+1$ and $\alpha = 3$, then we apply the general power rule to obtain

$$\frac{d}{dx}(4x+1)^3 = 3(4x+1)^2\frac{d}{dx}(4x+1) = 3(4x+1)^2\cdot 4 = 12(4x+1)^2,$$

which gives the same result as in Example 3.2.3. ♠

Example 3.2.7 Find $\frac{d}{dx}(4x+1)^{2000}$.

Solution. Let $g(x) = 4x+1$ and $\alpha = 2000$, then we apply the general power rule to obtain

$$\begin{aligned}\frac{d}{dx}(4x+1)^{2000} &= 2000(4x+1)^{1999}\frac{d}{dx}(4x+1) \\ &= 2000(4x+1)^{1999}\cdot 4 = 8000(4x+1)^{1999}.\end{aligned}$$

This is an example that probably nobody wants to check by multiplying $(4x+1)^{2000}$ out and then taking the derivative. ♠

Example 3.2.8 Find $\frac{d}{dx}(x-\sqrt{3x+2})^{2000}$.

Solution. This example indicates that the general power rule may be needed more than once. That is, we start with the general power rule and then apply it again to $\sqrt{3x+2}$ to obtain

$$\begin{aligned}\frac{d}{dx}(x-\sqrt{3x+2})^{2000} &= 2000(x-\sqrt{3x+2})^{1999}\frac{d}{dx}(x-\sqrt{3x+2}) \\ &= 2000(x-\sqrt{3x+2})^{1999}\cdot(1-\frac{1}{2}(3x+2)^{-1/2}\cdot 3) \\ &= 2000(x-\sqrt{3x+2})^{1999}(1-\frac{3}{2}(3x+2)^{-1/2}).\end{aligned}$$

♠

Example 3.2.9 Find $\frac{d}{dx}\frac{(3x+2)^3}{(5x^2-2)^{25}}$.

Solution. We use the quotient rule for this example and use the general power rule for $(3x+2)^3$ and $(5x^2-2)^{25}$.

Let $f(x) = (3x+2)^3$, $g(x) = (5x^2-2)^{25}$, then we have

$$\begin{aligned}&f'(x) = 3(3x+2)^2\cdot 3 = 9(3x+2)^2, \\ &g'(x) = 25(5x^2-2)^{24}(10x) = 250x(5x^2-2)^{24}.\end{aligned}$$

Now, we just put them into the quotient rule to obtain

$$\begin{aligned}&\frac{d}{dx}\frac{(3x+2)^3}{(5x^2-2)^{25}} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \\ &= \frac{9(3x+2)^2(5x^2-2)^{25} - 250x(3x+2)^3(5x^2-2)^{24}}{(5x^2-2)^{50}}.\end{aligned}$$

♠

Finally, let's look at some examples where all the rules (the product, quotient, and chain rules) are involved.

Example 3.2.10 Find $\frac{d}{dx}\frac{(3x+2)^{30}(8x^5-7)}{5x^2-2}$.

Solution. We use the quotient rule for this example and use the product rule when taking the derivative of the numerator. For the derivative of $(3x+2)^{30}$, we use the general power rule.

Let $f(x) = (3x+2)^{30}(8x^5-7)$, $g(x) = 5x^2-2$, then we have

$$\begin{aligned} f'(x) &= 30(3x+2)^{29} \cdot 3 \cdot (8x^5-7) + (3x+2)^{30} \cdot 40x^4 \\ &= 90(3x+2)^{29}(8x^5-7) + 40x^4(3x+2)^{30}, \\ g'(x) &= 10x. \end{aligned}$$

Now, we just put them into the quotient rule to obtain

$$\begin{aligned} &\frac{d}{dx}\frac{(3x+2)^{30}(8x^5-7)}{5x^2-2} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \\ &= \frac{1}{(5x^2-2)^2}\Big\{\Big[90(3x+2)^{29}(8x^5-7)+40x^4(3x+2)^{30}\Big](5x^2-2) \\ &\quad -10x(3x+2)^{30}(8x^5-7)\Big\}. \end{aligned}$$

♠

Example 3.2.11 Find $\frac{d}{dx}\frac{(3x+2)^{30}(8x^5-7)^{40}}{(5x^2-2)^{50}}$.

Solution. Similar to the above, we let $f(x) = (3x+2)^{30}(8x^5-7)^{40}$, $g(x) = (5x^2-2)^{50}$, so we have

$$\begin{aligned} f'(x) &= 30(3x+2)^{29} \cdot 3 \cdot (8x^5-7)^{40} + (3x+2)^{30}40(8x^5-7)^{39} \cdot 40x^4 \\ &= 90(3x+2)^{29}(8x^5-7)^{40} + 1600x^4(3x+2)^{30}(8x^5-7)^{39}, \\ g'(x) &= 50(5x^2-2)^{49}(10x) = 500x(5x^2-2)^{49}. \end{aligned}$$

Now, we just put them into the quotient rule to obtain

$$\begin{aligned} &\frac{d}{dx}\frac{(3x+2)^{30}(8x^5-7)^{40}}{(5x^2-2)^{50}} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \\ &= \frac{1}{(5x^2-2)^{100}}\Big\{\Big[90(3x+2)^{29}(8x^5-7)^{40} \\ &\quad +1600x^4(3x+2)^{30}(8x^5-7)^{39}\Big](5x^2-2)^{50} \\ &\quad -500x(3x+2)^{30}(8x^5-7)^{40}(5x^2-2)^{49}\Big\}. \end{aligned}$$

♠

Example 3.2.12 Find $\frac{d}{dx}\Big[(2x+7)^{30}(3x^7-5) - \frac{x^{-1/3}}{6x^3-2}\Big]^{40}$.

Solution. Now we start with the general power rule and then use the product and quotient rules. That is,

$$\frac{d}{dx}\Big[(2x+7)^{30}(3x^7-5) - \frac{x^{-1/3}}{6x^3-2}\Big]^{40}$$

$$= 40\Big[(2x+7)^{30}(3x^7-5) - \frac{x^{-1/3}}{6x^3-2}\Big]^{39}$$
$$\cdot\Big\{30(2x+7)^{29}(2)(3x^7-5) + (2x+7)^{30}(21x^6)$$
$$- \frac{-\frac{1}{3}x^{-4/3}(6x^3-2) - x^{-1/3}(18x^2)}{(6x^3-2)^2}\Big\}$$
$$= 40\Big[(2x+7)^{30}(3x^7-5) - \frac{x^{-1/3}}{6x^3-2}\Big]^{39}$$
$$\cdot\Big\{60(2x+7)^{29}(3x^7-5) + 21x^6(2x+7)^{30}$$
$$- \frac{-\frac{1}{3}x^{-4/3}(6x^3-2) - x^{-1/3}(18x^2)}{(6x^3-2)^2}\Big\}.$$

♠

Verification of the Chain Rule (Theorem 3.2.4).

The idea is to regard $f(g(x))$ as a new function and use definition (i.e., limit) to find its derivative. So we start with

$$\frac{d}{dx}f(g(x)) = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h}. \tag{2.8}$$

Let $a = g(x)$, then the tangent line of the function $f(z)$ at $(a, f(a)) = (g(x), f(g(x)))$ is given by

$$\begin{aligned} y &= f(a) + f'(a)(z-a) \\ &= f(g(x)) + f'(g(x))[z - g(x)]. \end{aligned}$$

From our knowledge of tangent lines, we know that

$$f(z) \approx f(g(x)) + f'(g(x))[z - g(x)] \quad \text{when} \quad z \approx a = g(x). \tag{2.9}$$

We also know that g is continuous since g has derivatives. Thus, as $h \to 0$, $g(x+h) \to g(x)$. Therefore, we can regard $g(x+h)$ as z in (2.9) to obtain

$$f(g(x+h)) \approx f(g(x)) + f'(g(x))[g(x+h) - g(x)] \quad \text{when} \quad h \approx 0.$$

Then,

$$\frac{f(g(x+h)) - f(g(x))}{h} \approx f'(g(x))\frac{g(x+h)-g(x)}{h} \quad \text{when} \quad h \approx 0.$$

Now, if we take a limit as $h \to 0$, then we get, from (2.8),

$$\begin{aligned} \frac{d}{dx}f(g(x)) &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} f'(g(x))\frac{g(x+h)-g(x)}{h} \\ &= f'(g(x))\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(g(x)) \cdot g'(x). \end{aligned}$$

Therefore, the chain rule is verified. ♠

Guided Practice 3.2

Find $f'(x)$ for the following functions $f(x)$.

1. $f(x) = (x^2 + 5x + 1)^4$.

2. $f(x) = 2\sqrt{\frac{1}{3}x^3 - 2}$.

3. $f(x) = 7x(5x + 4)^3$.

4. $f(x) = \frac{2x}{\sqrt{7x+1}}$

Exercises 3.2

1. Find a function $f(x)$ and a function $g(x)$ such that

$$\frac{d}{dx}f(g(x)) \neq f'(x) \cdot g'(x).$$

2. Simplify (multiply out) and then take the derivative of $(5x+1)^2$. Then compare it with the derivative of $(5x + 1)^2$ using the chain rule.

3. Simplify (multiply out) and then take the derivative of $(5x+1)^3$. Then compare it with the derivative of $(5x + 1)^3$ using the chain rule.

4. Derive the general power rule from the chain rule.

5. Find $f'(1)$ given that $g(1) = 2$ and $g'(1) = 3$ for the following functions f defined in terms of g.

(a) $f(x) = [g(x)]^5$.
(b) $f(x) = \sqrt{g(x)}$.
(c) $f(x) = \left(\frac{g(x)}{x}\right)^3$.
(d) $f(x) = \left(\frac{x}{g(x)}\right)^3$.

6. Find $f'(x)$ for

(a) $f(x) = \sqrt{3x + 1}$.
(b) $f(x) = (1 - 2x)^{-1}$.
(c) $f(x) = \left(x^2 + \frac{1}{x^2}\right)^2$.
(d) $f(x) = \left(x^3 + \frac{1}{x^3}\right)^3$.

(e) $f(x) = (x - x^3)^{2000}$.
(f) $f(x) = 3x + \sqrt{x - x^3}$.
(g) $f(x) = (x^2 - \sqrt{5x})^{2000}$.
(h) $f(x) = (3 + \sqrt{x - x^3})^{2000}$.
(i) $f(x) = (2x + 4)^{34}(7x^2 + 8x - 1)^{50}$.
(j) $f(x) = (2x^{-2/3} + 4)^{34}(7x^2 + \sqrt{8x - 1})^{50}$.
(k) $f(x) = \frac{(2x^3+4)^{34}}{(7x^2+8x-1)^5}$.
(l) $f(x) = \frac{(\sqrt{2x}+4)^{34}}{(7x^2+8x-1)^5}$.
(m) $f(x) = \frac{(3x+2)^{50}(3x^6-7x)}{8x^9-2}$.
(n) $f(x) = \frac{(3x+2)^{50}(3x^6-\sqrt{7x})}{8x^9+\sqrt{x-2}}$.
(o) $f(x) = \left(\frac{x}{x^2+1}\right)^4$.
(p) $f(x) = \left(\frac{x^3}{x+1}\right)^5$.
(q) $f(x) = [x^3 - 5 - \frac{x^{-3/2}}{7x^3+6x-1}]^{88}$.
(r) $f(x) = [(7x^2 + 1)^{40}(x^3 - 5) - \frac{x^{-3/2}}{7x^3+6x-1}]^{88}$.
(s) $f(x) = \frac{(2x+1)^{50}(3x^4-8)^{34}}{(2x^2+5x+3)^{60}}$.
(t) $f(x) = \frac{(\sqrt{2x}+1)^{50}(3x^4-8)^{34}}{(2x^2+\sqrt{5x}+3)^{60}}$.

7. Find the tangent line for

(a) $f(x) = (2x + 1)^{34}(7x^2 + 8x - 1)^{50},\ x = 0$.
(b) $f(x) = (2x + 1)^4(3x - 1)^5,\ x = 1$.
(c) $f(x) = \frac{(2x+1)^{34}}{(7x^2+8x-1)^5},\ x = 0$.
(d) $f(x) = \frac{(2x+1)^4}{(3x-1)^5},\ x = 1$.

8. Use a tangent line to approximate

(a) $(2 - \sqrt{9.1})^3(1 + \sqrt{9.1})^2$.
(b) $\frac{(3-\sqrt{16.2})^9}{(1+\sqrt{16.2})^2}$.

9. For a given material, at any time t the instantaneous electrical power $P(t)$ measured in *watts* is related to the electrical current $I(t)$ measured in *ampere* and the resistance constant R for the material measured in *ohms* by the formula $P(t) = R\ I(t)^2$. Suppose for a material with a resistance constant of $R = 15$ *ohms* that at $t = 7$ *seconds* the current $I(7)$ is 5 *amps* and is changing at a rate of $I'(7) = 2\ \frac{amps}{sec}$, what is the rate of change in the power P at this time? In other words what is $P'(7)$ in $\frac{watts}{sec}$?

3.3 Implicit Differentiation and Related Rates

When we take the derivative $(x^2)' = 2x$, we know the function x^2 and its derivative $2x$ *explicitly*, in this sense, this type of derivative can be called **explicit differentiation**.

When we are given an equation such as

$$y^2 + x^4 y + 2x - 7 = 0, \tag{3.1}$$

we can treat Eq. (3.1) as a quadratic equation for y, so we can use the quadratic formula

$$y = \frac{-x^4 \pm \sqrt{x^8 - 4(2x-7)}}{2} \tag{3.2}$$

to *explicitly* determine a y as a function of the independent variable x when we choose either $+$ or $-$ in (3.2). Then we can take derivative of this function y using derivative rules.

For other equations, such as

$$y^5 + 7y + 2x^4 + 5 = 0, \tag{3.3}$$

it can be verified (we simply accept this result here without proof) that we can still determine a y as a differentiable function of x, but the explicit formula for y in terms of x is not obtainable (try and see if you can find such a formula). Now, if we let $y = y(x)$ be a differentiable function determined from Eq. (3.3), then Eq. (3.3) becomes

$$y^5(x) + 7y(x) + 2x^4 + 5 = 0. \tag{3.4}$$

Without knowing the explicit formula of $y(x)$ in terms of x, we can still try to find $y'(x)$ in the following way: we take a derivative with respect to x on both sides of (3.4) and use the general power rule to obtain

$$5y^4(x)y'(x) + 7y'(x) + 8x^3 = 0.$$

Then we can solve for $y'(x)$ and obtain

$$y'(x)[5y^4(x) + 7] = -8x^3,$$

or

$$y'(x) = -\frac{8x^3}{5y^4(x) + 7}. \tag{3.5}$$

In (3.5), it *seems* we got a formula for $y'(x)$. But if you stare the formula for a moment, then you will realize that the formula also involves $y(x)$ which is unknown, so the formula is not explicit, or we still don't know anything about $y'(x)$. In this sense, this type of derivative is called **implicit differentiation**, which means taking a derivative of a function without knowing its formula explicitly, and the derivative is also not explicit.

Example 3.3.1 Assume that

$$x^5y^3 + 3xy + x^4 - 15 = 0$$

determines a y as a differentiable function of x, then find $y'(x)$ using implicit differentiation.

Solution. We regard $y = y(x)$ as a differentiable function in x, and then take a derivative with respect to x using derivative rules to obtain

$$5x^4y^3 + x^5 3y^2y' + 3y + 3xy' + 4x^3 = 0,$$

where $y = y(x)$, $y' = y'(x)$. Then

$$y'[3x^5y^2 + 3x] = -5x^4y^3 - 3y - 4x^3,$$

or

$$y' = \frac{-5x^4y^3 - 3y - 4x^3}{3x^5y^2 + 3x}.$$

♠

It seems that implicit differentiations are not useful as the derivatives are implicit. However, if we know, for example, that some point is on the curve of a function, then we can use implicit differentiation to find the tangent line at the point, so we can do local linear approximations even for some functions without formulas.

Example 3.3.2 Assume that

$$x^5y^3 + 3xy + x^4 - 15 = 0 \tag{3.6}$$

determines a y as a differentiable function of x. Find its tangent line at the point $(1, 2)$ and then approximate $y(1.01)$.

Solution. The equation is the same as in Example 3.3.1, so we get

$$y' = \frac{-5x^4y^3 - 3y - 4x^3}{3x^5y^2 + 3x}.$$

Next, let's check that the point $(1, 2)$ is really on the curve: $8+6+1-15 = 0$, so it is on the curve. Then, we can plug $x = 1$ and $y = 2$ to y' to get

$$y'(1) = \frac{(-5)(8) - 6 - 4}{(3)(4) + 3} = \frac{-50}{15} = -\frac{10}{3}.$$

Now, at the point $(1, 2)$, the slope is $y'(1) = -\frac{10}{3}$. Therefore, the tangent line at $(1, 2)$ is given by

$$y = 2 - \frac{10}{3}(x - 1).$$

To approximate $y(1.01)$, we can use the tangent line and obtain

$$y(1.01) \approx [2 - \frac{10}{3}(x - 1)]\Big|_{x=1.01} = 2 - \frac{10}{3}(1.01 - 1) = 2 - \frac{1}{30} = \frac{59}{30}.$$

♠

This indicates that implicit differentiations and tangent line approximations are very useful, because otherwise it would be very difficult to approximate y from Eq. (3.6) with $x = 1.01$, that is, from

$$1.05101y^3 + 3.03y + x^4 - 15 = 0.$$

The idea of implicit differentiations can also be used to find the derivative of an inverse function. Let $f(x)$ and $f^{-1}(x)$ be inverse functions of each other, then

$$f(f^{-1}(x)) = x,$$

so that using the idea of implicit differentiations, we get

$$f'(f^{-1}(x))(f^{-1})'(x) = 1.$$

Thus, we have

Theorem 3.3.3 *If we know f', then the derivative $(f^{-1})'$ is given by*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3.7}$$

Or equivalently, for $a = f(b)$ (so that $f^{-1}(a) = b$), we have

$$(f^{-1})'(a) = \frac{1}{f'(b)}. \tag{3.8}$$

Example 3.3.4 Consider $f(x) = x^3 + x$, $x \geq 0$, which has an inverse $f^{-1}(x)$. Find $(f^{-1})'(10)$.

Solution. For our purposes here, you can use a graphing calculator to see that $f(x)$ has an inverse. We have $f'(x) = 3x^2 + 1$, and we find that $10 = f(2)$. Thus, from (3.8), we get

$$(f^{-1})'(10) = \frac{1}{f'(2)} = \frac{1}{13}.$$

♠

Related Rates.

Recall that if $q(t)$ denotes a quantity, then $q'(t)$ gives the *rate* of change, or *time rate* of change when t denotes the time.

Sometimes, we need to deal with two quantities $f(t)$ and $g(t)$, and if we **know** $f'(t)$ (the rate of $f(t)$) and **want** $g'(t)$ (the rate of $g(t)$), then we need to find an **equation** (a **relationship**) about $f(t)$ and $g(t)$, so that we can take derivatives on the two sides of the equation and then solve $g'(t)$ using $f'(t)$. That is, we know one rate and use an equation to find another rate, therefore, this study is called **related rates**. When taking these derivatives, the formulas of these quantities as functions of t may not be known, so we use the idea of implicit differentiations.

Example 3.3.5 Assume that in producing and selling a certain product, the number of units ordered, x, and the corresponding unit price, u, are related by the following equation

$$u^3 + 2x^2 - 10xu - 1 = 0. \tag{3.9}$$

If the unit price is increasing at a rate of $\$\,2$ per month, then how fast is the number of units changing when the number x is 5 and the corresponding unit price u is $\$\,1$?

Solution. First, note that "how fast" means the (time) rate, or the first derivative. We regard x and u as functions of t (t is measured in months). Then we know that $u'(t) = 2$ ($/month), and we want $x'(t)$ when $x(t) = 5$ and $u(t) = 1$.

Now, the relationship about $u(t)$ and $x(t)$ is given in Eq. (3.9), so using implicit differentiation, we take derivatives in t on both sides of Eq. (3.9) to get

$$3u^2(t)u'(t) + 4x(t)x'(t) - 10x'(t)u(t) - 10x(t)u'(t) = 0,$$

then, plugging in $x(t) = 5$, $u(t) = 1$, and $u'(t) = 2$, we get

$$6 + 20x'(t) - 10x'(t) - 100 = 0,$$

or

$$x'(t) = 9.4 \ \text{(units/month)}.$$

That is, the increase in the number of units is approximately 9 units per month. ♠

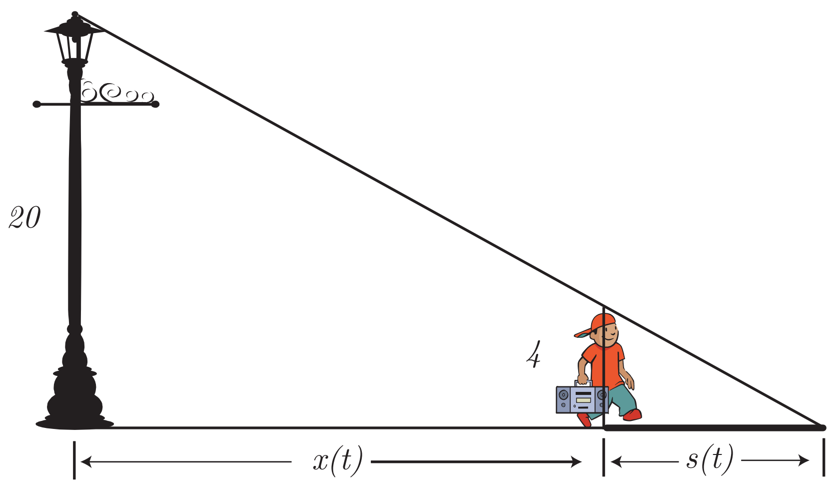

Figure 3.2: The boy walks away from a street light

Example 3.3.6 A 4-ft boy walks away at a rate of 2 ft/sec from a street light, which is 20 ft above the ground. Find the rate at which the length of his shadow is increasing.

Solution. We can use **Figure 3.2** to illustrate the situation. We let $x(t)$ be the distance between the boy and the base of the lamppost, and let $s(t)$ be the shadow of the boy, which are functions of time t. Then we *know* that $x'(t) = 2$ ft/sec and we *want* $s'(t)$, so we need to find an equation about $x(t)$ and $s(t)$. From the figure, we have two *similar* triangles, so that the ratios of the corresponding sides are the same (or you can now think of having the same slope for the same angle for the two similar triangles). Therefore,

$$\frac{4}{s(t)} = \frac{20}{x(t) + s(t)},$$

then,

$$20s(t) = 4x(t) + 4s(t),$$

hence, the relationship between $x(t)$ and $s(t)$ is given by

$$s(t) = \frac{1}{4}x(t).$$

Now, without knowing the formulas of s and x as functions of t, we can use implicit differentiation to take derivatives in t on both sides to obtain

$$s'(t) = \frac{1}{4}x'(t) = \frac{1}{4}(2) = \frac{1}{2} \text{ ft/sec},$$

that is, the shadow of the boy is increasing at a rate of $\frac{1}{2}$ ft/sec. ♠

Example 3.3.7 If the side of a square is increasing at a rate of 4 inches per minute, then how fast is the area of the square changing when the side is 10 inches?

Solution. Let $s(t)$ be the side of the square and let $A(t)$ be the area of the square, which are functions of time t. Then we know that $s'(t) = 4$ in/min and we want $A'(t)$, so we need to find an equation about $s(t)$ and $A(t)$. Now, we have a square, so we have

$$A(t) = s^2(t). \tag{3.10}$$

Without knowing the formulas of s and A as functions of t, we can use implicit differentiation to take derivatives in t on both sides of (3.10) to obtain

$$A'(t) = 2s(t)s'(t) = 2s(t) \cdot 4 = 8s(t).$$

Now, we want to find the rate of change of the area at the moment when the side is 10 inches, so we can plug in $s(t) = 10$ to obtain

$$A'(t) = 8s(t) = 8(10) = 80 \ \ (\text{in}^2/\text{min}),$$

that is, the area of the square is increasing at a rate of 80 in^2/min when the side is 10 inches. ♠

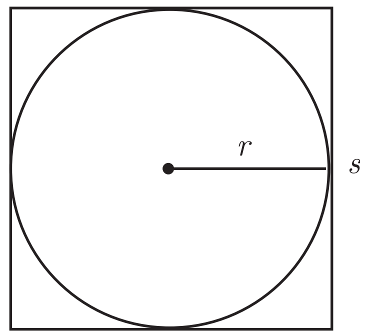

Figure 3.3: A circle is inscribed in a square

Example 3.3.8 A circle is inscribed in a square. If the side of the square is increasing at a rate of 2 in/sec, find how fast the area of the circle is increasing when the side is 20 inches.

Solution. The situation can be visualized in **Figure 3.3**, where r (radius of the circle) and s (side of the square) are all functions of t. We know $s'(t) = 2$ (in/sec), and we want $A'(t)$ when $s = 20$ where $A(t)$ is the area of the circle, so we need to find an equation about $s(t)$ and $A(t)$. Since $A = \pi r^2$ and $s = 2r$, we obtain

$$A = \pi(\frac{s}{2})^2 = \frac{\pi s^2}{4}.$$

Taking derivatives in t using implicit differentiation and plugging in $s = 20$ and $s'(t) = 2$, we get

$$A' = \frac{\pi 2ss'}{4} = \frac{\pi 2(20)(2)}{4} = 20\pi \quad (\text{in}^2/\text{sec}).$$

That is, the area of the circle is increasing at a rate of 20π in^2/sec when the side is 20 inches. ♠

Exercises 3.3

1. For the following equations, find $y'(x)$ using implicit differentiation (assuming that a differentiable function y of x can be determined).

(a) $y^3 + xy + x^3 - 5 = 0.$

(b) $x^3y^5 + 9x^2y^3 + x^5 - 10 = 0.$

(c) $[y + x + 6]^3 + 2(xy)^2 + y + x^4 - 6 = 0$.

2. Assume that

$$x^2y^3 + 3xy + x^3 - 18 = 0$$

determines a y as a differentiable function of x. Find its tangent line at the point $(2, 1)$ and then approximate $y(2.01)$.

3. Assume that

$$x^2y^5 + 3xy^3 + x^3 - 18 = 0$$

determines a y as a differentiable function of x. Find its tangent line at the point $(2, 1)$ and then approximate $y(2.01)$.

4. Consider $f(x) = x^5 + 2x$, $x \geq 0$, which has an inverse $f^{-1}(x)$. Find $(f^{-1})'(3)$.

5. Assume that in producing and selling a certain product, the number of units ordered, x, and the corresponding unit price, u, are related by the following equation

$$0.005u^3 + 2x^3 - 10xu + 245 = 0.$$

(a) If the unit price is increasing at a rate of \$ 2 per month, then how fast is the number of units changing when the number x is 5 and the corresponding unit price u is \$ 10?

(b) If the number of units is increasing at a rate of 3 units per month, then how fast is the unit price changing when the number x is 5 and the corresponding unit price u is \$ 10?

6. Let $P(x) = 1000x - \frac{x^2}{2}$ be a profit function. If the number of units (x) is increasing at a rate of 5 units per month, then how fast is the profit changing when the number x is 7?

7. Let $u(x) = 200 - 0.002x$ be a unit price function. If the number of units (x) is increasing at a rate of 5 units per month, then how fast is the revenue changing when the number x is 7?

8. A 5-ft girl walks away at a rate of 2.5 ft/sec from a street light, which is 30 ft above the ground. Find the rate at which the length of her shadow is increasing.

9. A 30-ft-long ladder leans against a lamppost. The lower end of the ladder begins to slip away horizontally at a rate of 2 ft/sec. Find how fast is the top of the ladder moving when the bottom of the ladder is 20 ft from the lamppost.

10. If the length of a side of an equilateral triangle is increasing at a rate of 3 in/min, find how fast is the area of the triangle increasing when the side is 10 inches.

11. A circle is inscribed in a square. If the side of the square is increasing at a rate of 4 in/sec, find how fast the area of the circle is increasing when the side is 10 inches.

12. A square is inscribed in a circle. If the radius of the circle is increasing at a rate of 4 in/sec, find how fast the area of the square is increasing when the radius is 20 inches.

13. If you pump air into a spherical balloon at a rate of 4 in^3/min, then find how fast the radius of the balloon is increasing when the radius is 3 inches. (Hint: the volume $V = \frac{4}{3}\pi r^3$.)

14. For a given material, at time t the instantaneous electrical power $P(t)$ measured in *watts* is related to the electrical current $I(t)$ measured in *amperes* and the resistance constant R for the material measured in *ohms* by the formula $P(t) = R\ I(t)^2$. Suppose for a material with a resistance constant of $R = 15$ *ohms* that at $t = 7$ *seconds* the current $I(7)$ is 5 *amps* and is changing at a rate of $I'(7) = 2\ \frac{amps}{sec}$. What is the rate of change in the power P at this time? In other words what is $P'(7)$ in $\frac{watts}{sec}$?

Chapter 4

Exponential and Logarithmic Functions

The derivatives of exponential and logarithmic functions are given in this chapter. Some applications, such as those in population dynamics or radioactive decay of chemical elements, are also given. The concept of differential equations is briefly introduced. A special technique of taking derivatives, called logarithmic differentiation, is also studied, which allows us to take derivatives of some complicated functions.

4.1 The Derivatives of Exponential Functions

Since e^x is a very important function and now we have learned something about derivatives, we ask the following

Question: *What is* $\frac{d}{dx}e^x$*?*

Make sure that you don't apply the power rule because e^x is an exponential function, not a power function. Since we don't have other things to use, we have to use definition (i.e., limit). That is, we need to use a limit to find $\frac{d}{dx}e^x$. Therefore, we start with

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}, \qquad (1.1)\end{aligned}$$

where we take e^x out of the limit as $h \to 0$ because e^x has nothing to do with h and hence can be regard as a constant.

Next, we have two ways to derive the limit in (1.1). The first way is to use the table in **Figure 4.1**, which we copied from Figure 2.11, from which we find that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

h	$\frac{e^h-1}{h}$
0.1	1.0517
0.01	1.0050
0.001	1.0005
-0.1	0.9516
-0.01	0.9950
-0.001	0.9995

Figure 4.1: A table for $\frac{e^h-1}{h}$

The second way is to use (2.10) in Section 2.2 and let $h = \frac{1}{m}$ so that

$$e = \lim_{m\to\infty} \Big(1 + \frac{1}{m}\Big)^m = \lim_{h\to 0}(1+h)^{\frac{1}{h}}, \tag{1.2}$$

or $e \approx (1+h)^{\frac{1}{h}}$ as $h \approx 0$. This implies that $e^h \approx (1+h)$ as $h \approx 0$, thus,

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{(1+h) - 1}{h} = 1.$$

Therefore, from (1.1), we obtain

Property 4.1.1 *For the exponential function e^x, we have*

$$\frac{d}{dx} e^x = e^x.$$

♠

Typically, taking a derivative would change a function, for example, $(x^2)' = 2x$. So the result in Property 4.1.1 is quite surprising in the sense that the derivative of e^x is the same as the original function e^x. It also indicates that the power rule (which would give xe^{x-1}) does not apply to exponential functions.

For $e^{g(x)}$ where the function g has derivatives, we can decompose it as

$$y = e^z, \quad z = g(x),$$

then, using the chain rule, we obtain

Property 4.1.2 *If $g(x)$ has derivatives, then*

$$\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x).$$

♠

Property 4.1.2 says that to take the derivative of $e^{g(x)}$, we copy $e^{g(x)}$ first, and then multiply by $g'(x)$.

Example 4.1.3 Find $\frac{d}{dx}e^{2x}$.

Solution. Using Property 4.1.2, we have

$$\frac{d}{dx}e^{2x} = e^{2x} \cdot 2 = 2e^{2x}.$$

Since $e^{2x} = e^x e^x$, this example can be checked using the product rule on $e^x e^x$. See exercises. ♠

Example 4.1.4 Find $\frac{d}{dx}e^{e^{2x}}$.

Solution. Using Property 4.1.2 (twice), we have

$$\frac{d}{dx}e^{e^{2x}} = e^{e^{2x}}\frac{d}{dx}e^{2x} = e^{e^{2x}}e^{2x} \cdot 2 = 2e^{e^{2x}}e^{2x}.$$

♠

Example 4.1.5 Find $\frac{d}{dx}(x^3+2)^{-3/4}e^{3x^4+x}$.

Solution. Using Property 4.1.2 and the product rule, we have

$$\begin{aligned}&\frac{d}{dx}(x^3+2)^{-3/4}e^{3x^4+x}\\ &= -\frac{3}{4}(x^3+2)^{-7/4}(3x^2)e^{3x^4+x} + (x^3+2)^{-3/4}e^{3x^4+x}(12x^3+1).\end{aligned}$$

♠

Example 4.1.6 Find $\frac{d}{dx}\frac{(x^3+2)^{-3/4}}{e^{3x^4+x}}$.

Solution. Using Property 4.1.2 and the quotient rule, we have

$$\begin{aligned}&\frac{d}{dx}\frac{(x^3+2)^{-3/4}}{e^{3x^4+x}}\\ &= \frac{-\frac{3}{4}(x^3+2)^{-7/4}(3x^2)e^{3x^4+x} - (x^3+2)^{-3/4}e^{3x^4+x}(12x^3+1)}{e^{6x^4+2x}}.\end{aligned}$$

♠

Example 4.1.7 Find $\frac{d}{dx}(x^{-3/4} + e^{3x^4+x})^{50}$.

Solution. We first start with general power rule and then use Property 4.1.2 to obtain

$$\frac{d}{dx}(x^{-3/4} + e^{3x^4+x})^{50}$$
$$= 50(x^{-3/4} + e^{3x^4+x})^{49}[-\frac{3}{4}x^{-7/4} + e^{3x^4+x}(12x^3 + 1)].$$

♠

Guided Practice 4.1

Find $f'(x)$ for the following functions $f(x)$.

1. $f(x) = e^{4x}$.

2. $f(x) = e^{x^2+3x+1}$.

3. $f(x) = xe^{2x}$.

4. $f(x) = \frac{e^x}{x^2}$

5. $f(x) = \sqrt{e^{x^2} + 1}$.

Exercises 4.1

1. Can we apply the power rule to e^x? Why or why not?

2. Note that $e^{2x} = e^x e^x$. Take the derivative of e^{2x} using Property 4.1.2 and then check it with the derivative of $e^x e^x$ using the product rule.

3. Note that $e^{3x} = e^x e^x e^x$. Take the derivative of e^{3x} using Property 4.1.2 and then check it with the derivative of $e^x e^x e^x$ using the product rule.

4. Find $f'(x)$ for

(a) $f(x) = e^{4x} + x^3 - e^x$.
(b) $f(x) = e^{4x^2-2x}$.
(c) $f(x) = x^4 e^{4x^2-2x}$.
(d) $f(x) = (x^2 + 2)^{-4/3} e^{4x^2-2x}$.
(e) $f(x) = \frac{e^x+e^{-x}}{2}$.
(f) $f(x) = \frac{e^x-e^{-x}}{2x}$.

(g) $f(x) = \frac{e^{5x^7}+3x}{x^3+2+x^{-3/4}}$.

(h) $f(x) = \frac{e^{5x^7+8x}}{\sqrt{x^3+2}+x^{-3/4}}$.

(i) $f(x) = \frac{x^4 e^{5x^7}}{x^3+2+x^{-3/4}}$.

(j) $f(x) = \frac{x^4 e^{5x^7+8x}}{\sqrt{x^3+2}+x^{-3/4}}$.

(k) $f(x) = (x^{-5/3} + e^{7x^3+5x})^{40}$.

(l) $f(x) = (x^{-5/3} + x^6 e^{7x^3+5x})^{40}$.

4.2 Exponential Growth and Exponential Decay

Question: *Assume that the population of a certain college was 8,300 in 1980 and 10,400 in 1992. Assume further that the population of the college grows at a rate proportional to its population. Then can we find a mathematical model to predict the future population growth of the college?*

In this section, we will address this issue and learn some applications of exponential functions. This issue actually goes back to about 1800, at which time people analyzed data from many case studies of certain quantities in real situations, such as population dynamics or radioactive decay of chemical elements, and discovered that the changes of these quantities fitted so well into the following model:

"For some quantity $P(t)$ of time t, the rate of change of $P(t)$ is **proportional** to the quantity $P(t)$ with some proportional constant k."

Mathematically, since the rate of $P(t)$ is $P'(t)$, we derive the following mathematical model

$$P'(t) = kP(t). \tag{2.1}$$

Nowadays, more advanced models have been discovered/proposed to derive more accurate results. However, the model (2.1) is still used very often when people just want to get a rough estimate/predict in certain applications, including population dynamics and radioactive decay of chemical elements. That is why we need to learn something about the model (2.1) here.

To introduce a concept, let's recall that

$$x^2 - 3x + 2 = 0 \tag{2.2}$$

is called an *algebraic equation* in the sense that it is an *equation* involving an *unknown number* x, and the operations in Eq. (2.2) are *algebraic operations* (multiplication, subtraction, and addition). When you solve the equation, you will find that $x = 1$ and $x = 2$ are the *solutions* in the sense that Eq. (2.2) holds true if you plug in $x = 1$ or $x = 2$. It also indicates that Eq. (2.2) may not hold true if you plug in an arbitrary x value. For example, if you plug in $x = 3$, then $x^2 - 3x + 2 = 2 \neq 0$, so Eq. (2.2) does not hold true for $x = 3$.

With this as a background, we call Eq. (2.1) a **differential equation** in the sense that it is an *equation* involving the *derivative* of an unknown function $P(t)$. Our goal here is to *solve* the differential equation (2.1), that is, to find *solutions* for Eq. (2.1).

To begin, we start with some simple examples.

Example 4.2.1 Find $P'(t)$ for $P(t) = 2e^{0.03t}$.

Solution. We have

$$P'(t) = 2e^{0.03t} \cdot 0.03 = 0.03(2e^{0.03t}) = 0.03P(t),$$

which says that the rate of the quantity $P(t)$ (i.e., $P'(t)$) is proportional to the quantity $P(t)$ with a proportional constant 0.03. ♠

Example 4.2.2 Find $P'(t)$ for $P(t) = ce^{kt}$ where c and k are constants.

Solution. We have

$$P'(t) = ce^{kt} \cdot k = k(ce^{kt}) = kP(t),$$

which says that the rate of the quantity $P(t)$ ($P'(t)$) is proportional to the quantity $P(t)$ with a proportional constant k. ♠

Note that in $P(t) = ce^{kt}$, when we evaluate at $t = 0$, we get $c = P(0)$, so that $P(t) = P(0)e^{kt}$. Accordingly, we have

Property 4.2.3 *Let $P(t) = P(0)e^{kt}$ where k is a constant, then $P(t)$ satisfies (2.1). That is, $P(t) = P(0)e^{kt}$ is a solution of the differential equation (2.1).* ♠

Question: *Can the differential equation (2.1) have other forms of solutions?*

The good news is that the answer is No. That is, the following result says that the converse of Property 4.2.3 is also true: if $P(t)$ is a solution of Eq. (2.1), then $P(t)$ must be given by $P(t) = P(0)e^{kt}$, no other choices.

Property 4.2.4 *If $P(t)$ is a solution of Eq. (2.1), that is, if $P'(t) = kP(t)$, then $P(t)$ must be given by $P(t) = P(0)e^{kt}$.*

Verification. Let $P(t)$ be a solution of Eq. (2.1), then $P'(t) = kP(t)$, or $P'(t) - kP(t) = 0$, so we obtain

$$P'(t)e^{-kt} - kP(t)e^{-kt} = 0,$$

which, using the product rule, implies

$$\frac{d}{dt}[P(t)e^{-kt}] = 0.$$

From geometry, if a function is such that its derivative is zero everywhere on its domain, then the function is a constant, or the curve is a horizontal line (this result can be proved using something called the "mean value theorem" outlined in Section 5.1). Therefore, for some constant c,

$$P(t)e^{-kt} = c,$$

or

$$P(t) = ce^{kt} = P(0)e^{kt}.$$

♠

From now on, if a quantity $P(t)$ of time t can be modeled using Eq. (2.1), then we will follow Property 4.2.4 and write down $P(t)$ as $P(t) = P(0)e^{kt}$. Moreover, in applications, we assume $t \geq 0$ and regard $t = 0$ as the **initial time** and call $P(0)$ the **initial quantity** or **initial value**.

Next, consider $P(t) = P(0)e^{kt}$ with $P(0) > 0$ for $t \geq 0$. First, note that the proportional constant k is assumed to be nonzero because otherwise $P(t) = P(0)e^0 = P(0)$ becomes a constant, which is not interesting. Now, if $k > 0$, then $P(t) = P(0)e^{kt}$ increases as t increases. So, following a similar discussion in Chapter 1, we call it *exponential growth*. If $k < 0$, then $P(t) = P(0)e^{kt}$ decreases as t increases, so we call it *exponential decay*. That is,

1. $P(t) = P(0)e^{kt}$ grows exponentially if $k > 0$,
2. $P(t) = P(0)e^{kt}$ decays exponentially if $k < 0$,

where the proportional constant k is also called the **growth constant** or **decay constant** respectively. Now, we look at some examples.

Example 4.2.5 Solve the differential equation

$$P'(t) = \frac{P(t)}{2}, \quad P(0) = 5.$$

Solution. Comparing with Eq. (2.1), we get $k = \frac{1}{2}$. Thus, from properties 4.2.3 and 4.2.4, we obtain

$$P(t) = P(0)e^{kt} = 5e^{t/2}.$$

♠

Example 4.2.6 (Growth of bacteria) Assume that the population of a certain colony of bacteria was 600 initially, and that the colony grows at a rate proportional to its population with the proportional constant 0.03, where t is measured in hours. Find the function $P(t)$ representing the population of this colony according to the model. Then find, using the model, the population after 9 hours, and when the population will reach 1,000.

Solution. Since $P'(t)$ gives the rate of change of the population and the proportional constant is 0.03, we get

$$P'(t) = 0.03P(t), \quad P(0) = 600.$$

Then, from Property 4.2.4, we obtain

$$P(t) = P(0)e^{kt} = 600e^{0.03t}.$$

To find the population after 9 hours using the model, we let $t = 9$ and obtain

$$P(9) = P(0)e^{0.03 \cdot 9} = 600e^{0.27} \approx 785.98 \approx 786.$$

To find when the population reaches $1,000$, we need to find t such that $P(t) = 1,000$. According to the model, $P(t) = 600e^{0.03t}$, thus we get

$$600e^{0.03t} = 1,000,$$

which can be simplified to become $e^{0.03t} = \frac{5}{3}$, then $0.03t = \ln \frac{5}{3}$, or

$$t = \frac{\ln(5/3)}{0.03} \approx 17,$$

that is, it takes about 17 hours for the population to reach 1,000. ♠

The following example answers the question raised at the beginning of this section.

Example 4.2.7 (Population dynamics) Assume that the population of a certain college was $8,300$ in 1980 and $10,400$ in 1992. Assume further that the population of the college grows at a rate proportional to its population. Find the function $P(t)$ representing the population of this college according to the model. Then find, using the model, what the population will be in 1998, and in what year the population reaches $14,000$.

Solution. We treat the year 1980 as the initial time $t = 0$, so that the initial population is $P(0) = 8300$. From Property 4.2.4,

$$P(t) = P(0)e^{kt} = 8300e^{kt},$$

where k is the proportional constant and t is the number of years after 1980. To determine $P(t)$, we need to find k. Now, "the population was $10,400$ in

1992" means that it took $t = 12$ $(1992 - 1980 = 12)$ years (from 1980) for the population to reach 10,400, so we get $P(12) = 10,400$, which means, according to the model,

$$8300e^{k\cdot 12} = 10400.$$

Next, we can simplify to get $e^{12k} \approx 1.253$ and then $12k \approx \ln 1.253$, or

$$k \approx \frac{\ln 1.253}{12} \approx 0.0188.$$

Therefore,

$$P(t) = P(0)e^{kt} = 8300e^{0.0188t}. \tag{2.3}$$

To find the population in 1998 using the model, we let $t = 18$ $(1998 - 1980 = 18)$ to obtain

$$P(18) = P(0)e^{0.0188\cdot 18} = 8300e^{0.3384} \approx 11642.$$

To find the year the population reaches 14,000, we need to find t such that $P(t) = 14,000$. According to the model, $P(t) = 8300e^{0.0188t}$, thus we get

$$8300e^{0.0188t} = 14000,$$

which can be simplified to become $0.0188t = \ln \frac{14000}{8300} \approx \ln 1.687$, or

$$t \approx \frac{\ln 1.687}{0.0188} \approx 28,$$

that is, it takes about 28 years, or in the year of 2008, for the population to reach 14,000. ♠

Note that in Example 4.2.7, you can also write $P(t)$ as

$$P(t) = 10400e^{0.0188t}, \tag{2.4}$$

where t is the number of years after 1992. Because $8300e^{0.0188\cdot 12} = 10400$, (2.4) becomes

$$P(t) = 10400e^{0.0188t} = 8300e^{0.0188\cdot 12}e^{0.0188t} = 8300e^{0.0188(12+t)},$$

which is the same as (2.3) as $12+t$ now gives the number of years after 1980. In fact, this idea holds true for general dynamical systems.

Example 4.2.8 (Radioactive decay) The decay constant for a certain chemical element is -0.04, where t is measured in years. How long will it take for a quantity of this element to decay to one half of its original size?

Solution. The word "decay" indicates that we need to use Eq. (2.1). So we have

$$P(t) = P(0)e^{kt} = P(0)e^{-0.04t},$$

where $P(0)$ denotes the original size. We need to find t such that $P(t) = \frac{1}{2}P(0)$, so we obtain

$$P(0)e^{-0.04t} = \frac{1}{2}P(0).$$

Now, we can cancel $P(0)$ (this is the reason why the precise value of $P(0)$ is not given here, because it is not needed), and get $e^{-0.04t} = \frac{1}{2}$, or $-0.04t = \ln\frac{1}{2} \approx -0.693$, and then $t \approx 17$. Therefore, it takes about 17 years. ♠

The t value in Example 4.2.8 is called the **half-life** of that element, giving the time the element takes to reduce its size by one half. See **Figure 4.2**.

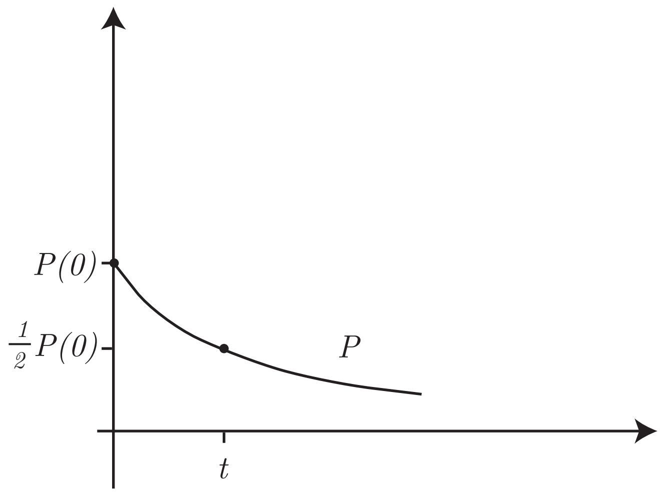

Figure 4.2: Half-life for the element

Example 4.2.8 indicates that given the k value, we can find the half-life. The next example indicates that the converse is also true. That is, given a half-life, we can find the k value, hence we can answer the related questions.

Example 4.2.9 (Radioactive decay) Given that the half-life of a certain chemical element is 20 years, how long will it take for a quantity of this element to decay to one fifth of its original size?

Solution. Let's find k first. We have $P(t) = P(0)e^{kt}$ from the model, and, as the half-life is 20 years, we obtain $P(20) = \frac{1}{2}P(0)$. Thus,

$$P(0)e^{20k} = \frac{1}{2}P(0),$$

which can be solved to give

$$k = \frac{\ln(1/2)}{20} \approx -0.0347,$$

so that

$$P(t) = P(0)e^{-0.0347t}.$$

Next, knowing the value of k, we can answer the question by finding t such that $P(t) = \frac{1}{5}P(0)$, that is,

$$P(0)e^{-0.0347t} = \frac{1}{5}P(0),$$

which can be solved to give

$$t = \frac{\ln(1/5)}{-0.0347} \approx 46 \text{ years}.$$

♠

These examples indicate that for these types of applications, you can write down

$$P(t) = P(0)e^{kt}$$

directly, where $P(0)$ is typically given, or the precise value of $P(0)$ is not needed. Then based on the information given, you can solve for k or for t.

Exercises 4.2

1. Solve the differential equation

$$P'(t) = 0.03P(t), \quad P(0) = 76.$$

2. Solve the differential equation

$$P'(t) = \frac{2P(t)}{5}, \quad P(0) = 2.$$

3. Find all functions whose derivatives are the same as their original functions.

4. Assume that the population of a certain colony of bacteria was 750 initially, and that the colony grows at a rate proportional to its population with the proportional constant 0.05, where t is measured in hours. Find the function $P(t)$ representing the population of this colony according to the model. Then find, using the model, the population after 12 hours, and when the population will reach 1,500.

5. Assume that the population of a certain city was 55,300 in 1961 and 75,400 in 1980. Assume further that the city grows at a rate proportional to its population. Find the function $P(t)$ representing the population of this city according to the model. Then find, using the model, what the population will be in 1990, and in what year the population reaches 99,000.

6. The decay constant for a certain chemical element is -0.02, where t is measured in years. How long will it take for a quantity of this element to decay to one half of its original size? (That is, find the half-life of the element.)

7. The decay constant for a certain chemical element is -0.02, where t is measured in years. How long will it take for a quantity of this element to decay to one third of its original size?

8. Given that the half-life of a certain chemical element is 250 years, how long will it take for a quantity of this element to decay to one fourth of its original size?

9. Given that the half-life of a certain chemical element is 100 years, how long will it take for a quantity of this element to decay to one third of its original size?

4.3 The Derivatives of Logarithmic Functions

Since $\ln x$ is also a very important function, we ask the following

Question: *For $x > 0$, what is $\frac{d}{dx}\ln x$?*

In this case, instead of using definition and carrying out a limit (which can be done), we have an easier way to find $\frac{d}{dx}\ln x$. That is, we know that

$$x = e^{\ln x}, \quad (x > 0),$$

so we can take derivatives on both sides by using the chain rule on the right-hand side to obtain

$$\begin{aligned} 1 &= e^{\ln x} \cdot \frac{d}{dx}\ln x \\ &= x\frac{d}{dx}\ln x. \end{aligned}$$

Then we can simplify (divide by x on both sides) and obtain

Property 4.3.1 *For $\ln x$, $x > 0$, we have*

$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad x > 0.$$

♠

For the composition $\ln(g(x))$ where $g(x) > 0$ has derivatives, we can use the chain rule (write $y = \ln z$, $z = g(x)$) to obtain

Property 4.3.2 *If $g(x) > 0$ has derivatives, then*

$$\frac{d}{dx}\ln(g(x)) = \frac{1}{g(x)}g'(x) = \frac{g'(x)}{g(x)}.$$

♠

Property 4.3.2 says that the derivative of $\ln(g(x))$ is given by 1 over what you see inside the "ln" function, then multiplied by the derivative of what is inside the "ln" function.

Example 4.3.3 Find $\frac{d}{dx}\ln(x^2 - 4x + 5)$.

Solution. Using Property 4.3.2, we have

$$\frac{d}{dx}\ln(x^2 - 4x + 5) = \frac{2x - 4}{x^2 - 4x + 5}.$$

♠

Example 4.3.4 Find $\frac{d}{dx}\ln|x|,\ x \neq 0$.

Solution. We look at case by case for $|x|$. For $x > 0$, we have

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}.$$

For $x < 0$, we have, using Property 4.3.2,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}.$$

Thus, we obtain

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0.$$

♠

Example 4.3.5 Find $\frac{d}{dx}\frac{e^{2x+3}}{x\ln(x^6+4)}$.

Solution. Using Property 4.3.2 and the quotient and product rules, we have

$$\frac{d}{dx}\frac{e^{2x+3}}{x\ln(x^6+4)}$$
$$= \frac{1}{[x\ln(x^6+4)]^2}\Big\{2e^{2x+3}x\ln(x^6+4) - e^{2x+3}[\ln(x^6+4) + x\frac{6x^5}{x^6+4}]\Big\}.$$

♠

In some cases, logarithmic functions with other bases are needed, such as the common logarithmic function when doing certain laboratory reports. Thus, let's find the derivative of $\log_b x, x > 0$, where $b > 0$ and $b \neq 1$. Again, we have an easier way to achieve this because of the following result.

Property 4.3.6 *Let $b > 0$ and $b \neq 1$, then*

$$\log_b x = \frac{\ln x}{\ln b}, \quad x > 0.$$

Verification. Write

$$\log_b x = y, \quad \ln x = w, \quad \ln b = z,$$

then we get

$$x = b^y, \quad x = e^w, \quad b = e^z.$$

Thus,

$$e^w = x = b^y = (e^z)^y = e^{zy},$$

so that $w = zy$ and therefore $y = \frac{w}{z}$, that is, $\log_b x = \frac{\ln x}{\ln b}$. ♠

Property 4.3.6 says that the difference between $\log_b x$ and $\ln x$ is to divide by $\ln b$. So when taking the derivative of $\log_b x$, we simply take the derivative of $\ln x$ and then divide by $\ln b$. Consequently, we get

Property 4.3.7 *Let $b > 0$ and $b \neq 1$. Then*

$$\frac{d}{dx} \log_b x = \frac{1}{x \ln b}.$$

If $g(x) > 0$ has derivatives, then

$$\frac{d}{dx} \log_b(g(x)) = \frac{g'(x)}{g(x) \ln b}.$$

♠

Example 4.3.8 Find $\frac{d}{dx} \log_3(x^2 - 4x + 5)$.

Solution. Using Property 4.3.7, we have

$$\frac{d}{dx} \log_3(x^2 - 4x + 5) = \frac{2x - 4}{(x^2 - 4x + 5) \ln 3}.$$

♠

The following three examples indicate that sometimes it is a good idea to use properties of logarithmic functions to simplify first and then take derivatives.

Example 4.3.9 Find $\frac{d}{dx} \ln(x^4 - 4x + 5)^4$.

Solution 1. Before taking the derivative, we use a property of logarithmic functions ($\ln A^a = a \ln A$) to simplify and then take the derivative. Then we get

$$\begin{aligned}\frac{d}{dx}\ln(x^4-4x+5)^4 &= \frac{d}{dx}4\ln(x^4-4x+5)\\ &= \frac{4(4x^3-4)}{x^4-4x+5}.\end{aligned}$$

Solution 2. Using Property 4.3.2 directly, we get

$$\begin{aligned}\frac{d}{dx}\ln(x^4-4x+5)^4 &= \frac{4(x^4-4x+5)^3(4x^3-4)}{(x^4-4x+5)^4}\\ &= \frac{4(4x^3-4)}{x^4-4x+5}.\end{aligned}$$

♠

Example 4.3.10 Find $\frac{d}{dx}\ln(x^2+2)^2(x^4+4)^4$.

Solution 1. Simplifying using an additional property of logarithmic functions ($\ln(AB) = \ln A + \ln B$) and then taking the derivative, we get

$$\begin{aligned}&\frac{d}{dx}\ln(x^2+2)^2(x^4+4)^4 = \frac{d}{dx}[\ln(x^2+2)^2+\ln(x^4+4)^4\,]\\ &= \frac{d}{dx}[2\ln(x^2+2)+4\ln(x^4+4)\,]\\ &= 2\frac{2x}{x^2+2}+4\frac{4x^3}{x^4+4} = \frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}.\end{aligned}$$

Solution 2. Using Property 4.3.2 directly which involves the product rule, we get

$$\begin{aligned}&\frac{d}{dx}\ln(x^2+2)^2(x^4+4)^4 = \frac{[(x^2+2)^2(x^4+4)^4\,]'}{(x^2+2)^2(x^4+4)^4}\\ &= \frac{1}{(x^2+2)^2(x^4+4)^4}\Big[2(x^2+2)2x(x^4+4)^4+(x^2+2)^2 4(x^4+4)^3 4x^3\Big]\\ &= \frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}.\end{aligned}$$

♠

Example 4.3.11 Find $\frac{d}{dx}\ln\frac{(x^2+2)^2}{(x^4+4)^4}$.

Solution 1. Simplifying using an additional property of logarithmic functions ($\ln\frac{A}{B} = \ln A - \ln B$) and then taking the derivative, we get

$$\begin{aligned}&\frac{d}{dx}\ln\frac{(x^2+2)^2}{(x^4+4)^4} = \frac{d}{dx}[\ln(x^2+2)^2-\ln(x^4+4)^4\,]\\ &= \frac{d}{dx}[2\ln(x^2+2)-4\ln(x^4+4)\,]\\ &= 2\frac{2x}{x^2+2}-4\frac{4x^3}{x^4+4} = \frac{4x}{x^2+2}-\frac{16x^3}{x^4+4}.\end{aligned}$$

Solution 2. Using Property 4.3.2 directly which involves the quotient rule, we get

$$\begin{aligned}
&\frac{d}{dx}\ln\frac{(x^2+2)^2}{(x^4+4)^4} = \frac{1}{\frac{(x^2+2)^2}{(x^4+4)^4}}\frac{d}{dx}\frac{(x^2+2)^2}{(x^4+4)^4}\\
&= \frac{(x^4+4)^4}{(x^2+2)^2}\frac{1}{(x^4+4)^8}\left[2(x^2+2)2x(x^4+4)^4 - (x^2+2)^2 4(x^4+4)^3 4x^3\right]\\
&= \frac{4x}{x^2+2} - \frac{16x^3}{x^4+4}.
\end{aligned}$$

♠

You should see that the first solutions in examples 4.3.9 − 4.3.11 are much easier when properties of logarithmic functions are used to simplify first before taking derivatives.

In fact, properties of logarithmic functions are crucially useful when taking derivatives of some very complicated functions. To motivate this topic, we ask

Question: *For $x > 0$, what is $\frac{d}{dx}x^x$?*

The function x^x is so strange that it is not a power function and not an exponential function, since both the base and exponent are variables. Thus the rules for power and exponential functions do not apply.

From the above examples, we have seen that logarithmic functions can be used to simplify things, so let's apply them here. From the property $\ln A^a = a \ln A$ of logarithmic functions, we obtain

$$\ln x^x = x \ln x = (x)(\ln x),$$

which *changes the situation completely*, that is, $(x)(\ln x)$ is now a **product** whose derivative can be derived using the product rule. Now, the question is whether the derivative of $\ln x^x$ can be used to find the derivative of x^x. In general, the question is whether the derivative of $\ln g(x)$ can be used to find the derivative of $g(x)$.

Note that Property 4.3.2 says that

$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)},$$

so if we multiply by $g(x)$ on both sides, then we get the following **property of logarithmic differentiation**,

Property 4.3.12 (Logarithmic differentiation) *For $g(x) > 0$ and differentiable,*

$$\frac{d}{dx}g(x) = g(x)\frac{d}{dx}\ln(g(x)). \tag{3.1}$$

♠

Property 4.3.12 can also be derived from $g(x) = e^{\ln g(x)}$ and then take a derivative:

$$\frac{d}{dx}g(x) = \frac{d}{dx}e^{\ln g(x)} = e^{\ln g(x)}\frac{d}{dx}\ln(g(x)) = g(x)\frac{d}{dx}\ln(g(x)).$$

We can check (3.1) for some simple functions. For example, for $g(x) = 1$, the left-hand side of (3.1) is zero, and the right-hand side of (3.1) is $\frac{d}{dx}\ln 1 = \frac{d}{dx}0 = 0$. For $g(x) = x$, the left-hand side of (3.1) is 1, and the right-hand side of (3.1) is $x\frac{d}{dx}\ln x = x\frac{1}{x} = 1$.

The property of logarithmic differentiation indicates that if we know $\frac{d}{dx}\ln(g(x))$, then we can find $\frac{d}{dx}g(x)$. It is especially useful when $\frac{d}{dx}g(x)$ is not easy to obtain directly but $\frac{d}{dx}\ln(g(x))$ can be obtained, such as in the following examples.

Example 4.3.13 Find $\frac{d}{dx}x^x$, $x > 0$.

Solution. Using Property 4.3.12, we obtain

$$\begin{aligned}\frac{d}{dx}x^x &= x^x\frac{d}{dx}\ln x^x = x^x\frac{d}{dx}(x\ln x)\\ &= x^x(\ln x + x\frac{1}{x}) = x^x(\ln x + 1).\end{aligned}$$

We can also do it using $x^x = e^{\ln x^x} = e^{x\ln x}$ and obtain the same result.

♠

Example 4.3.14 Find $\frac{d}{dx}(3x^2+1)^{x^3}$.

Solution. Using Property 4.3.12, we obtain

$$\begin{aligned}\frac{d}{dx}(3x^2+1)^{x^3} &= (3x^2+1)^{x^3}\frac{d}{dx}\ln(3x^2+1)^{x^3} = (3x^2+1)^{x^3}\frac{d}{dx}[x^3\ln(3x^2+1)]\\ &= (3x^2+1)^{x^3}[3x^2\ln(3x^2+1) + x^3\frac{6x}{3x^2+1}].\end{aligned}$$

We can also do it using $(3x^2+1)^{x^3} = e^{\ln(3x^2+1)^{x^3}} = e^{x^3\ln(3x^2+1)}$ and obtain the same result. ♠

Example 4.3.15 Find $\frac{d}{dx}\frac{(x^2+2)^2(x^4+4)^4(x^6+6)^6}{(x^8+8)^8(x^{10}+10)^{10}}$.

Solution. Due to experience, you should not try to take the derivative using the quotient and product rules because, for one thing, the product rule would be needed three times, which would be really messy. Instead, let's use Property 4.3.12. Let

$$g(x) = \frac{(x^2+2)^2(x^4+4)^4(x^6+6)^6}{(x^8+8)^8(x^{10}+10)^{10}},$$

then,

$$\begin{aligned}\ln g(x) &= \ln \frac{(x^2+2)^2(x^4+4)^4(x^6+6)^6}{(x^8+8)^8(x^{10}+10)^{10}} \\ &= 2\ln(x^2+2)+4\ln(x^4+4)+6\ln(x^6+6)-8\ln(x^8+8)-10\ln(x^{10}+10),\end{aligned}$$

and hence

$$\begin{aligned}&\frac{d}{dx}\ln g(x) \\ &= \frac{d}{dx}[2\ln(x^2+2)+4\ln(x^4+4)+6\ln(x^6+6)-8\ln(x^8+8)-10\ln(x^{10}+10)\,] \\ &= \frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}+\frac{36x^5}{x^6+6}-\frac{64x^7}{x^8+8}-\frac{100x^9}{x^{10}+10}.\end{aligned}$$

Therefore, using Property 4.3.12, we obtain

$$\begin{aligned}g'(x) &= g(x)\frac{d}{dx}\ln g(x) \\ &= g(x)[\frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}+\frac{36x^5}{x^6+6}-\frac{64x^7}{x^8+8}-\frac{100x^9}{x^{10}+10}].\end{aligned}$$

♠

Example 4.3.16 Find $\frac{d}{dx}3^x$.

Solution. Using Property 4.3.12, we obtain

$$\frac{d}{dx}3^x = 3^x\frac{d}{dx}\ln 3^x = 3^x\frac{d}{dx}(x\ln 3) = 3^x\ln 3.$$

♠

If the base 3 in Example 4.3.16 is replaced by a general base b ($b > 0$, $b \neq 1$), then the same will go through. We state it as

Property 4.3.17 *Let $b > 0$ and $b \neq 1$. Then*

$$\frac{d}{dx}b^x = b^x \ln b.$$

If $g(x)$ has derivatives, then

$$\frac{d}{dx}b^{g(x)} = b^{g(x)}(\ln b)g'(x).$$

♠

Note that when $b = e$, the formulas $(e^x)' = e^x$ and $\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$ become special cases of Property 4.3.17 because $\ln e = 1$. This also indicates that for all exponential functions with base b, we can treat b as e and follow the derivatives of e^x and $e^{g(x)}$, and then multiply by $\ln b$.

Remark 4.3.18 A common mistake here is to take the derivative of x^x or 3^x using the power rule. The power rule applies only to functions where the powers are constants, such as the 3 in x^3; it does not apply to other cases. For example, applying the power rule to x^x would give $xx^{x-1} = x^x$, a wrong answer. ♠

Example 4.3.19 Find $\frac{d}{dx}3^{x^3}$.

Solution. Using Property 4.3.17 we obtain

$$\frac{d}{dx}3^{x^3} = 3^{x^3}(\ln 3)\frac{d}{dx}x^3 = 3x^2 3^{x^3}\ln 3.$$

♠

Example 4.3.20 Find $\frac{d}{dx}3^{3^x}$.

Solution. Using Property 4.3.17 (twice) we obtain

$$\frac{d}{dx}3^{3^x} = 3^{3^x}(\ln 3)\frac{d}{dx}3^x = 3^{3^x}(\ln 3)3^x(\ln 3) = 3^{3^x}3^x(\ln 3)^2.$$

♠

We need to remark here that to apply logarithmic functions, a function must be positive. Also, it is beneficial to use logarithmic differentiation only when the derivatives are not easy to obtain directly, such as the above examples. Otherwise, just use the derivative rules to derive derivatives directly.

Finally, we show that the property of logarithmic differentiation can be used to prove the power rule.

Theorem 4.3.21 (Power Rule) *For any fixed real number α, the derivative of the power function x^α is given by*

$$\frac{d}{dx}x^\alpha = \alpha x^{\alpha-1} \tag{3.2}$$

if $x^{\alpha-1}$ results in a real value.

Verification. We assume $x > 0$ here. Using Property 4.3.12, we obtain

$$\begin{aligned}\frac{d}{dx}x^\alpha &= x^\alpha \frac{d}{dx}\ln x^\alpha = x^\alpha \frac{d}{dx}(\alpha \ln x)\\ &= x^\alpha \alpha \frac{1}{x} = \alpha x^{\alpha-1}.\end{aligned}$$

♠

Guided Practice 4.3

Find $f'(x)$ for the following functions $f(x)$.

1. $f(x) = \ln(4x + 1)$.

2. $f(x) = \log_5(7x)$.

3. $f(x) = 5^x$.

4. $f(x) = e^x \ln(2x)$.

5. $f(x) = \frac{2^x}{x^3+4}$.

6. $f(x) = \frac{\log_2(2x)}{2^x}$.

7. $f(x) = (x + 1)^{x^2}$.

Exercises 4.3

1. Find $f'(x)$ for

(a) $f(x) = x^3 \ln x$.
(b) $f(x) = x^3 \log_5 x$.
(c) $f(x) = \ln(\ln x)$.
(d) $f(x) = \ln(x + \ln x)$.
(e) $f(x) = \frac{x-\ln 3x}{\ln x - x}$.
(f) $f(x) = \frac{x-\log_6 3x}{\log x - x}$.
(g) $f(x) = \frac{e^x+\ln 3x}{x^2-2x}$.
(h) $f(x) = \frac{e^{-x}-4\ln 5x}{x^2+2x}$.
(i) $f(x) = \frac{e^x \ln 3x}{x^2-2x}$.
(j) $f(x) = \frac{e^x \log_8 3x}{x^4+4x}$.
(k) $f(x) = \ln(3x + e^{x^2} - 2)^7$.
(l) $f(x) = \log(3x + e^{x^2} - 2)^7$.
(m) $f(x) = x^2 \ln(3x + e^{x^2} - 2)^7$.
(n) $f(x) = 3x^4 \log(3x + e^{x^2} - 2)^7$.
(o) $f(x) = \ln x^6(3x + e^{x^2})^7$.
(p) $f(x) = \ln \frac{x^6}{(3x+e^{x^2})^7}$.

2. Find $f'(x)$ using logarithmic differentiation for

(a) $f(x) = 4^x$.
(b) $f(x) = 5^x$.
(c) $f(x) = (x^3 + x)^x,\ x > 0$.

(d) $f(x) = (x^3 + x)^{-x},\ x > 0.$

(e) $f(x) = (4x^2 + 3x)^{2x^3-x},\ x > 0.$

(f) $f(x) = (x^3 + x)^{-x^2-7x+8},\ x > 0.$

(g) $f(x) = 4^{4^x}.$

(h) $f(x) = 4^{x^4},\ x > 0.$

(i) $f(x) = 4^{x^x},\ x > 0.$

(j) $f(x) = x^{4^x},\ x > 0.$

(k) $f(x) = x^{x^4},\ x > 0.$

(l) $f(x) = x^{x^x},\ x > 0.$

(m) $f(x) = (x^2 + 1)^{10}(x^2 + 2)^{20}(x^2 + 3)^{30}(x^2 + 4)^{40}(x^2 + 5)^{50}.$

(n) $f(x) = \frac{(x^2+1)^{10}(x^2+2)^{20}(x^2+3)^{30}}{(x^2+4)^{40}(x^2+5)^{50}}.$

3. Verify the following property: If $f(x) > 0$ and $g(x)$ have derivatives, then

$$\frac{d}{dx}f(x)^{g(x)} = f(x)^{g(x)}[g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}].$$

Chapter 5

Derivative Tests and Applications

We will study two derivative tests: the first derivative test and the second derivative test. The first derivative test can be used to check for increasing or decreasing of functions and to find local or global maximum or minimum values of functions. The second derivative test can be used to check the concavities of functions and to find local or global maximum or minimum values of functions. Based on the information obtained using these derivative tests, we are able to sketch functions.

Then, we will learn how to apply these derivative tests to solve optimization problems in real world applications, such as minimizing the cost and maximizing the revenue or profit in business applications. The idea is to use the first or second derivative test to determine where the global minimum or global maximum values are achieved on the corresponding domains of the functions.

5.1 The First Derivative Test

In many applications, the independent variables of functions denote certain quantities that are increasing. For example, for the exponential growth/decay model $P(t) = P(0)e^{kt}$ ($P(0) > 0$) studied earlier, the independent variable is the *time* t that only goes **on** (increases). And as t is increased, $P(t) = P(0)e^{kt}$ is increasing if $k > 0$, and decreasing if $k < 0$. Thus, we generalize this idea and say that $P(t) = P(0)e^{kt}$ is an *increasing function* if $k > 0$, and say that $P(t) = P(0)e^{kt}$ is a *decreasing function* if $k < 0$.

In general, we make the following definition.

Definition 5.1.1 *A function $f(x)$ defined on an interval I is said to be an* **increasing function** *if for $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. $f(x)$ is said to be a* **decreasing function** *if for $x_1, x_2 \in I$, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.*

That is, we always let the independent variable increase and then look at the changes in the function values: if the function values also increase, then the function is increasing; if the function values decrease, then the function is decreasing. Some typical increasing and decreasing functions are shown in **Figure 5.1**.

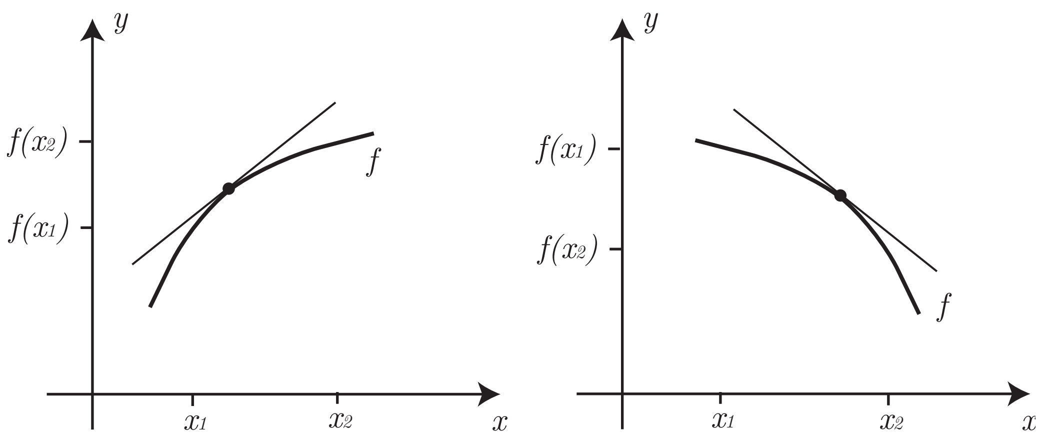

Figure 5.1: Some typical increasing and decreasing functions

According to Definition 5.1.1, we know that straight lines with positive slopes are increasing functions, and straight lines with negative slopes are decreasing functions.

Next, let's use **Figure 5.2** to introduce some notions. In Figure 5.2, $f(m_1)$ is the biggest value among *all* function values on the interval $[a, b)$, so we call $f(m_1)$ the **global maximum value** of $f(x)$. Similarly, $f(m_2)$ is called the **global minimum value** of $f(x)$. The value of $f(m_3)$ is not the biggest among all function values on $[a, b)$, but if you only consider those x values that are very close to m_3, or look at a very small piece of the curve around the point $(m_3,\ f(m_3))$, that is, if you consider $f(m_3)$ *locally*, then $f(m_3)$ is the biggest among *those* function values. Accordingly, we call $f(m_3)$ a **local maximum value** of $f(x)$. Similarly, $f(m_4)$ is called a **local minimum value** of $f(x)$. Since f is defined at $x = a$, we see from Figure 5.2 that $f(a)$ also gives a local minimum value. Since f is not defined at $x = b$, no local maximum value at b.

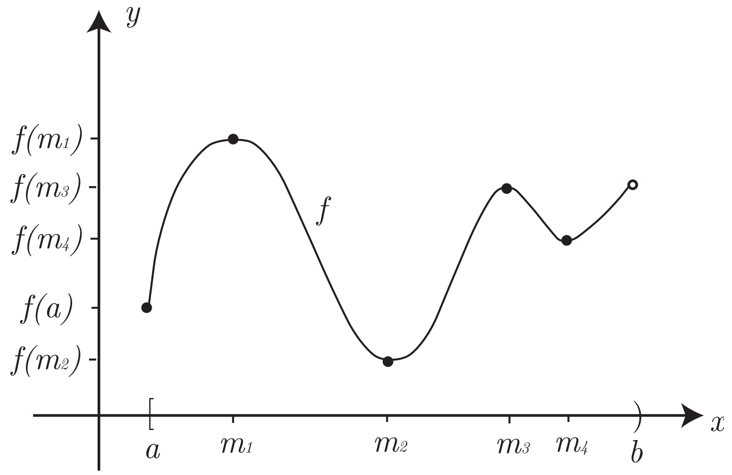

Figure 5.2: Local or global maximum or minimum values

Note that the global maximum (or global minimum) value can be obtained at more than one place, such as for a constant function where any function value is the global maximum value and also the global minimum value. Note also that the global maximum (or global minimum) value is also a local maximum (or local minimum) value, but a local maximum (or local minimum) value may or may not be the global maximum (or global minimum) value.

Sometimes, we use **extreme value** to denote either a local minimum value, a local maximum value, a global minimum value, or a global maximum value.

From Figure 5.1 and **Figure 5.3**, we find that if a function is such that its derivative is positive at every point on some interval, then the function should be increasing on that interval; if a function is such that its derivative is negative at every point on some interval, then the function should be decreasing on that interval. Also, if a function is such that its derivative is positive on the left-hand side of a point c and negative on the right-hand side of c, then $f(c)$ is a maximum value, at least locally. A similar argument about minimum values can also be made. That is, it indicates that the first derivative can be used to find increasing or decreasing and maximum or minimum values.

Now, we generalize these ideas as the following result, whose verification will be outlined at the end of this section.

Theorem 5.1.2 (The first derivative test) *Let $f(x)$ be a continuous function on its domain (a, b).*

1. *If $f'(x) > 0$ on (a, b), then $f(x)$ is increasing on (a, b).*

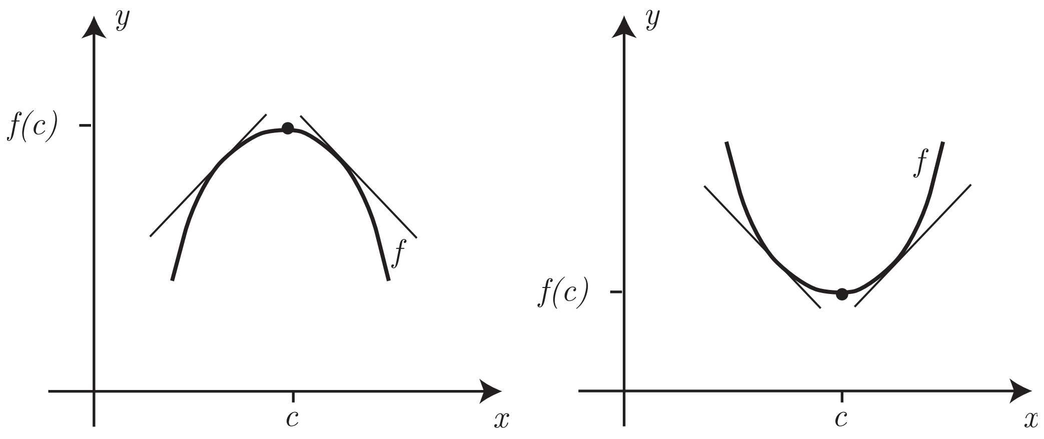

Figure 5.3: Maximum or minimum values using the first derivatives

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2. *If $f'(x) < 0$ on (a, b), then $f(x)$ is decreasing on (a, b).*

3. *Let c be a point in (a, b). If $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is the global maximum value. If these happen only near c, that is, if $f'(x) > 0$ for $x < c$ and near c, and $f'(x) < 0$ for $x > c$ and near c, then $f(c)$ is a local (maybe global) maximum value.*

4. *Let c be a point in (a, b). If $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is the global minimum value. If these happen only near c, that is, if $f'(x) < 0$ for $x < c$ and near c, and $f'(x) > 0$ for $x > c$ and near c, then $f(c)$ is a local (maybe global) minimum value.*

♠

Remark 5.1.3 Items 3 and 4 of the first derivative test says that if $f'(x)$ changes signs only at c, then $f(c)$ must be the global maximum or minimum value. However, if $f'(x)$ changes signs at a point c for x near c, then, since the information of $f(x)$ for x far away from c is unknown, we can only say that $f(c)$ is a local maximum or minimum value, which may or may not be the global maximum or minimum value, as can be inferred using Figure 5.3.

There are other cases that the first derivative test provides information about the global maximum or minimum values. For example, for a function $f(x)$ defined on $(-\infty, \infty)$, if its local maximum values occur at finite places, such as at $x = a, b$, and c, and if $\lim_{x\to\pm\infty} f(x) = -\infty$ (so that for large values of x, $f(x)$ do not contribute to the global maximum value), then we can compare $f(a), f(b)$, and $f(c)$ so that the biggest value will be the global maximum value. Similar things can also be done for the global minimum values.

These indicate that the first derivative test can be used to check for increasing or decreasing, and to find local or global maximum or minimum values. ♠

Corollary 5.1.4 *If $f'(x) > 0$ or $f'(x) < 0$ on (a, b), then $f(x)$ has an inverse function.* ♠

In Figure 5.2, the function f is defined at a, so that $f(a)$ also gives a local minimum value. In general, we have the following result concerning functions on closed intervals.

Corollary 5.1.5 (Closed interval test) *For a function defined on a closed interval, the function values at the end points must be used to compare so as to determine extreme values of the function on the interval.* ♠

Now we look at some examples, covering different types of functions.

Example 5.1.6 For $f(x) = x^2 - 2x - 8$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = 2x - 2 = 2(x - 1),$$

and the sign chart is given in **Figure 5.4**.

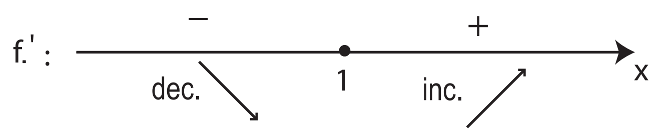

Figure 5.4: Sign chart of $f'(x)$ for $f(x) = x^2 - 2x - 8$

Now, $f(x)$ is decreasing for $x < 1$ and increasing for $x > 1$. Therefore, from the first derivative test (see Figure 5.4), we see that $f(1)$ must be the global minimum value.

Since $\lim_{x\to\pm\infty} f(x) = \infty$, we conclude that the function $f(x)$ has no local or global maximum values. ♠

In fact, the function $f(x) = x^2 - 2x - 8$ is a quadratic function, so its curve is a parabola that opens up because the coefficient of x^2, 1, is positive. Thus, the solution of Example 5.1.6 matches with the results of quadratic functions.

Example 5.1.7 For $f(x) = x^3 - 3x^2 + 5$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = 3x^2 - 6x = 3x(x-2),$$

and the sign chart is given in **Figure 5.5**.

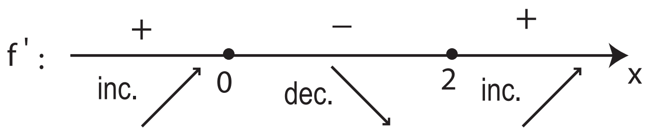

Figure 5.5: Sign chart of $f'(x)$ for $f(x) = x^3 - 3x^2 + 5$

Now, $f(x)$ is increasing for $x < 0$ or $x > 2$ and decreasing on $(0, 2)$. Therefore, from the first derivative test (see Figure 5.5), we see that $f(0)$ is a local maximum value and $f(2)$ is a local minimum value.

To determine whether these local extreme values are also global ones, we look at the limits: $\lim_{x\to\infty} f(x) = \infty$ (see Example 2.2.17) and $\lim_{x\to-\infty} f(x) = -\infty$, from which we see that the function $f(x)$ has no global maximum or global minimum values. ♠

Example 5.1.8 For $f(x) = (x-1)^3 + 6$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = 3(x-1)^2,$$

and the sign chart is given in **Figure 5.6**.

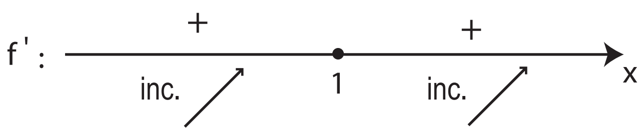

Figure 5.6: Sign chart of $f'(x)$ for $f(x) = (x-1)^3 + 6$

Now, $f(x)$ is increasing for $x < 1$ or $x > 1$ and $f(x)$ is continuous, therefore, from the first derivative test (see Figure 5.6) and the continuity, we see that $f(x)$ is increasing on $(-\infty, \infty)$, and hence has no maximum or minimum values. (Note that if $f(x)$ is not continuous, then it may not be increasing on $(-\infty, \infty)$, see Figure 5.45). ♠

Example 5.1.9 For $f(x) = 3 - (x-1)^4$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = -4(x-1)^3,$$

and the sign chart is given in **Figure 5.7**.

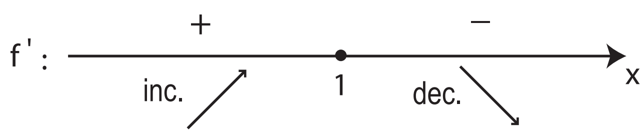

Figure 5.7: Sign chart of $f'(x)$ for $f(x) = 3 - (x-1)^4$

Now, $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$. Therefore, from the first derivative test (see Figure 5.7), we see that $f(1)$ must be the global maximum value.

Since $\lim_{x\to\pm\infty} f(x) = -\infty$, the function $f(x)$ has no local or global minimum values. ♠

Example 5.1.10 For $f(x) = x - \ln x,\ x > 0$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x},$$

and the sign chart is given in **Figure 5.8**.

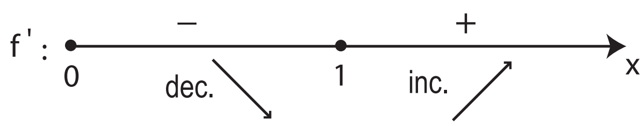

Figure 5.8: Sign chart of $f'(x)$ for $f(x) = x - \ln x$

Now, $f(x)$ is decreasing on $(0, 1)$ and increasing for $x > 1$. Therefore, from the first derivative test (see Figure 5.8), we see that $f(1)$ must be the global minimum value.

Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x - \ln x) = 0 - (-\infty) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$ (for example, for $x = e^{10}$, $x - \ln x = e^{10} - \ln e^{10} = e^{10} - 10$, a very big number, indicating why the limit is infinity), we know that the function $f(x)$ has no local or global maximum values.

Note that in this case, the global minimum value is $f(1) = 1 - \ln 1 = 1 - 0 = 1 > 0$, indicating that $x - \ln x > 0$, or the straight line $y = x$ is above the curve of $\ln x$, which can be seen from geometry. ♠

Example 5.1.11 For $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = \frac{x + 1 - (x - 1)}{(x + 1)^2} = \frac{2}{(x + 1)^2},$$

and the sign chart is given in **Figure 5.9**.

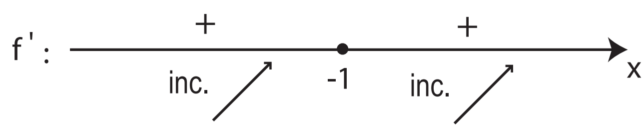

Figure 5.9: Sign chart of $f'(x)$ for $f(x) = \frac{x-1}{x+1}$

Now, $f(x)$ is increasing for $x < -1$ or $x > -1$, thus, $f(x)$ has no local or global maximum or minimum values. Note that $f(x)$ is not defined at $x = -1$, so it is not continuous there, hence the function may not be increasing on $(-\infty, \infty)$, see Figure 5.45. ♠

Example 5.1.12 For $f(x) = \frac{5}{1+2e^{-x}}$, $x \geq 0$, find where it is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = \frac{d}{dx}5(1 + 2e^{-x})^{-1} = -5(1 + 2e^{-x})^{-2}2e^{-x}(-1) = \frac{10e^{-x}}{(1 + 2e^{-x})^2},$$

and the sign chart is given in **Figure 5.10**.

Now, $f(x)$ is increasing for $x \geq 0$. Since $f(x)$ is defined at the left end point of the domain $[0, \infty)$, we see (use Corollary 5.1.5) that $f(0)$ must be the global minimum value, and $f(x)$ has no other local minimum values or maximum values. ♠

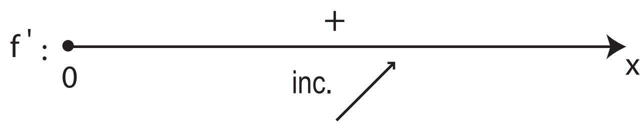

Figure 5.10: Sign chart of $f'(x)$ for $f(x) = \frac{5}{1+2e^{-x}}$

Verification of the first derivative test (Theorem 5.1.2).

Before we verify the first derivative test, let's look at **Figure 5.11**, where a function f is defined on $[a, b]$, and L denotes the straight line passing through the points $(a, f(a))$ and $(b, f(b))$, so the slope of L is $\frac{f(b)-f(a)}{b-a}$.

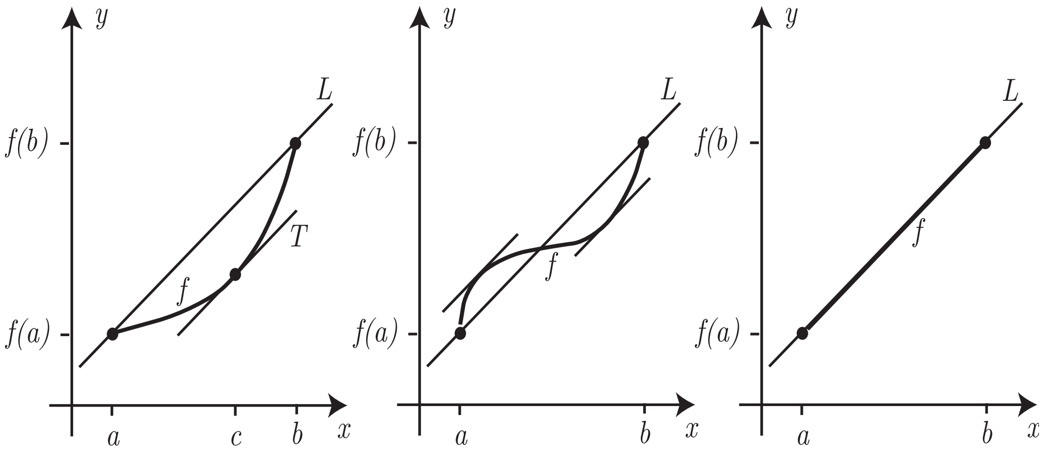

Figure 5.11: The geometry of the mean value theorem

In the first picture, we can find one place, c, where the tangent line, T, is parallel to L, so that $f'(c) = \frac{f(b)-f(a)}{b-a}$. In the second picture, we find two places where the tangent lines are parallel to L. In the third picture, the function is a straight segment, so that the tangent line at every point is parallel to L. Based on these, we infer the following result.

Theorem 5.1.13 (The mean value theorem) *Let f be continuous on $[a, b]$ and be differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

or

$$f(b) - f(a) = f'(c)(b - a).$$

♠

This theorem is called the *mean value* theorem since $\frac{f(b)-f(a)}{b-a}$ gives the *average* (also called *mean value* sometimes) velocity when f denotes the position of a moving object.

Now, to verify the first derivative test (Theorem 5.1.2), we let $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Applying the mean value theorem to the function f on the interval $[x_1, x_2]$, we get a point $c \in (x_1, x_2) \subset (a,b)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Note that $x_2 - x_1$ is positive, so that if $f'(x) > 0$ on (a,b), then $f'(c) > 0$ and hence $f(x_1) < f(x_2)$. If $f'(x) < 0$ on (a,b), then $f'(c) < 0$ and hence $f(x_1) > f(x_2)$. This verifies parts 1 and 2 of the first derivative test (Theorem 5.1.2), and other parts are their consequences. ♠

Guided Practice 5.1

For the following, find where a function is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

1. $f(x) = x^2 + 4x + 5$.

2. $f(x) = \frac{2x}{x+3}$.

3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 1$.

4. $f(x) = e^{-\frac{x^2}{2}}$.

Exercises 5.1

1. Find a value a and a value b such that b^{ax} is an increasing function. If it is given that b^{ax} is an increasing function, then what are the restrictions on a and b?

2. Find a value a and a value b such that b^{ax} is a decreasing function. If it is given that b^{ax} is a decreasing function, then what are the restrictions on a and b?

3. For the following, find where a function is increasing or decreasing. Then find where it takes local or global maximum or minimum values.

(a) $f(x) = x^2 + 2x - 3$.

(b) $f(x) = x^2 - 2x + 3$.

(c) $f(x) = x^3 - 4.5x^2 + 6x + 8$.

(d) $f(x) = -2x^3 + 9x^2 - 12x + 15$.

(e) $f(x) = 4(x - 3)^3 + 9$.

(f) $f(x) = 3(x - 2)^3 - 7$.

(g) $f(x) = 4(x - 1)^4 - 3$.

(h) $f(x) = 3(x - 2)^4 + 5$.

(i) $f(x) = e^x$.

(j) $f(x) = \ln x,\ (x > 0)$.

(k) $f(x) = e^x - x$.

(l) $f(x) = e^x + x$.

(m) $f(x) = \frac{x+1}{x-1},\ (x \neq 1)$.

(n) $f(x) = \frac{x+1}{x+2},\ (x \neq -2)$.

(o) $f(x) = \frac{x}{(x+1)^2},\ (x \neq -1)$.

(p) $f(x) = \frac{x}{(x-1)^2},\ (x \neq -1)$.

(q) $f(x) = \frac{\ln x}{x},\ x \in (0,\, e^{10}]$.

(r) $f(x) = \frac{x}{\ln x},\ x \in (1,\, e^{10}]$.

5.2 The Second Derivative Test

Paul's story: On one of our many childhood snow days my brother discovered that he could pack one of our enormous 55 gallon trash cans full of snow, flip it over gently, remove the trash can and create a 55 gallon tightly packed mound of snow, which he termed a *snow castle*. He also came up with the idea of putting snow castles on the roof of our house with the snow already up there. We would place several 55 gallon snow castles on the roof on the overhang above the front door and my brother would hide in the front yard and give me a *go* signal indicating that a brother or sister was in range. Then I would shove one snow castle off the roof, which would flatten the sibling to the sidewalk and then fire a second one to bury the sibling. Looking down I would happily see only feet protruding from a pile of snow. One time I heard the front door open and my brother quickly gave me the go and with a big grin on my face I fired the first snow castle and then the second, but oddly I didn't hear the familiar thump. I promptly looked across the yard to see my brother lying beside the tree laughing hysterically. Then I heard the horrifying sound that I guess most would equate to that of a trapped bear, but that I sickeningly recognized as an enraged father. He had been late heading into the office that morning. "*ARRRGGHHH! What in the World?! Who's Up There?! Get DOWN Here, NOW!*" My brother informed me later that in an instant the rosy happy color in my face changed to winter white, see **Figure 5.12**.

The crucial point of this story is that at the moment I realized I had hit dad, and not a sibling, my smile instantly turned into a frown. I was joyfully happy until that moment and miserably sad for a long time afterwards. Graphs of functions can be broken into pieces where they are *smiling*, *frowning*, or neither *smiling* nor *frowning*. We will attach the idea of a graph *smiling* to the concept of the graph being **concave up**, and attach the idea of a graph *frowning* to the concept of a

Figure 5.12: Dumping Snow on Dad

graph being **concave down**. That point when my *smiling* turned into a *frowning* we will relate to the notion of an **inflection point**. In **Figure 5.13**, we can easily visually distinguish when the curve of the mouth is concave up, when it is concave down, and the inflection point where the change in concavity occurs.

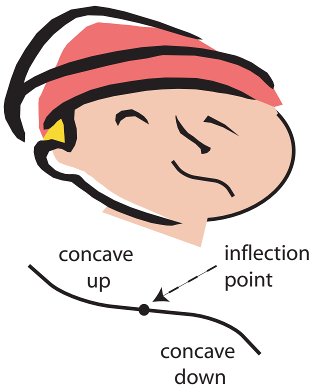

Figure 5.13: A *Smile* Turning into a *Frown*

The first face in **Figure 5.14** is to represent my face after dumping snow on a brother or sister. I am always smiling and using our simple analogy the curve of my mouth here is always concave up. The second face shows my dread waiting for my father to return from work and deliver my punishment. I of course had put on multiple pairs of underwear to use as padding in preparation for my rear to meet the *board of correction*. Clearly here I am constantly frowning and again following our analogy the curve of my mouth is always concave down. The third face represents my realization that somehow during the course of a busy day my father had

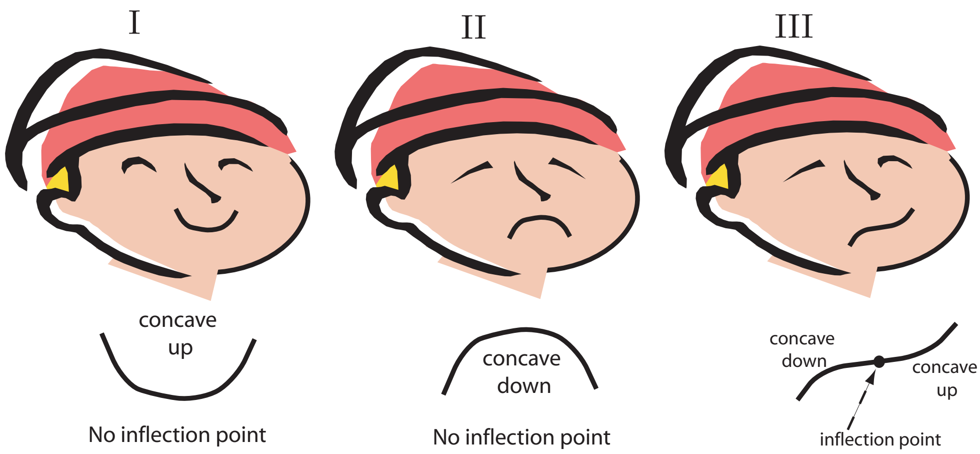

Figure 5.14: Three Other *Moments* That Day

actually managed to forget that he intended to punish me. Here the curve of my mouth starts out concave down and then changes to concave up. Again the location on the curve where the curve changes from being concave down to concave up is the inflection point.

Given the graph of a curve it is a fairly simple task to estimate where the curve is concave up, concave down, and the location of the inflection points, see **Figure 5.15**. Note that the part of the curve that is neither concave up nor concave down has the shape of a straight line, this is typical for a curve that it neither concave up or concave down. The inflection points are located at the points A,B,C, as the curve changes concavity, frowning to smiling or smiling to frowning, across these points. We will determine later that the points D, E, and F do **NOT** meet the criteria to be inflection points, as the curve doesn't change from frowning to smiling or smiling to frowning across these points.

Having an intuitive graphical feel for concavity is, however, not sufficient for applications. One reason is that up to this point we have just eyeballed the actual locations of the inflection points. Next we show how the second derivative can be used to determine these exactly, and then proceed to show how concavity can be helpful in determining the important values that are maximum values and minimum values of functions. **End of the story**.

Now, we start by introducing a notion. Look at **Figure 5.16**, where both functions are increasing, but they are different.

Question: *How do we describe the differences between the functions in Figure 5.16?*

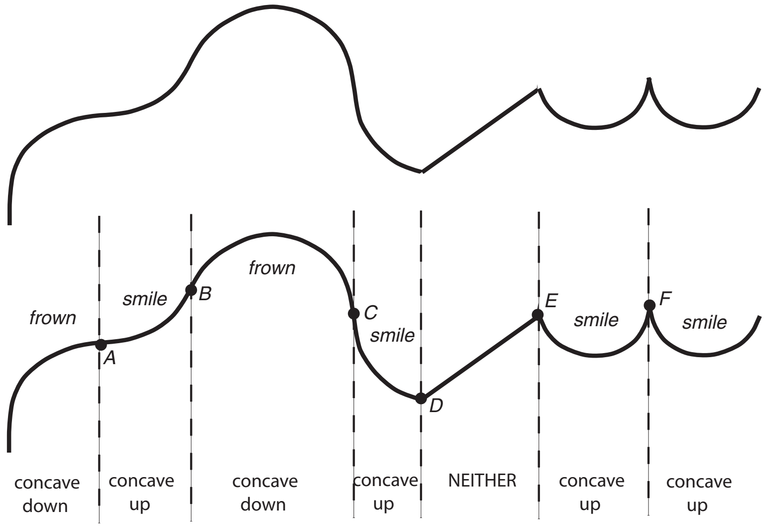

Figure 5.15: Dividing a Curve by Concavity

For this purpose, let's look at the tangent lines of the curve given in **Figure 5.17**. You will see that the curve is always *below* its tangent lines. In this sense, we say that the curve in Figure 5.17 is **concave down**. Similarly, the curve in **Figure 5.18** is always *above* its tangent lines, so we say that it is **concave up**.

In **Figure 5.19**, the curve of the function $f(x)$ changes its concavity at $x = c$. In this sense, we say that c is an **inflection point**.

Another way to understand a concave down curve is to look at **Figure 5.20**, where a few slopes (m values) are shown to indicate that as x is increased, the slope $f'(x)$ of the tangent line of $f(x)$ is decreasing.

That is, if the slope f' is decreasing, then the curve of the function f would be below its tangent lines, or concave down. Now, if we let $g(x) = f'(x)$, then from the first derivative test, if $g' < 0$, then g is decreasing. Going back from g to f, we see that if $(f')' < 0$ (that is, $f'' < 0$), then f' is decreasing and hence the function f would be concave down. Summarizing, it seems that $f'' < 0$ implies that the function is concave down.

Similarly, for the curve of the function given in **Figure 5.21**, we see that as x is increased, the slope $f'(x)$ is increasing. Using a similar argument as above, we see that it happens when $f'' > 0$. That is, it seems that $f'' > 0$ implies concave up.

Next, let's use these ideas and figures to determine when maximum or minimum values can be achieved. If we assume $f''(c) < 0$ at some point c and assume that $f''(x)$ is continuous at c, then it is true (not to be proved here) that $f''(x) < 0$ for x near c, so that $f(x)$ will be concave down for x near c. Now, if we also assume $f'(c) = 0$, then, from **Figure 5.22**, the

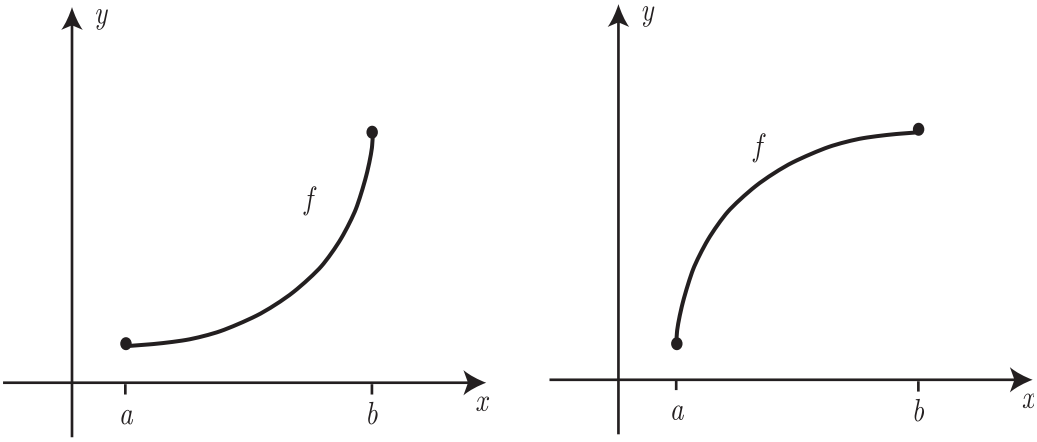

Figure 5.16: Two different increasing functions

slope of the tangent line at c is zero (or the tangent line is a horizontal line), which indicates in geometry that $f(c)$ is the biggest value, at least locally. That is, in this case, $f(c)$ looks like a local (maybe global) maximum value.

Similarly, if we assume $f''(c) > 0$ and $f'(c) = 0$, then, we get **Figure 5.23**, which indicates that $f(c)$ would be a local (maybe global) minimum value.

Question: *What can we conclude about $f(c)$ if $f'(c) = 0$ and $f''(c) = 0$?*

To answer this question, let's look at some examples.

Example 5.2.1 For $f(x) = x^4$, find $f'(0)$ and $f''(0)$, and examine $f(x)$ at $x = 0$.

Solution. We have

$$f'(x) = 4x^3, \quad f''(x) = 12x^2,$$

thus, at $c = 0$, we get

$$f'(0) = 0, \quad f''(0) = 0.$$

Since $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, and $f(x) = x^4$ is symmetric about $x = 0$, we conclude that $f(0) = 0$ is the global minimum value of $f(x)$, which can be seen from the first graph in **Figure 5.24**. ♠

Example 5.2.2 For $f(x) = -x^4$, find $f'(0)$ and $f''(0)$, and examine $f(x)$ at $x = 0$.

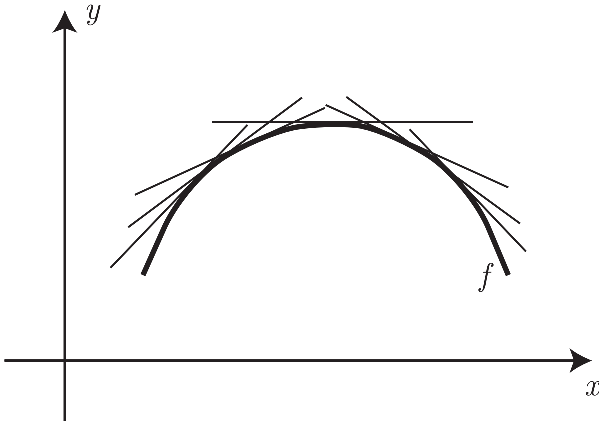

Figure 5.17: A concave down curve

Solution. We have

$$f'(x) = -4x^3, \quad f''(x) = -12x^2,$$

thus,

$$f'(0) = 0, \quad f''(0) = 0.$$

Since $f(0) = 0$, $f(x) < 0$ for $x \neq 0$, and $f(x) = -x^4$ is symmetric about $x = 0$, we conclude that $f(0) = 0$ is the global maximum value of $f(x)$, which can be seen from the second graph in **Figure 5.24**. ♠

Example 5.2.3 For $f(x) = x^3$, find $f'(0)$ and $f''(0)$, and examine $f(x)$ at $x = 0$.

Solution. We have

$$f'(x) = 3x^2, \quad f''(x) = 6x,$$

thus,

$$f'(0) = 0, \quad f''(0) = 0.$$

Since $f(0) = 0$, $f(x) < 0$ for $x < 0$, $f(x) > 0$ for $x > 0$, and $f(x) = x^3$ is anti-symmetric about $x = 0$, we conclude that $f(0) = 0$ is neither a local maximum value nor a local minimum value of $f(x)$, which can be seen from the third graph in **Figure 5.24**. ♠

We now see from examples 5.2.1 – 5.2.3 that under the condition $f'(0) = f''(0) = 0$, all different things can happen for $f(0)$, so that no *general* conclusions can be made. For example, the conclusion that $f(0) = 0$ is the global minimum value made in Example 5.2.1 is only valid for the function in Example 5.2.1, not for the other functions in examples 5.2.2 – 5.2.3.

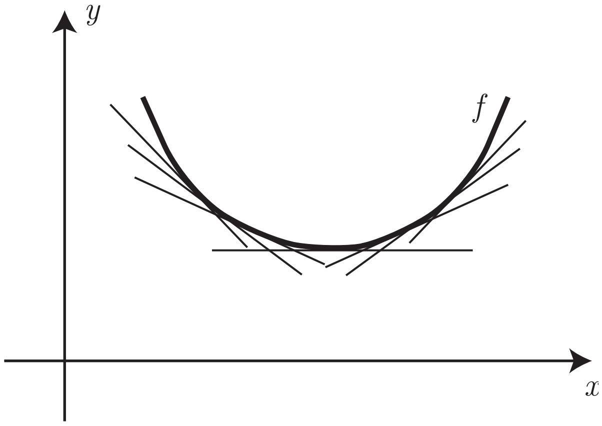

Figure 5.18: A concave up curve

Summarizing what we have seen, we find that the second derivative can be used to check concavities and maximum or minimum values. We state these as follows, whose verification will be given at the end of this section.

Theorem 5.2.4 (The second derivative test) *Let $f(x)$ be a continuous function on its domain (a, b).*

1. *If $f''(x) < 0$ on (a, b), then $f(x)$ is concave down on (a, b).*

2. *If $f''(x) > 0$ on (a, b), then $f(x)$ is concave up on (a, b).*

3. *Assume that $f''(x)$ is continuous on (a, b) and let c be a point in (a, b). If $f'(c) = 0$ and $f''(x) < 0$ on (a, b), then $f(c)$ is the global maximum value. If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local (maybe global) maximum value.*

4. *Assume that $f''(x)$ is continuous on (a, b) and let c be a point in (a, b). If $f'(c) = 0$ and $f''(x) > 0$ on (a, b), then $f(c)$ is the global minimum value. If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local (maybe global) minimum value.*

5. *If $f'(c) = 0$ and $f''(c) = 0$, then no general conclusions can be made about $f(c)$.*

♠

The item 5 of Theorem 5.2.4 means that if $f'(c) = 0$ and $f''(c) = 0$, then we cannot say "$f(c)$ must be such and such". Instead, we analyze $f(x)$ and get a conclusion that is only valid for the given function $f(x)$. Examples 5.2.1 – 5.2.3 are such cases.

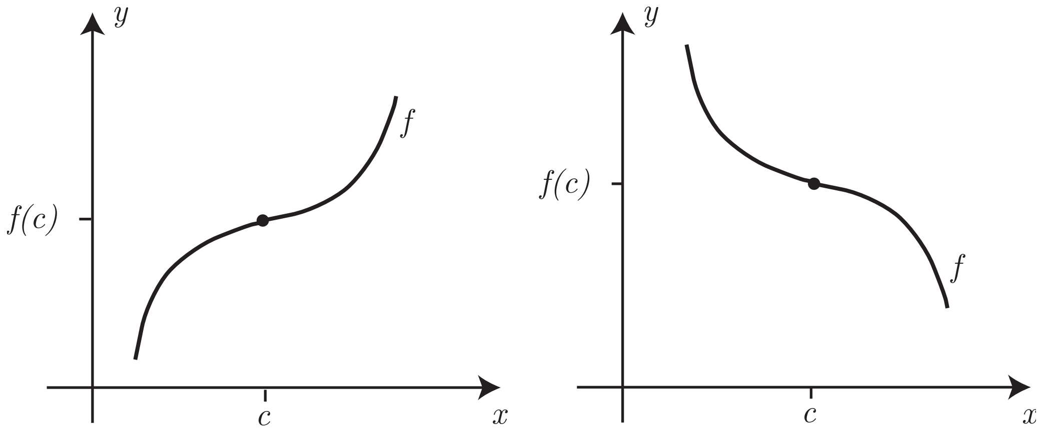

Figure 5.19: An inflection point

For the items 3 and 4, a remark similar to Remark 5.1.3 can also be made here. That is, if the concavity is known on the entire domain, then we can conclude about the global maximum or global minimum values. However, if the concavity is known only for x near a point c, then, since the information of $f(x)$ for x far away from c is unknown, we can only conclude about local maximum or local minimum value, which may or may not be the global maximum or global minimum value, as can be inferred using figures 5.22 and Figure 5.23. Also, if local extreme values occur at finite places, then the global maximum (or global minimum) value may be obtained by comparing these local extreme values. On closed intervals, the function values at the end points must also be considered.

These indicate that the second derivative test can be used to check for concavities, and to find local or global maximum or minimum values.

Now, let's revisit those examples of Section 5.1, and see how to apply the second derivative test to check concavities and determine extreme values.

Example 5.2.5 For $f(x) = x^2 - 2x - 8$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$\begin{aligned} f'(x) &= 2x - 2 = 2(x-1), \\ f''(x) &= 2, \end{aligned}$$

and the sign chart is given in **Figure 5.25**.

Now, $f(x)$ is concave up on $(-\infty, \infty)$ and hence has no inflection points. To find maximum/minimum values, we look at places with $f'(x) = 0$, and

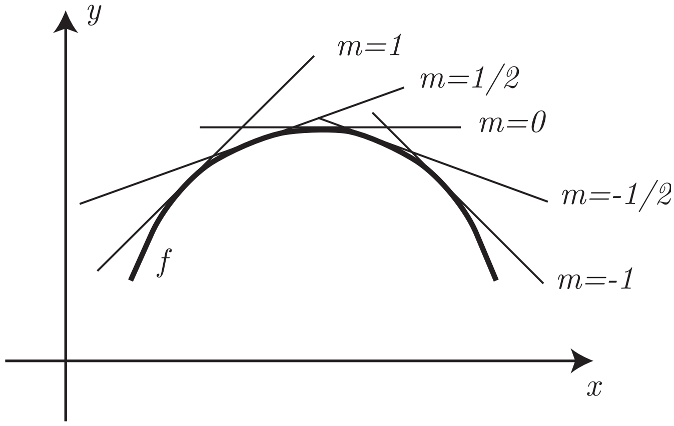

Figure 5.20: Changes in slopes for a concave down curve

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we have $x = 1$. Hence we see from the second derivative test (see Figure 5.25) that $f(1)$ must be the global minimum value. Then, taking limits in the same way as in Example 5.1.6, we know that the function $f(x)$ has no local or global maximum values. ♠

Example 5.2.6 For $f(x) = x^3 - 3x^2 + 5$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$\begin{aligned} f'(x) &= 3x^2 - 6x = 3x(x-2), \\ f''(x) &= 6x - 6 = 6(x-1), \end{aligned}$$

and the sign chart is given in **Figure 5.26**.

Now, $f(x)$ is concave up for $x > 1$ and concave down for $x < 1$ and hence $x = 1$ is the only inflection point. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we have $x = 0, 2$. For $x = 0$, since $f'(0) = 0$ and $f''(0) < 0$, we see from the second derivative test (see Figure 5.26) that $f(0)$ is a local maximum value. For $x = 2$, as $f'(2) = 0$ and $f''(2) > 0$, we see from the second derivative test that $f(2)$ is a local minimum value.

Then, taking limits in the same way as in Example 5.1.7, we know that the function $f(x)$ has no global maximum or minimum values. ♠

Comparing Examples 5.1.7 and 5.2.6, we find that for most polynomial functions, the second derivative test is easier to use because derivatives on polynomial functions result in simpler forms and then all we need to do is to evaluate the second derivative at a few x values, whereas to use the first derivative test, the sign chart of $f'(x)$ must be made.

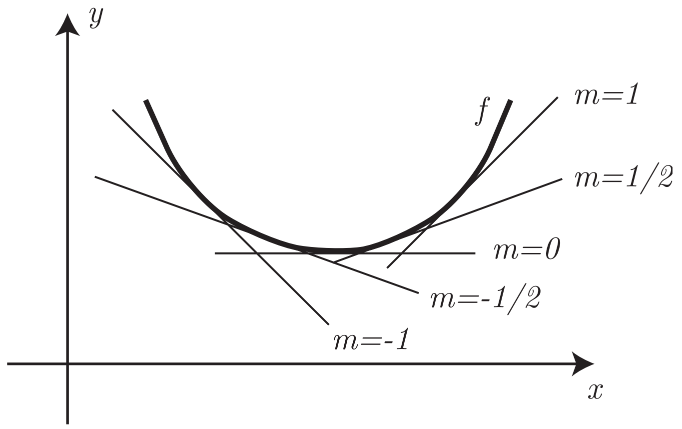

Figure 5.21: Changes in slopes for a concave up curve

Example 5.2.7 For $f(x) = (x-1)^3 + 6$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$\begin{aligned} f'(x) &= 3(x-1)^2, \\ f''(x) &= 6(x-1), \end{aligned}$$

and the sign chart is given in **Figure 5.27**.

Now, $f(x)$ is concave up for $x > 1$ and concave down for $x < 1$ and hence $x = 1$ is the only inflection point. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we have $x = 1$. Note that $f'(1) = 0$ and $f''(1) = 6(1-1) = 0$, so the second derivative test is inconclusive about $f(1)$. In this case, we use the first derivative test to conclude (as did in Example 5.1.8) that $f(x)$ is increasing on $(-\infty, \infty)$ and hence has no maximum or minimum values. ♠

In fact, the second derivative test can still be used, but it is somewhat tedious in this case. For example, we can argue that since $f(x)$ is concave up for $x > 1$ and $\lim_{x\to\infty} f(x) = \infty$, $f(x)$ must be increasing for $x > 1$. Otherwise, $f(x)$ would decrease from $x = 1$ and then increase (as $\lim_{x\to\infty} f(x) = \infty$). Then we would get a place other than $x = 1$ such that the derivative of $f(x)$ is zero. But $f'(x) = 3(x-1)^2 = 0$ only at $x = 1$. Similarly, we can argue that $f(x)$ must be increasing for $x < 1$, so we can achieve the same conclusion as above.

Remark 5.2.8 The second derivative test can become inconclusive sometimes, then try the first derivative test in those cases, as indicated by Example 5.2.7. ♠

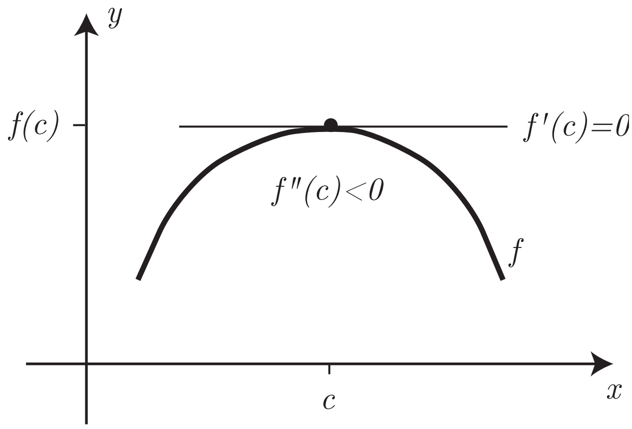

Figure 5.22: A maximum value

Example 5.2.9 For $f(x) = 3 - (x-1)^4$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$\begin{aligned} f'(x) &= -4(x-1)^3, \\ f''(x) &= -12(x-1)^2, \end{aligned}$$

and the sign chart is given in **Figure 5.28**.

Now, $f(x)$ is concave down for $x < 1$ or $x > 1$ and hence has no inflection points. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we have $x = 1$. Note that $f'(1) = 0$ and $f''(1) = 0$, so the second derivative test is inconclusive about $f(1)$. In this case, we use the first derivative test to conclude (as did in Example 5.1.9) that $f(1)$ is the global maximum value. Then, taking limits in the same way as in Example 5.1.9, we know that the function $f(x)$ has no local or global minimum values.

The second derivative test can also be used in the following way: the tangent line at $x = 1$ is $y = 3$, and since $f(x) = 3 - (x-1)^4 < 3$ for $x \neq 1$, we see that the curve of $f(x)$ is below its tangent line also at $x = 1$, so $f(x)$ is concave down on $(-\infty, \infty)$. Hence we see from the second derivative test that $f(1)$ is the global maximum value. ♠

Example 5.2.10 For $f(x) = x - \ln x,\ x > 0$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x},$$

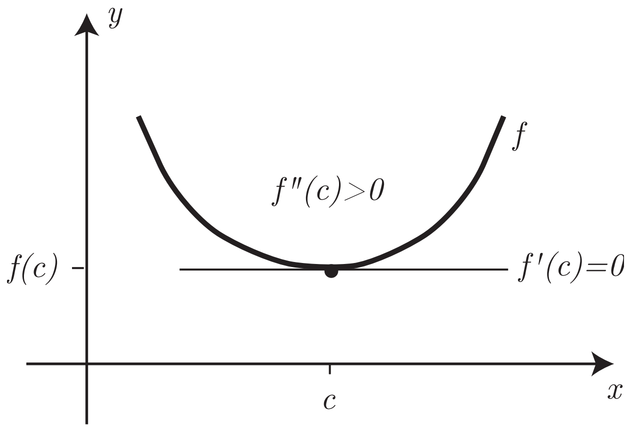

Figure 5.23: A minimum value

$$f''(x) \quad = \quad \frac{d}{dx}[-x^{-1}] = x^{-2} = \frac{1}{x^2},$$

and the sign chart is given in **Figure 5.29**.

Now, $f(x)$ is concave up on $(0, \infty)$ and hence has no inflection points. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we have $x = 1$. Then we see from the second derivative test that $f(1)$ must be the global minimum value.

Next, taking limits as we did in Example 5.1.10, we see that $f(x)$ has no local or global maximum values. ♠

Example 5.2.11 For $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$\begin{aligned} f'(x) &= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}, \\ f''(x) &= \frac{d}{dx}[2(x+1)^{-2}] = -4(x+1)^{-3} = \frac{-4}{(x+1)^3}, \end{aligned}$$

and the sign chart is given in **Figure 5.30**.

Now, $f(x)$ is concave up for $x < -1$ and concave down for $x > -1$, and $f(x)$ has no inflection points since $f(x)$ is not defined at $x = -1$. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we don't have solutions. In this case, the second derivative test is not easy to use, so let's use the first derivative test. Then, as $f(x)$ is increasing for $x < -1$ or $x > -1$, the function $f(x)$ has no local or global maximum or minimum values. ♠

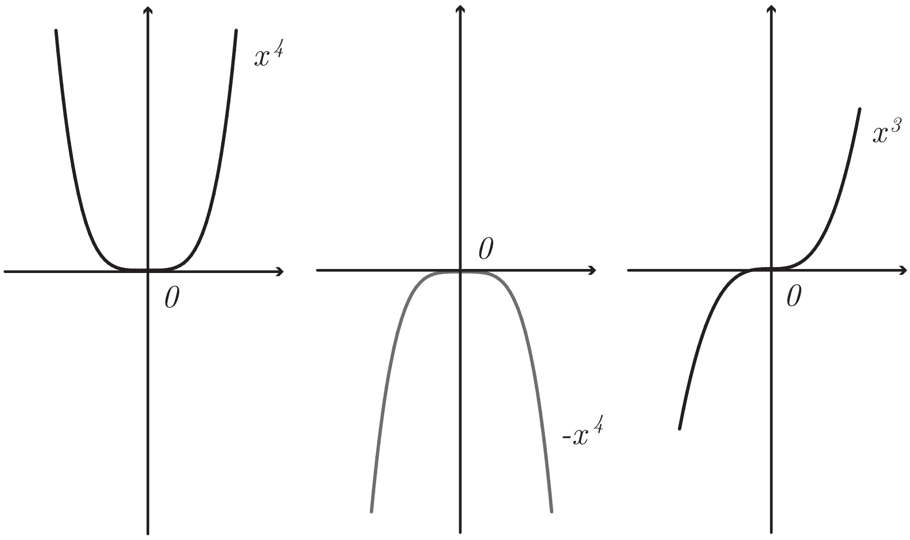

Figure 5.24: Three cases for $f'(0) = f''(0) = 0$

Figure 5.25: Sign chart of $f''(x)$ for $f(x) = x^2 - 2x - 8$

Example 5.2.12 For $f(x) = \frac{5}{1+2e^{-x}}$, $x \geq 0$, find where it is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

Solution. We have

$$f'(x) = \frac{d}{dx}5(1+2e^{-x})^{-1} = -5(1+2e^{-x})^{-2}2e^{-x}(-1) = \frac{10e^{-x}}{(1+2e^{-x})^2},$$
$$f''(x) = 10\frac{-e^{-x}(1+2e^{-x})^2 + e^{-x}2(1+2e^{-x})2e^{-x}}{(1+2e^{-x})^4} = \frac{20e^{-x}(e^{-x}-\frac{1}{2})}{(1+2e^{-x})^3},$$

and the sign chart is given in **Figure 5.31**.

Now, $f(x)$ is concave up on $[0, \ln 2)$ and concave down for $x > \ln 2$ and hence $x = \ln 2$ is the only inflection point. To find maximum/minimum values, we look at places with $f'(x) = 0$, and we don't have solutions. In this case, the second derivative test is not easy to use, so let's use the first derivative test. Then, as $f(x)$ is increasing for $x \geq 0$ and $f(x)$ is defined at $x = 0$, the value $f(0)$ must be the global minimum value and $f(x)$ has no

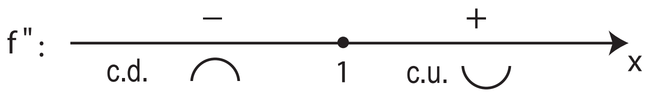

Figure 5.26: Sign chart of $f''(x)$ for $f(x) = x^3 - 3x^2 + 5$

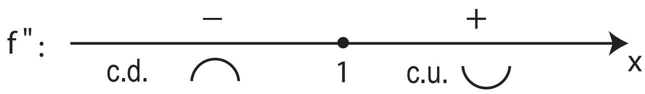

Figure 5.27: Sign chart of $f''(x)$ for $f(x) = (x-1)^3 + 6$

other local minimum values or local or global maximum values. ♠

These examples and the examples in Section 5.1 indicate that when asked for increasing or decreasing, use the first derivative test; when asked for concave up or concave down, use the second derivative test; and when asked for maximum or minimum values, use either the first or the second derivative test. That is, be flexible, if a derivative test is not easy to work with, then try another derivative test. When using the second derivative test, keep in mind that it may be inconclusive sometimes.

Verification of the second derivative test (Theorem 5.2.4).

We will only verify part 1 because others will be similar. Now, we know that $f''(x) < 0$ on (a, b), and we need to verify that for any point $c \in (a, b)$, the curve of f is below the tangent line, T, at $x = c$.

We will use *proof by contradiction*. If the curve of f is not below the tangent line T, then on the right-hand side of c (the case on the left-hand side is similar), there exists a point c_1 such that $f(c_1)$ is on or above the tangent line T. See **Figure 5.32**.

Applying the mean value theorem to f on the interval $[c, c_1]$, we get a point $c^* \in (c, c_1)$ such that $f'(c^*) \geq f'(c)$ (the slope is equal to or bigger than the slope of the tangent line T). But, from the first derivative test (Theorem 5.1.2), $(f'(x))' = f''(x) < 0$ implies that $f'(x)$ is decreasing, so that $f'(c^*) < f'(c)$ since $c < c^*$. This is a contradiction, thus, the curve of f must be below the tangent line T, that is, $f(x)$ is concave down. ♠

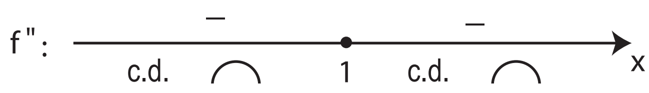

Figure 5.28: Sign chart of $f''(x)$ for $f(x) = 3 - (x-1)^4$

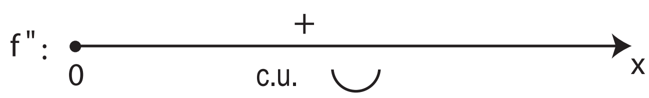

Figure 5.29: Sign chart of $f''(x)$ for $f(x) = x - \ln x$

Guided Practice 5.2

For the following, find where the function is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

1. $f(x) = x^2 + 4x + 5$.

2. $f(x) = \frac{2x}{x+3}$.

3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 1$.

4. $f(x) = e^{-\frac{x^2}{2}}$.

Exercises 5.2

1. How do you describe the differences between the functions in Figure 5.16?

2. For the following, find where the function is concave up or concave down, and find any inflection points. Then find where it takes local or global maximum or minimum values.

(a) $f(x) = x^2 + 2x - 3$.

(b) $f(x) = x^2 - 2x + 3$.

(c) $f(x) = x^3 - 4.5x^2 + 6x + 8$.

(d) $f(x) = -2x^3 + 9x^2 - 12x + 15$.

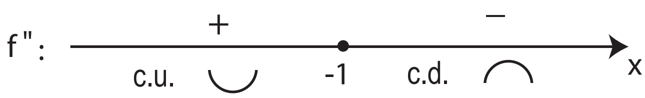

Figure 5.30: Sign chart of $f''(x)$ for $f(x) = \frac{x-1}{x+1}$

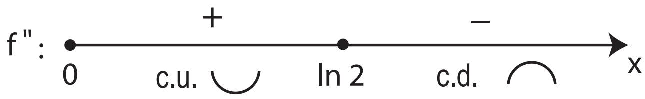

Figure 5.31: Sign chart of $f''(x)$ for $f(x) = \frac{5}{1+2e^{-x}}$

(e) $f(x) = 4(x-3)^3 + 9$.

(f) $f(x) = 3(x-2)^3 - 7$.

(g) $f(x) = 4(x-1)^4 - 3$.

(h) $f(x) = 3(x-2)^4 + 5$.

(i) $f(x) = e^x$.

(j) $f(x) = \ln x,\ (x > 0)$.

(k) $f(x) = e^x - x$.

(l) $f(x) = e^x + x$.

(m) $f(x) = \frac{x+1}{x-1},\ (x \neq 1)$.

(n) $f(x) = \frac{x+1}{x+2},\ (x \neq -2)$.

(o) $f(x) = \frac{x}{(x+1)^2},\ (x \neq -1)$.

(p) $f(x) = \frac{x}{(x-1)^2},\ (x \neq 1)$.

(q) $f(x) = \frac{\ln x}{x},\ x \in (0, e^{10}]$.

(r) $f(x) = \frac{x}{\ln x},\ x \in (1, e^{10}]$.

5.3 Curve Sketching

In the previous two sections, we have seen how to use derivatives to determine increasing or decreasing, concave up or concave down, and maximum or minimum values. Here, we will learn how to "picture these notions", that is, how to *visualize* these notions so as to sketch functions. Let's first introduce some other notions by using **Figure 5.33**.

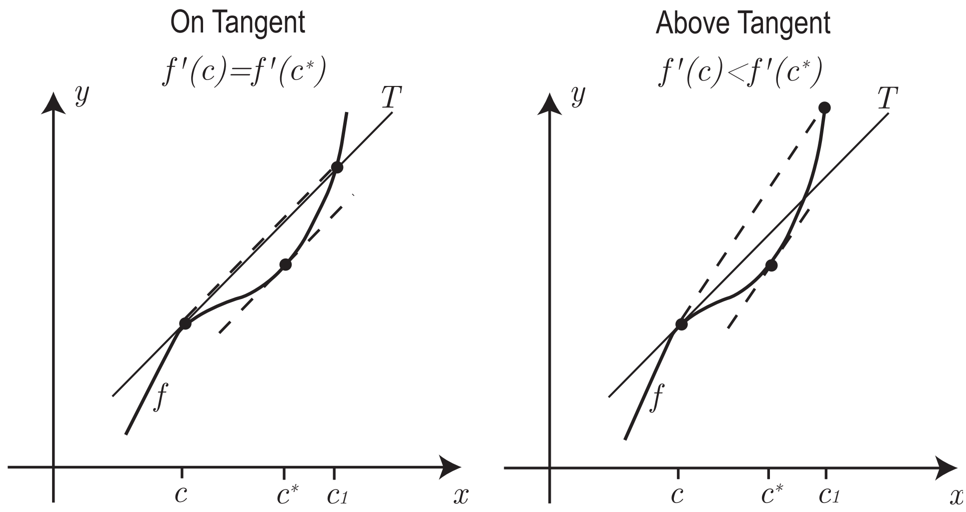

Figure 5.32: $f(c_1)$ is above or on the tangent line T

To sketch the curve of the function $f(x)$ in Figure 5.33, you see that the points P_1, P_2, P_3, P_4, and P_5 are *critically important*, because if you know these points, then you can link these points using concavity so as to sketch the curve. Accordingly, we call the points P_1, P_2, P_3, P_4, and P_5 **critical points**. Note from geometry that $f(x)$ has derivatives at P_1, P_2, P_3, and P_4, and these derivatives are zero; note also that $f(x)$ has no derivative at P_5 because of the sharp corner there (refer to the discussions on continuity and differentiability in Chapter 2). Thus we have

Definition 5.3.1 *If $f'(k) = 0$ or if $f'(k)$ does not exist, then k is called a* **critical point** *of the function $f(x)$.*

In Figure 5.33, the curve of $f(x)$ intercepts the x-axis at $x = a$ and $x = b$, thus, a and b are called the **x-intercepts** of $f(x)$. Similarly, $y = c$ is called the **y-intercept** of $f(x)$. Note that a function can have more than one x-intercept, but a function has at most one y-intercept (otherwise it will not be a function). If a function $f(x)$ is defined at $x = 0$, then $f(0)$ is the y-intercept. To find x-intercepts, set $f(x) = 0$ and then solve for the values of x. Note here that sometimes the x values of $f(x) = 0$ are not solvable, such as when $f(x)$ is a polynomial of a high degree, then we use other things, such as limits, to locate x-intercepts. We sometimes also use "intercepts" to denote x-intercept(s) and the y-intercept.

For functions such as $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$, one has

$$\lim_{x\to -1^-} \frac{x-1}{x+1} = \infty, \quad \lim_{x\to -1^+} \frac{x-1}{x+1} = -\infty, \quad \lim_{x\to \pm\infty} \frac{x-1}{x+1} = 1,$$

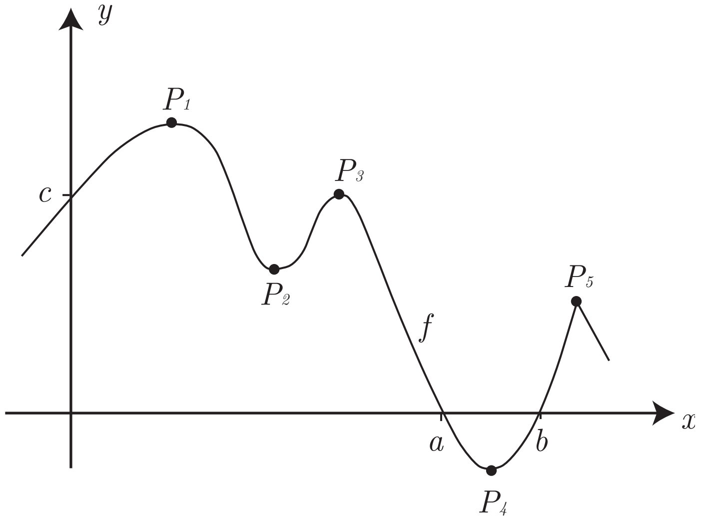

Figure 5.33: Critical points

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so in geometry the function gets very close to (but never intercept) the horizontal straight line $y = 1$ and the vertical straight line $x = -1$ *asymptotically*. In this sense, the straight line $y = 1$ is called a **horizontal asymptote** of $f(x)$, and the straight line $x = -1$ is called a **vertical asymptote** of $f(x)$. The typical cases having horizontal/vertical asymptotes are rational functions, such as $f(x) = \frac{x-1}{x+1}$. (Recall that a function is called a rational function if it is a polynomial divided by another polynomial.)

The Figure 5.33 and the function $\frac{x-1}{x+1}$ also indicates the following steps to follow when sketching functions.

1. Use the first derivative (with a sign chart) to determine increasing, decreasing, and critical points.

2. Use the second derivative (with a sign chart) to determine concave up, concave down, and inflection points.

3. Use the first or second derivative test to check whether the critical points are local or global maximum or minimum values.

4. Use $\lim_{x\to\pm\infty} f(x)$ to determine the behavior of $f(x)$ for large values of x (if $f(x)$ is defined there), which may result in horizontal asymptotes. If $f(x)$ is not defined at a point of interest, then use the limit(s) as x approaches that point to determine the behavior of $f(x)$ near that point, which may result in vertical asymptotes.

5. If $f(x)$ is defined at $x = 0$, then $f(0)$ is the y-intercept. If $f(x) = 0$ can be solved for x values, then they are x-intercepts. Otherwise, use

other things, such as limits, to locate x-intercepts.

6. Sketch using the information from the above steps.

Now, we revisit those examples in sections 5.1 and 5.2 and sketch them. That is, we will learn how to *visualize* the information obtained from the first and second derivatives.

Example 5.3.2 Sketch $f(x) = x^2 - 2x - 8$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= 2x - 2 = 2(x-1), \\ f''(x) &= 2, \end{aligned}$$

and the sign charts are given in **Figure 5.34**.

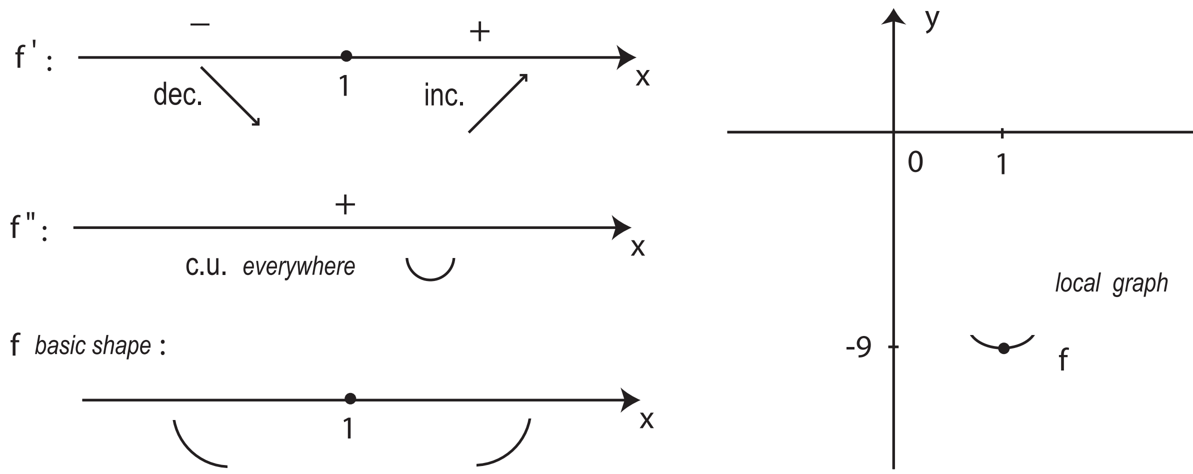

Figure 5.34: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = x^2 - 2x - 8$

Now, $x = 1$ is the only critical point and $f(x)$ is decreasing for $x < 1$ and increasing for $x > 1$. Also, $f(x)$ is concave up on $(-\infty, \infty)$ and hence has no inflection points. Therefore, from the first or second derivative test (see Figure 5.34), we see that $f(1) = -9$ must be the global minimum value, so we get the right picture in Figure 5.34.

The y-intercept is $y = f(0) = -8$. Since $\lim_{x\to\pm\infty} f(x) = \infty$, we conclude that the curve of $f(x)$ crosses the x-axis at two points so that $f(x)$ has two x-intercepts (in this simple case, they can be found to be at $x = -2, 4$). It also indicates that $f(x)$ has no local or global maximum values. Now we can sketch $f(x)$, given in **Figure 5.35**, which matches the knowledge of quadratic functions. ♠

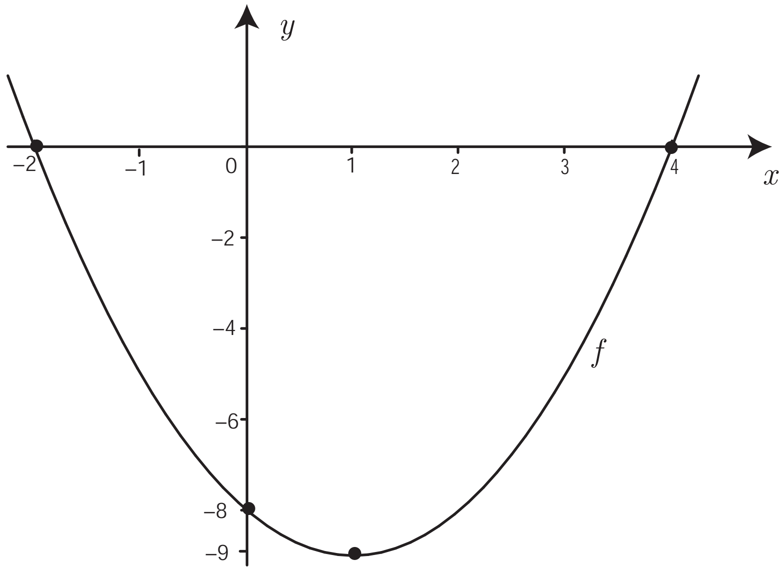

Figure 5.35: A sketch of $f(x) = x^2 - 2x - 8$

Example 5.3.3 Sketch $f(x) = x^3 - 3x^2 + 5$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= 3x^2 - 6x = 3x(x-2), \\ f''(x) &= 6x - 6 = 6(x-1), \end{aligned}$$

and the sign charts are given in **Figure 5.36**.

Now, $x = 0$ and $x = 2$ are critical points and $f(x)$ is increasing for $x < 0$ or $x > 2$ and decreasing on $(0, 2)$. Also, $f(x)$ is concave up for $x > 1$ and concave down for $x < 1$ and hence $x = 1$ is the only inflection point. Therefore, from the first or second derivative test (see Figure 5.36), we see that $f(0) = 5$ is a local maximum value and $f(2) = 1$ is a local minimum value, so we get the right picture in Figure 5.36.

The y-intercept is $y = f(0) = 5$. Since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, the function $f(x)$ has one x-intercept, and has no global maximum or minimum values. Now we can sketch $f(x)$, given in **Figure 5.37**. ♠

Note that the x-intercepts in Example 5.3.2 can be easily solved since the function is a polynomial of degree 2. For the function in Example 5.3.3 and similar cases, the functions are polynomials of higher degrees or other forms, where the x-intercepts are extremely difficult to find. That is why we only *indicate* or *locate* x-intercepts when sketching functions, as the precise values of x-intercepts are typically unknown.

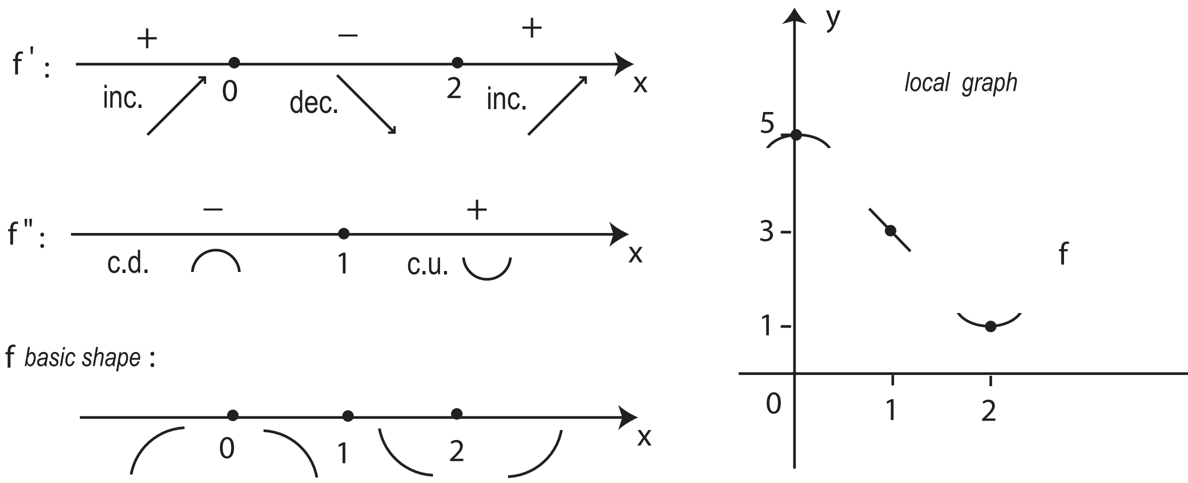

Figure 5.36: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = x^3 - 3x^2 + 5$

Example 5.3.4 Sketch $f(x) = (x-1)^3 + 6$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= 3(x-1)^2, \\ f''(x) &= 6(x-1), \end{aligned}$$

and the sign charts are given in **Figure 5.38**.

Now, $x = 1$ is the only critical point and $f(x)$ is increasing for $x < 1$ or $x > 1$. Also, $f(x)$ is concave up for $x > 1$ and concave down for $x < 1$ and hence $x = 1$ is the only inflection point. Note that the second derivative test is inconclusive about $f(1) = 6$ since $f''(1) = 6(1-1) = 0$. Thus we use the first derivative test (see Figure 5.38) and conclude that $f(x)$ is increasing on $(-\infty, \infty)$ (as the curve is continuous) and hence has no maximum or minimum values, and we also get the right picture in Figure 5.38.

The y-intercept is $y = f(0) = 5$. Since $\lim_{x\to\pm\infty} f(x) = \pm\infty$, $f(x)$ has one x-intercept. Now we can sketch $f(x)$, given in **Figure 5.39**. ♠

Example 5.3.5 Sketch $f(x) = 3 - (x-1)^4$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= -4(x-1)^3, \\ f''(x) &= -12(x-1)^2, \end{aligned}$$

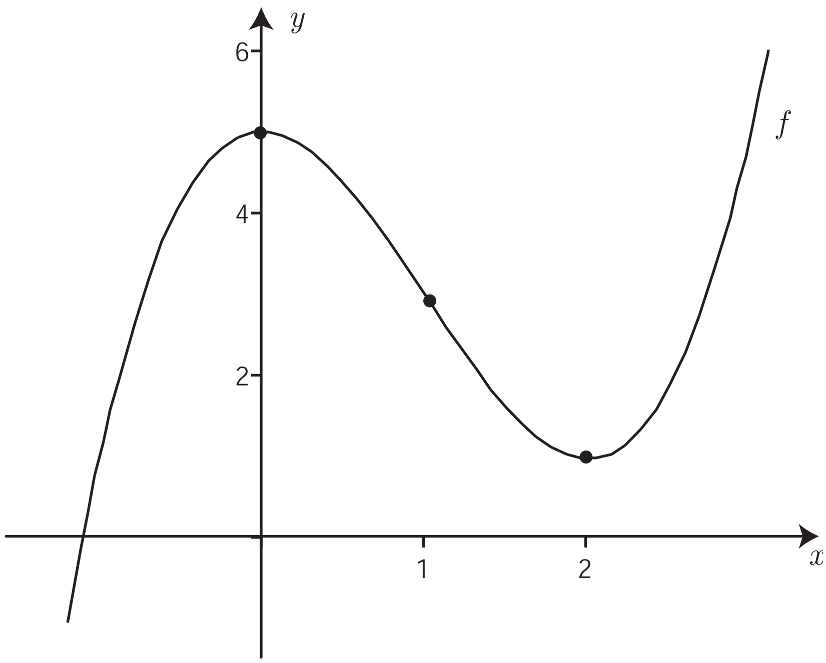

Figure 5.37: A sketch of $f(x) = x^3 - 3x^2 + 5$

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and the sign charts are given in **Figure 5.40**.

Now, $x = 1$ is the only critical point and $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$. Also, $f(x)$ is concave down for $x < 1$ or $x > 1$ and hence has no inflection points. These also indicate that $f(1) = 3$ must be the global maximum value.

The y-intercept is $y = f(0) = 2$, and since $\lim_{x\to\pm\infty} f(x) = -\infty$, the function $f(x)$ has two x-intercepts, and has no local or global minimum values. Now we can sketch $f(x)$, given in **Figure 5.41**. ♠

Example 5.3.6 Sketch $f(x) = x - \ln x,\ x > 0$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= 1 - \frac{1}{x} = \frac{x-1}{x}, \\ f''(x) &= \frac{d}{dx}[-x^{-1}] = x^{-2} = \frac{1}{x^2}, \end{aligned}$$

and the sign charts are given in **Figure 5.42**.

Now, $x = 1$ is the only critical point and $f(x)$ is decreasing on $(0, 1)$ and increasing for $x > 1$. Also, $f(x)$ is concave up on $(0, \infty)$ and hence has

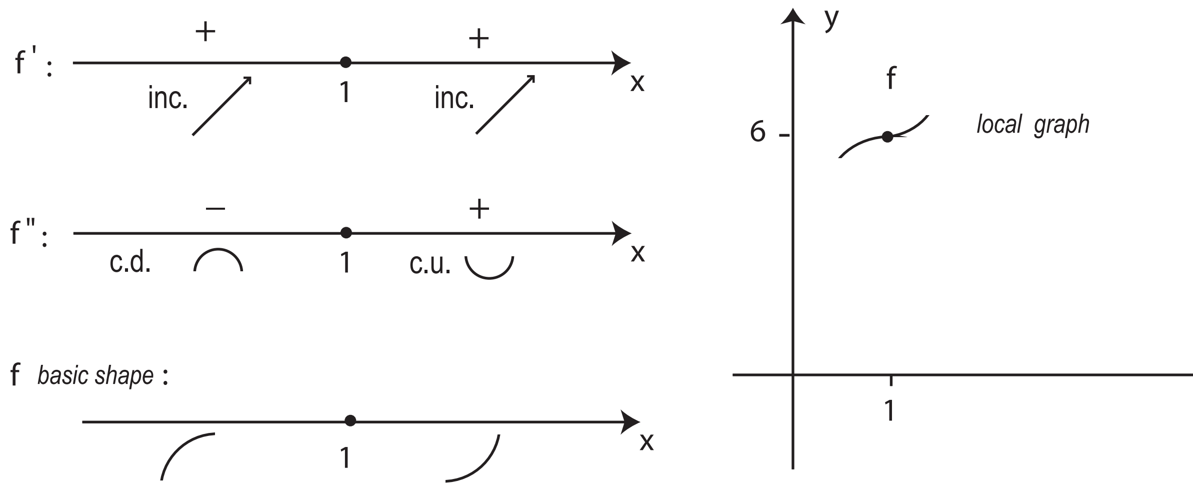

Figure 5.38: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = (x-1)^3 + 6$

no inflection points. These also indicate that $f(1) = 1$ must be the global minimum value.

There is no y-intercept in this case because $f(x)$ is defined for $x > 0$, and there are no x-intercepts because the smallest function value is 1. Next, using limits as we did in Example 5.1.10, we see that $f(x)$ has no local or global maximum values, and that the straight line $x = 0$ is a vertical asymptote. Now we can sketch $f(x)$, given in **Figure 5.43**. ♠

Example 5.3.7 Sketch $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}, \\ f''(x) &= \frac{d}{dx}[2(x+1)^{-2}] = -4(x+1)^{-3} = \frac{-4}{(x+1)^3}, \end{aligned}$$

and the sign charts are given in **Figure 5.44**.

Since $f(x)$ is not defined at $x = -1$, the function $f(x)$ has no critical points or inflection points. Now, $f(x)$ is increasing for $x < -1$ or $x > -1$, also, $f(x)$ is concave up for $x < -1$ and concave down for $x > -1$.

The y-intercept is $y = f(0) = -1$, and the x-intercept is $x = 1$ from $f(x) = \frac{x-1}{x+1} = 0$. To determine the behavior of $f(x)$ for x near $\pm\infty$ or

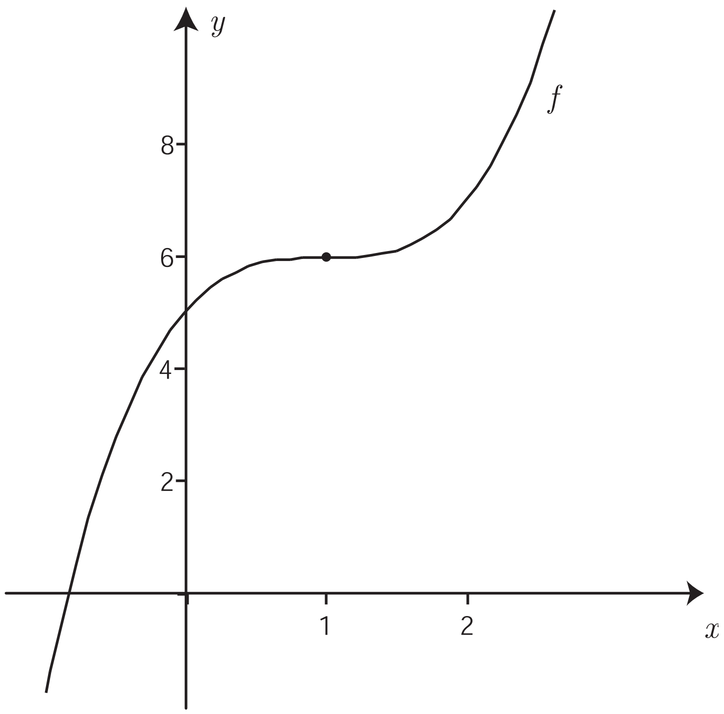

Figure 5.39: A sketch of $f(x) = (x-1)^3 + 6$

$x = -1$ (as $f(x)$ is not defined at $x = -1$), we take the following limits:

$$\lim_{x\to\pm\infty} \frac{x-1}{x+1} = 1, \quad \lim_{x\to -1^-} \frac{x-1}{x+1} = \infty, \quad \lim_{x\to -1^+} \frac{x-1}{x+1} = -\infty,$$

so we see that $y = 1$ is a horizontal asymptote and $x = -1$ is a vertical asymptote. Now we can sketch $f(x)$, given in **Figure 5.45**. ♠

Example 5.3.8 Sketch $f(x) = \frac{5}{1+2e^{-x}}$, $x \geq 0$. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

Solution. We have

$$\begin{aligned} f'(x) &= \frac{d}{dx} 5(1+2e^{-x})^{-1} = -5(1+2e^{-x})^{-2} 2e^{-x}(-1) = \frac{10e^{-x}}{(1+2e^{-x})^2}, \\ f''(x) &= 10\frac{-e^{-x}(1+2e^{-x})^2 + e^{-x}2(1+2e^{-x})2e^{-x}}{(1+2e^{-x})^4} = \frac{20e^{-x}(e^{-x} - \frac{1}{2})}{(1+2e^{-x})^3}, \end{aligned}$$

and the sign charts are given in **Figure 5.46**.

Now, $f(x)$ has no critical points and $f(x)$ is increasing for $x \geq 0$. Also, $f(x)$ is concave up on $[0, \ln 2)$ and concave down for $x > \ln 2$ and hence

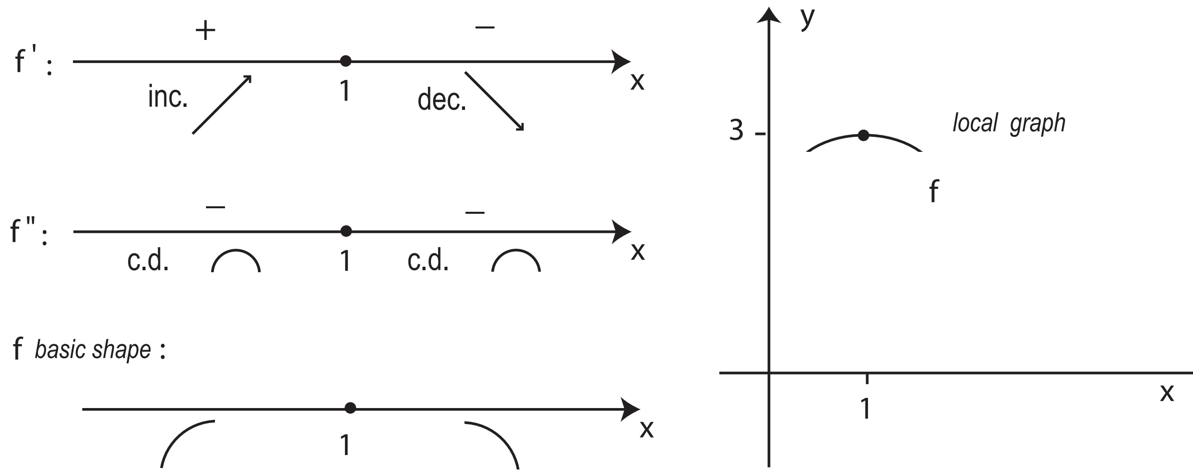

Figure 5.40: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = 3 - (x-1)^4$

$x = \ln 2$ is the only inflection point. These also indicate that the y-intercept $f(0) = \frac{5}{3}$ must be the global minimum value and $f(x)$ has no x-intercepts. Next, as

$$\lim_{x\to\infty} \frac{5}{1+2e^{-x}} = \frac{5}{1+0} = 5,$$

we see that $y = 5$ is a horizontal asymptote. Now we can sketch $f(x)$, given in **Figure 5.47**. ♠

Corresponding Characteristics of the Graphs of f, f', and f''.

Here, we will learn how to sketch f' when f is only given by a curve, not by a formula. Repeating the procedure, we can sketch f'' based on the curve of f'. Similarly, we will also learn how to sketch f when f' is only given by a curve, not by a formula. Again, Repeating the procedure, we can start with a curve of f'' and sketch f' and then sketch f.

Example 5.3.9 The curve of a function $f(x)$ is given in **Figure 5.48**. Sketch $f'(x)$.

Solution. The curve is decreasing on $[a, b)$, so $f'(x)$ is negative on $[a, b)$. Next, the slope is increasing on $[a, b]$ and the slope is zero at b, so $f'(x)$ is increasing on $[a, b]$ and $f'(b) = 0$.

We will do the same on $[b, c]$ and $[c, d]$. On (b, c), the curve is increasing, so $f'(x)$ is positive on (b, c), and $f'(c) = 0$ since the slope is zero at c. Next, the slope is increasing on $[b, 0]$ and then decreasing on $[0, c]$, so $f'(x)$

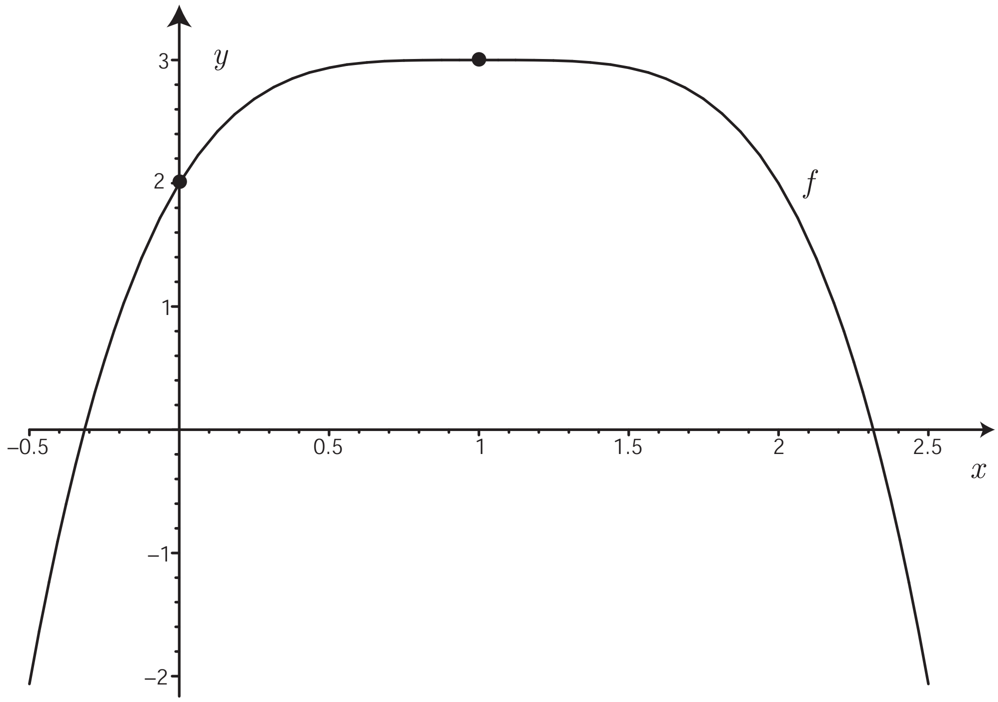

Figure 5.41: A sketch of $f(x) = 3 - (x-1)^4$

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is increasing on $[b, 0]$ and then decreasing on $[0, c]$. Similar to the analysis on $[a, b]$, we see that $f'(x)$ is negative and decreasing on $[c, d]$. Now we can sketch $f'(x)$, given in **Figure 5.49**. ♠

Example 5.3.10 The curve of a function $f'(x)$ is given in **Figure 5.50**. Sketch $f(x)$.

Solution. Note that for any constant C, $(f(x) + C)' = f'(x)$. So now we have many functions with derivative $f'(x)$. Thus, we only need to sketch one of them.

The derivative is negative on $[a, b)$, so $f(x)$ is decreasing on $[a, b)$. Next, the derivative is increasing on $[a, b]$, so using the second derivative test, $f(x)$ is concave up on $[a, b]$, and the slope is zero at b since $f'(b) = 0$.

We will do the same on $[b, c]$ and $[c, d]$. On (b, c), the derivative is positive, so $f(x)$ is increasing on (b, c). Next, the derivative is increasing on $[b, 0]$ and then decreasing on $[0, c]$, so using the second derivative test, $f(x)$ is concave up on $[b, 0]$ and then concave down on $[0, c]$. Similar to the analysis on $[a, b]$, we see that $f(x)$ is decreasing and concave down on $[c, d]$. Now we can sketch $f(x)$, given in **Figure 5.51**. ♠

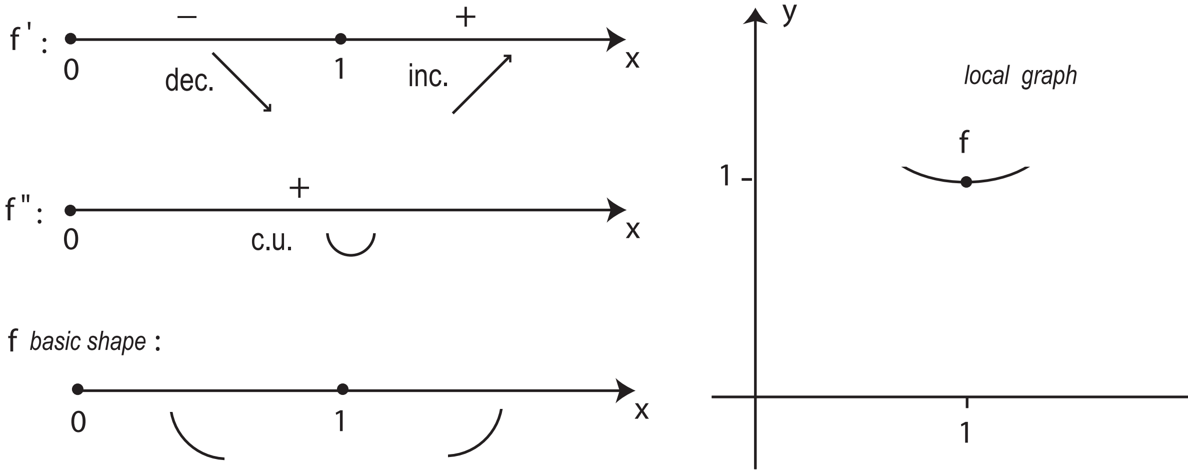

Figure 5.42: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = x - \ln x$

Guided Practice 5.3

Sketch the following functions. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

1. $f(x) = x^2 + 4x + 5$.

2. $f(x) = \frac{2x}{x+3}$.

3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 1$.

4. $f(x) = e^{-\frac{x^2}{2}}$.

Exercises 5.3

1. Sketch the following functions. Indicate critical points, concavity, inflection points, intercepts, and asymptotes, if any.

(a) $f(x) = x^2 + 2x - 3$.
(b) $f(x) = x^2 - 2x + 3$.
(c) $f(x) = x^3 - 4.5x^2 + 6x + 8$.
(d) $f(x) = -2x^3 + 9x^2 - 12x + 15$.
(e) $f(x) = 4(x-3)^3 + 9$.
(f) $f(x) = 3(x-2)^3 - 7$.
(g) $f(x) = 4(x-1)^4 - 3$.

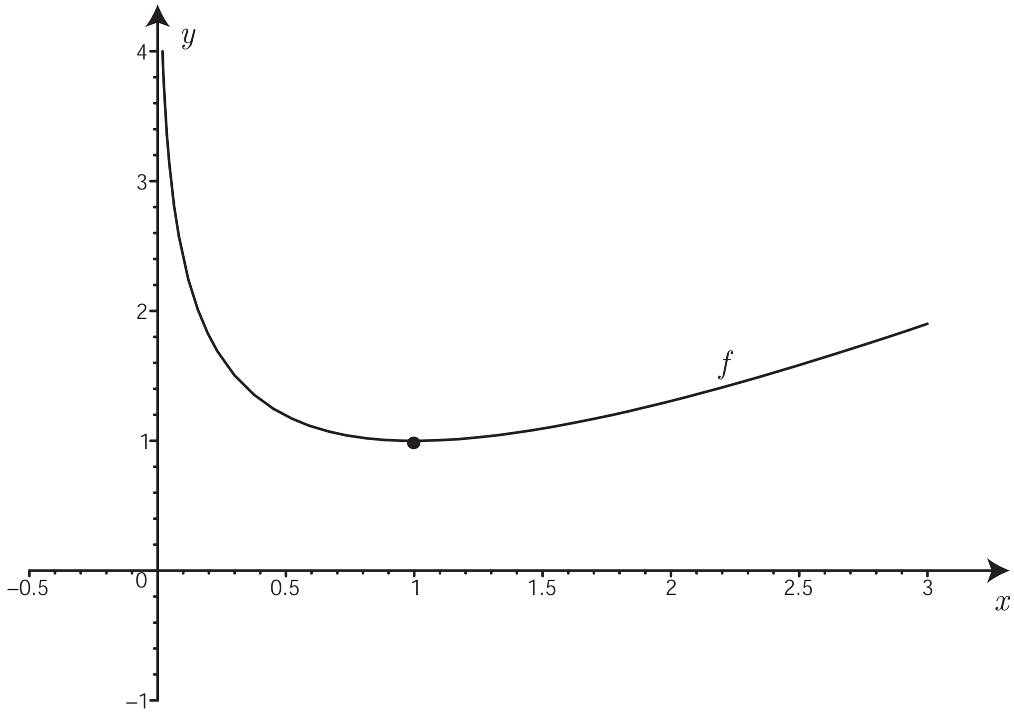

Figure 5.43: A sketch of $f(x) = x - \ln x,\ x > 0$

(h) $f(x) = 3(x-2)^4 + 5$.

(i) $f(x) = e^x$.

(j) $f(x) = \ln x,\ (x > 0)$.

(k) $f(x) = e^x - x$.

(l) $f(x) = e^x + x$.

(m) $f(x) = \frac{x+1}{x-1},\ (x \neq 1)$.

(n) $f(x) = \frac{x+1}{x+2},\ (x \neq -2)$.

(o) $f(x) = \frac{x}{(x+1)^2},\ (x \neq -1)$.

(p) $f(x) = \frac{x}{(x-1)^2},\ (x \neq -1)$.

(q) $f(x) = \frac{\ln x}{x},\ x \in (0, e^{10}]$.

(r) $f(x) = \frac{x}{\ln x},\ x \in (1, e^{10}]$.

2. For each curve in **Figure 5.52**, sketch its derivative.

3. If each curve in **Figure 5.52** represents the derivative of a function, then sketch such a function.

4. For the curve in **Figure 5.53**, sketch its derivative.

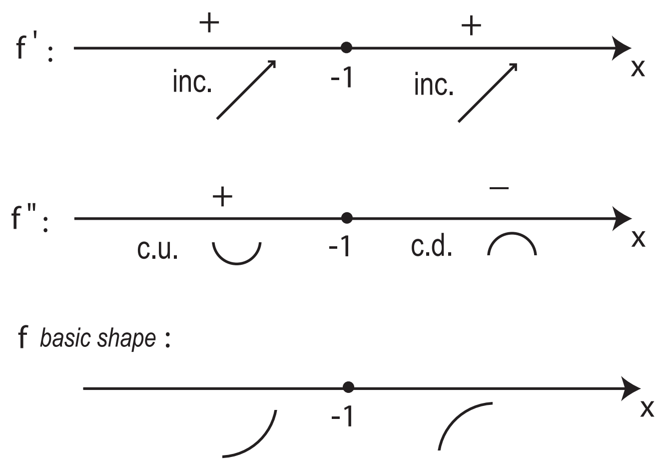

Figure 5.44: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = \frac{x-1}{x+1}$

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5. If the curve in **Figure 5.53** represents the derivative of a function, then sketch such a function.

5.4 Optimizations

In this section, we will learn how to use derivatives to solve optimizations, such as maximizing profit and minimizing cost in business applications.

For a typical problem here, the situation is described only by words, and then an optimization question is asked. To solve such a problem, we start by asking *what is the objective*? This will help us to use conditions/relations to formulate a function, called an **objective function**, and then we can use the first or second derivative test to determine the global maximum or global minimum value of the objective function on the corresponding domain (here, the curve sketching is not needed). Note that in some applications, such as when determining how many tables to produce to maximize the profit, if a solution is not an integer, then we round it to an integer to make the answer meaningful.

First, we look at some examples concerning revenue and profit in business applications. Recall that we have studied *price functions* (or *demand equations*) $u(x)$, *cost functions* $C(x)$, *revenue functions* $R(x)$, and *profit functions* $P(x)$. Their relationships are given by

$$R(x) = xu(x), \quad P(x) = R(x) - C(x) = xu(x) - C(x).$$

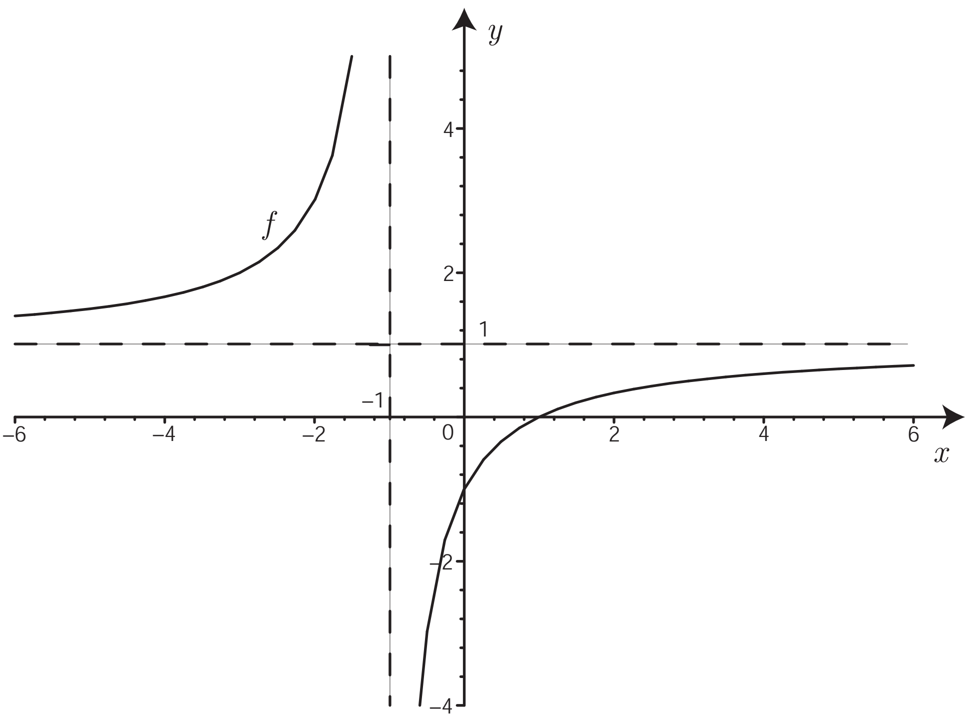

Figure 5.45: A sketch of $f(x) = \frac{x-1}{x+1}$, $(x \neq -1)$

Example 5.4.1 The price function for a certain product is $u(x) = 6 - \frac{1}{2}x$. Find the level of production that results in the maximum revenue.

Solution. First, note that "the level of production" means the number of units, or the x value. The objective here is to maximize the revenue. Thus, the objective function is now given by

$$R(x) = xu(x) = x(6 - \frac{1}{2}x) = 6x - \frac{1}{2}x^2,$$

with domain $x \in (0, 12)$ (in order for $u(x) = 6 - \frac{1}{2}x > 0$).

Then,

$$R'(x) = 6 - x, \quad R''(x) = -1 < 0.$$

Thus, $x = 6$ is the only critical point, and from the second derivative test, we know that $R(x)$ takes the global maximum value at $x = 6$ (the first derivative test can also be used to derive the same conclusion). Therefore, the level of production is to produce and sell 6 units of the product in order to maximize the revenue. ♠

Example 5.4.2 Suppose you are running a business for which the yearly rental fee is \$10, 000, and it costs \$50 to produce a certain product, and the

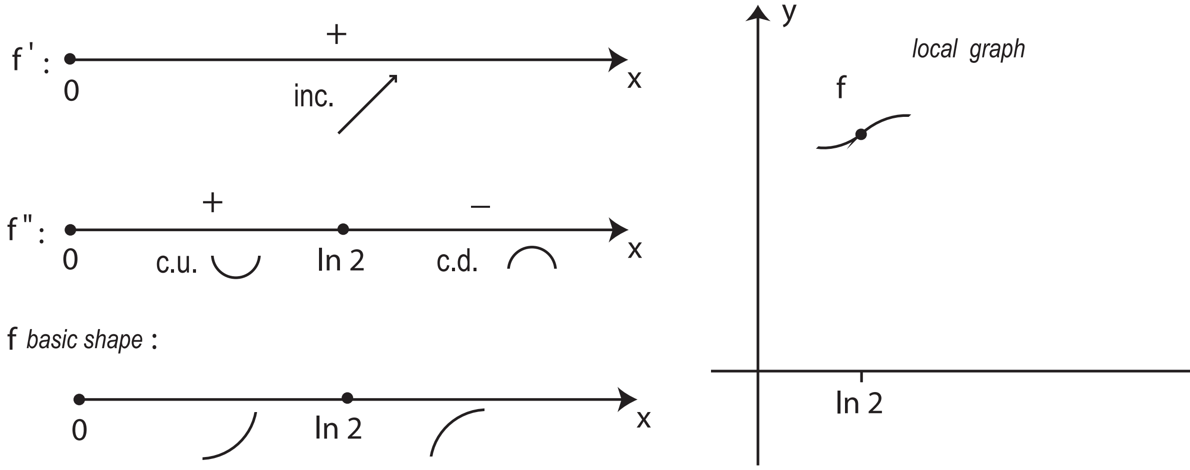

Figure 5.46: Sign charts of $f'(x)$ and $f''(x)$ for $f(x) = \frac{5}{1+2e^{-x}}$

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price function is $u(x) = 100 - 0.01x$. Find the level of production that results in the maximum yearly profit. Then find the corresponding unit price and yearly profit.

Solution. The $10,000 is the fixed cost, and if you produce x units of the product, the additional cost is 50x$, thus the cost function in this case is given by

$$C(x) = 50x + 10,000.$$

The objective here is to maximize the profit. Thus, the objective function is now given by

$$\begin{aligned} P(x) &= R(x) - C(x) = xu(x) - C(x) \\ &= x(100 - 0.01x) - (50x + 10,000) \\ &= -0.01x^2 + 50x - 10,000, \end{aligned}$$

with domain $x \in (0, 10000)$ (in order for $u(x) = 100 - 0.01x > 0$).

Then,

$$P'(x) = -0.02x + 50, \quad P''(x) = -0.02 < 0.$$

Thus, $x = 2500$ is the only critical point, and from the second derivative test, we know that $P(x)$ takes the global maximum value at $x = 2500$ (the first derivative test can also be used to derive the same conclusion). Therefore, the level of production is to produce and sell 2500 units of the product. The corresponding unit price (at the level of $x = 2500$) is

$$u(2500) = 100 - 0.01(2500) = \$75,$$

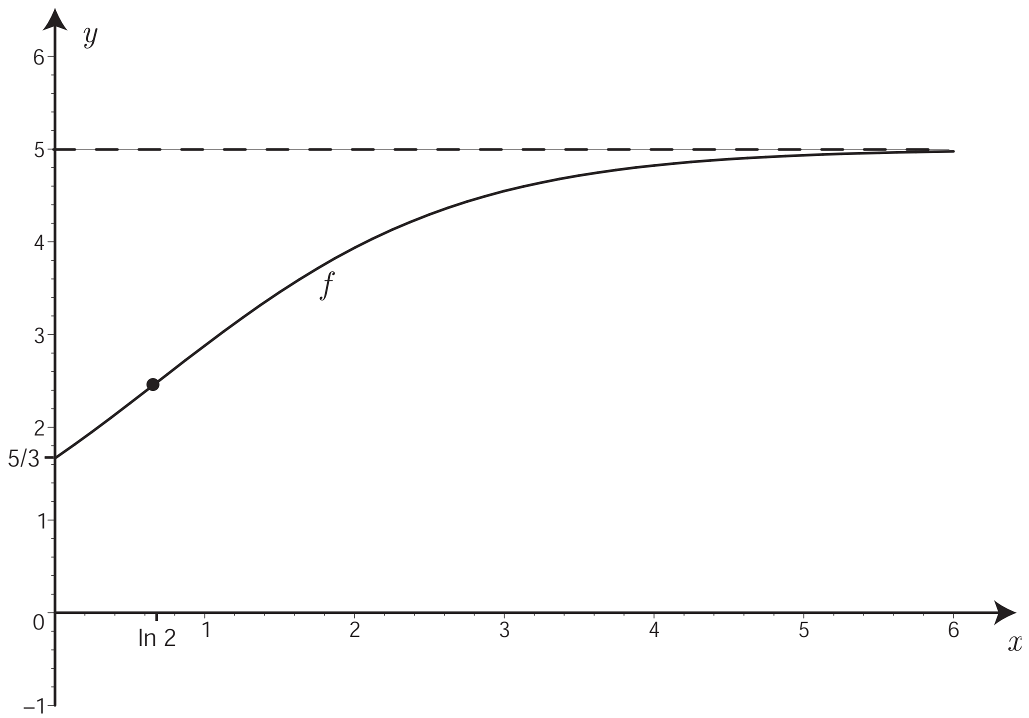

Figure 5.47: A sketch of $f(x) = \frac{5}{1+2e^{-x}}$, $x \geq 0$

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and the corresponding yearly profit (at the level of $x = 2500$) is

$$P(2500) = -0.01(2500)^2 + 50(2500) - 10,000 = \$52500.$$

♠

Example 5.4.3 Suppose you sell a certain T-shirt on campus, and find that if you charge \$10 each, then you can sell 90 T-shirts per week, but if you charge \$8 each, then you can sell 100 T-shirts per week. Assume that the price function is a linear function, then determine the price (per T-shirt) that can maximize the weekly revenue.

Solution. The assumption that the price function is a linear function means that it is a straight line. In $u(x)$, x means how many units. So we let x be the number of T-shirts and y be the corresponding unit price, then in the xy-plane, the price function (straight line) passes through the two points $(90, 10)$ and $(100, 8)$. Therefore, using the knowledge of straight lines with the slope $m = \frac{8-10}{100-90} = -\frac{1}{5}$ and the point $(90, 10)$, the price function is given by

$$y = 10 - \frac{1}{5}(x - 90) = 28 - \frac{1}{5}x,$$

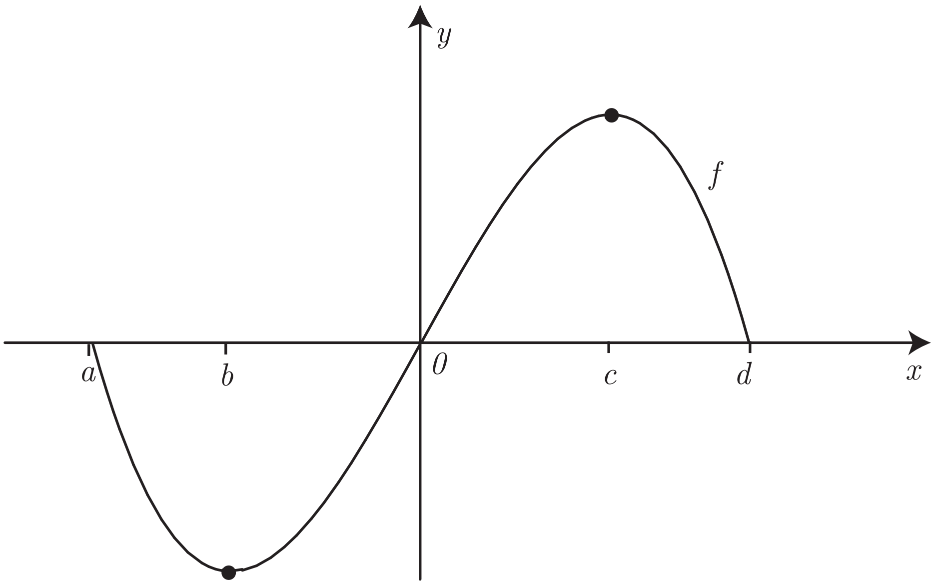

Figure 5.48: The curve of $f(x)$

or

$$u(x) = 28 - \frac{1}{5}x.$$

The objective here is to maximize the revenue. Thus, the objective function is now given by

$$R(x) = xu(x) = x(28 - \frac{1}{5}x) = 28x - \frac{1}{5}x^2,$$

with domain $x \in (0, 140)$ (in order for $u(x) = 28 - \frac{1}{5}x > 0$).

From

$$R'(x) = 28 - \frac{2}{5}x, \quad R''(x) = -\frac{2}{5} < 0,$$

we see that $x = 70$ is the only critical point, and we know from the second derivative test that $R(x)$ takes the global maximum value at $x = 70$ (the first derivative test can also be used to derive the same conclusion). That is, the weekly revenue is maximized if you sell $x = 70$ T-shirts. To determine the corresponding unit price, we plug $x = 70$ into $u(x) = 28 - \frac{1}{5}x$ and get

$$u(70) = 28 - \frac{1}{5}(70) = 28 - 14 = \$14.$$

Therefore, to maximize the weekly revenue, you should charge \$14 per T-shirt and sell 70 T-shirts per week. ♠

In the previous examples, optimizations are carried out for revenue and profit where objective functions are relatively easy to understand and to formulate.

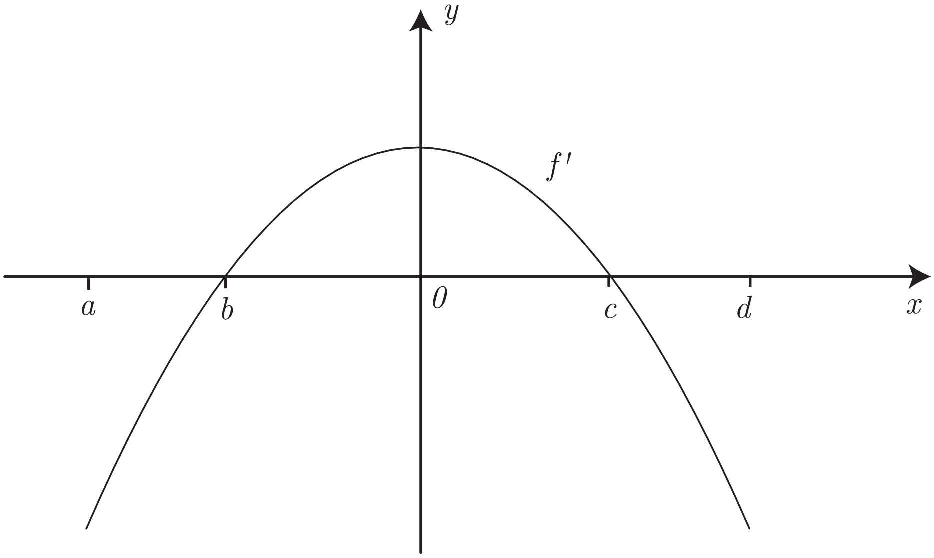

Figure 5.49: A sketch of $f'(x)$ for Example 5.3.9

In many other cases, such as the ones we will see in the following, objective functions are not easy to understand or to formulate. So that the algebra and geometry skills are very crucial here in order to understand and formulate the objective functions so as to solve the optimizations using derivative tests.

We first start with two generic examples to get the basic ideas, since these ideas are also applicable to many other related optimization problems.

Example 5.4.4 Find two positive numbers whose sum is 36 and whose product is as large as possible.

Solution. Let x and w be the two positive numbers. Our objective here is to maximize the product, given by

$$P = xw. \tag{4.1}$$

Since x and w are both unknown, we need to replace one in terms of the other one in order to construct an objective function of a single variable.

We know that the sum of x and w is 36, that is,

$$x + w = 36,$$

so we can express w in terms of x and get

$$w = 36 - x.$$

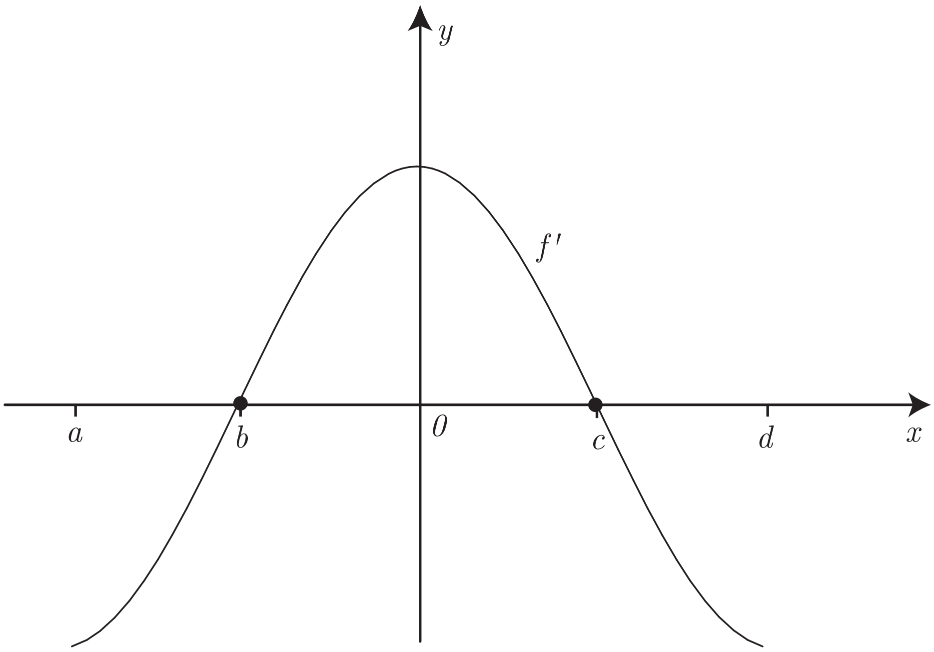

Figure 5.50: The curve of $f'(x)$

Now, we plug $w = 36 - x$ into (4.1) and derive the objective function given by

$$P(x) = x(36 - x) = 36x - x^2, \tag{4.2}$$

with domain $x \in (0, 36)$ (in order for $w = 36 - x > 0$).

Next, we use derivative tests to find, in this case, the global maximum value. From

$$P'(x) = 36 - 2x, \quad P''(x) = -2 < 0,$$

we see that $x = 18$ is the only critical point, and we know from the second derivative test that $P(x)$ takes the global maximum value at $x = 18$ (the first derivative test can also be used to derive the same conclusion).

Since $w = 36 - x = 36 - 18 = 18$, we conclude that the product is the largest when the two numbers are 18 and 18. ♠

If you try to solve this problem by *guessing and checking* numbers, such as letting $x = 1.3$, $w = 34.7$ and getting $xw = 45.11$; letting $x = 4.6$, $w = 31.4$ and getting $xw = 144.44$; etc., then you can never finish because there are too many (in fact uncountably many) numbers for you to check. However, after formulating it as a calculus problem, it can be solved easily using derivative tests.

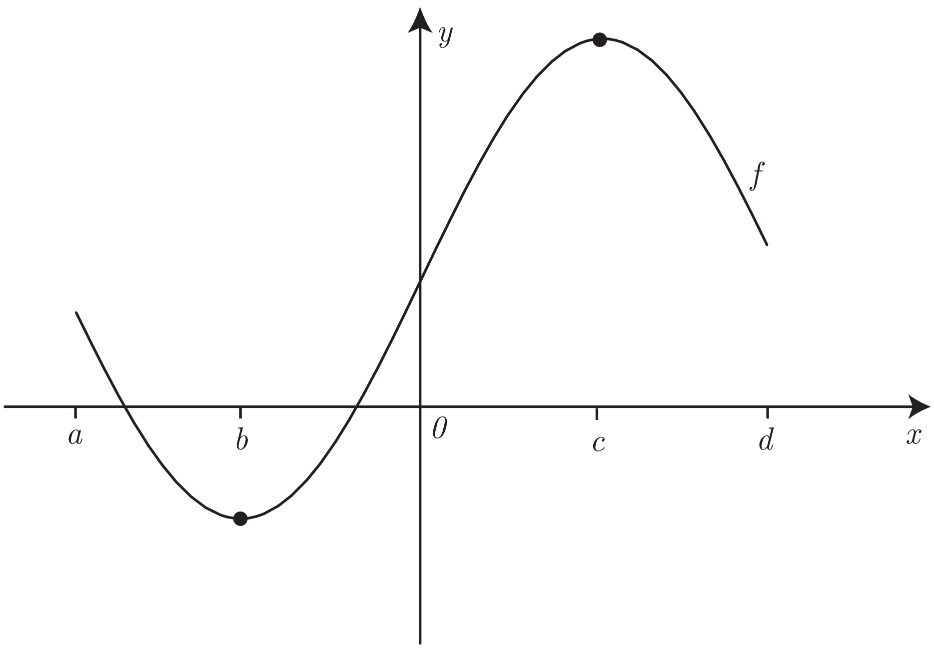

Figure 5.51: A sketch of $f(x)$ for Example 5.3.10

Example 5.4.5 Find two positive numbers whose product is 36 and whose sum is as small as possible.

Solution. Let x and w be the two positive numbers. Our objective here is to minimize the sum, given by

$$S = x + w, \tag{4.3}$$

which involves two unknowns, x and w.

Similar to Example 5.4.4, we use the information that the product is 36, that is, $xw = 36$, to express w in terms of x, given by $w = \frac{36}{x}$.

The objective function is now given by

$$S(x) = x + \frac{36}{x},$$

with domain $x > 0$ because we only consider positive numbers.

From

$$S'(x) \quad = \quad \frac{d}{dx}[x + 36x^{-1}] = 1 - 36x^{-2} = 1 - \frac{36}{x^2},$$

we get the only critical point at $x = 6$ because the domain is $x > 0$ (i.e. $S(x)$ is only defined for $x > 0$). Using the condition of $x > 0$ again, we get

$$S''(x) = \frac{d}{dx}[1 - 36x^{-2}] = 72x^{-3} = \frac{72}{x^3} > 0, \quad x > 0,$$

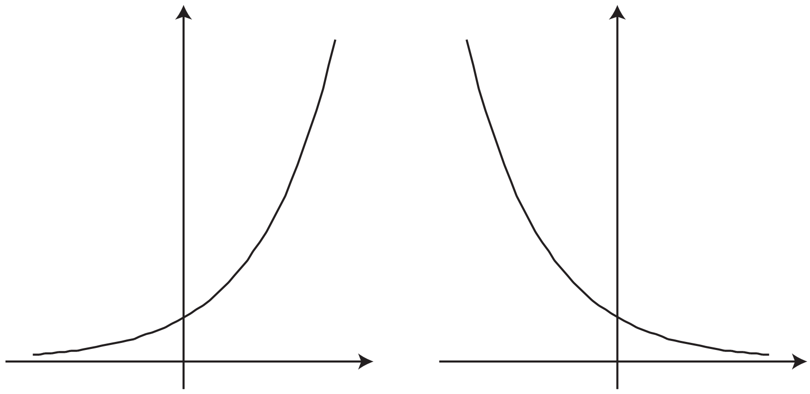

Figure 5.52: Two curves

so we know from the second derivative test that $S(x)$ defined on $(0, \infty)$ takes the global minimum value at $x = 6$ (the first derivative test can also be used to derive the same conclusion).

Since $w = \frac{36}{x} = \frac{36}{6} = 6$, we conclude that the sum is the smallest when the two numbers are 6 and 6. ♠

Remark 5.4.6 In Example 5.4.5, the global minimum value is achieved for $x > 0$. That is, regarding $S(x) = x + \frac{36}{x}$ as a function with the domain $(0, \infty)$, the function *does* have a global minimum value. However, if $x + \frac{36}{x}$ is regarded as a function on $(0, \infty)$ *and* on $(-\infty, 0)$, then, since $\lim_{x \to -\infty}(x + \frac{36}{x}) = -\infty$, the function $x + \frac{36}{x}$ *has no* global minimum value. This remark indicates that it is very important to understand that in applications we must only consider $x > 0$, so that the *global* maximum or minimum values can be achieved. ♠

From examples 5.4.4 and 5.4.5, we see that the following are the general steps for solving the related optimization problems.

1. Label unknowns and understand their relationships,

2. Determine the objective,

3. Express unknowns in terms of a single unknown so as to formulate an objective function of a single variable,

4. Use derivative tests to determine global maximum or minimum values on the corresponding domains.

The following examples look more *real*, where the most important and difficult step is to determine objective functions, for which algebra and geometry skills are very helpful.

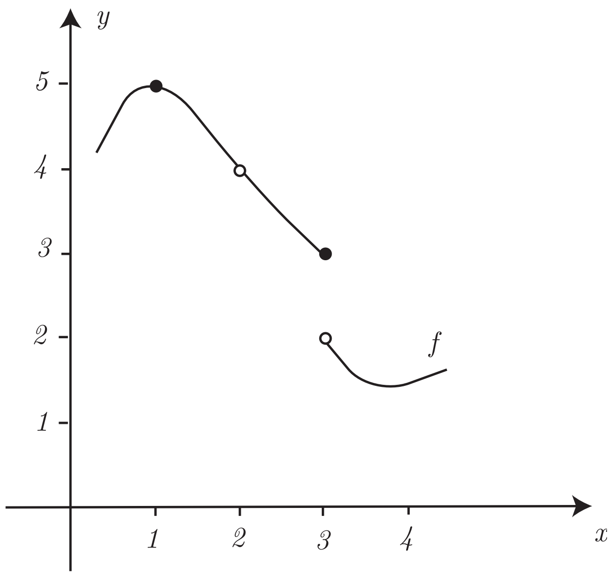

Figure 5.53: A curve

Example 5.4.7 If you want to plant a rectangular garden along one side of your house with a picket fence on the other three sides of the garden, find the dimensions of the largest garden possible if you only have 40 feet of fencing.

Solution. Let's use **Figure 5.54** to visualize the situation. Let x and w be the dimensions of the garden, which is a rectangle. Then our objective here is to maximize the area of the rectangle, given by

$$A = xw,$$

which involves two unknowns, x and w. Since we will use all 40 feet of fencing in order to get the largest garden, we have

$$2x + w = 40, \quad \text{or } w = 40 - 2x.$$

The objective function is now given by

$$A(x) = x(40 - 2x) = 40x - 2x^2, \tag{4.4}$$

with domain $x \in (0, 20)$ (in order for $w = 40 - 2x > 0$).

From

$$A'(x) = 40 - 4x, \quad A''(x) = -4 < 0,$$

we see that $x = 10$ is the only critical point, and we know from the second derivative test that $A(x)$ takes the global maximum value at $x = 10$ (the first derivative test can also be used to derive the same conclusion).

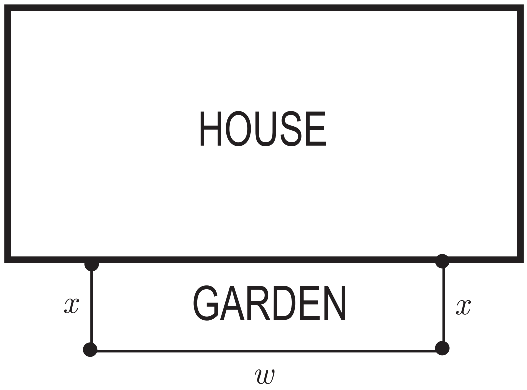

Figure 5.54: Dimensions for the garden

Since $w = 40 - 2x = 40 - 20 = 20$, we conclude that the garden is the largest when the dimensions are $x = 10$ and $w = 20$. ♠

Note that the solutions for Example 5.4.4 and Example 5.4.7 are similar: a sum is fixed, and a product is maximized.

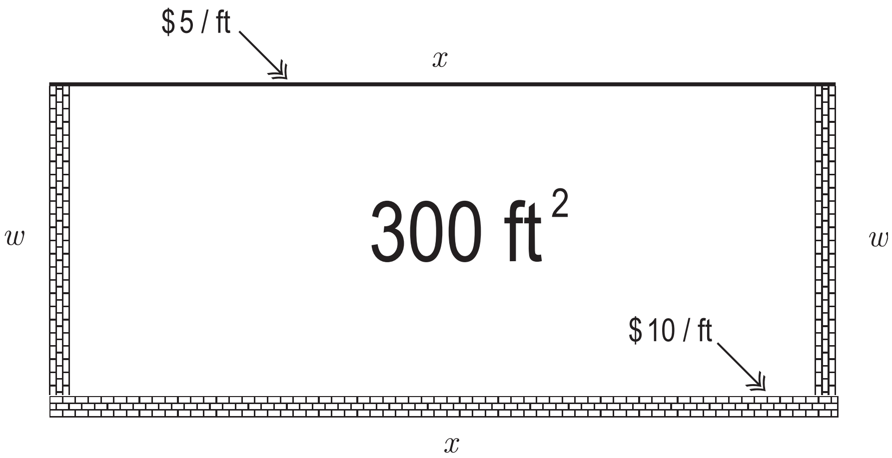

Figure 5.55: Dimensions for the garden and the costs

Example 5.4.8 A rectangular garden of area 300 square feet is to be surrounded on three sides by a brick wall costing \$10 per foot and on one side by a fence costing \$5 per foot. Find the dimensions of the garden such that the cost of materials is minimized.

Solution. Let's use **Figure 5.55** to visualize the situation. Our objective here is to minimize the cost of materials, given by

$$C = 5x + 10w + 10x + 10w = 15x + 20w,$$

which involves two unknowns, x and w. Since the area of the garden is 300, we have

$$xw = 300, \quad \text{or} \;\; w = \frac{300}{x}.$$

The objective function is now given by

$$C(x) = 15x + 20 \cdot \frac{300}{x} = 15x + \frac{6000}{x}, \tag{4.5}$$

with domain $x > 0$ because we only consider positive numbers.

From

$$C'(x) \;\; = \;\; \frac{d}{dx}[15x + 6000x^{-1}] = 15 - 6000x^{-2} = 15 - \frac{6000}{x^2},$$

we get the only critical point at $x = 20$ because the domain is $x > 0$. From

$$C''(x) = \frac{d}{dx}[15 - 6000x^{-2}] = 12000x^{-3} = \frac{12000}{x^3} > 0, \;\; x > 0,$$

we know from the second derivative test that $C(x)$ takes the global minimum value at $x = 20$ (the first derivative test can also be used to derive the same conclusion).

Since $w = \frac{300}{x} = \frac{300}{20} = 15$, we conclude that the cost is minimized when the dimensions are $x = 20$ and $w = 15$. ♠

Note that the solutions for Example 5.4.5 and Example 5.4.8 are similar: a product is fixed, and a sum is minimized.

Example 5.4.9 The U.S. parcel post regulations state that packages must have length plus girth of no more than 108 inches. Find the dimensions of a square-based rectangular package that has the largest volume that is mailable by parcel post.

Solution. Let's use **Figure 5.56** to visualize the situation. We let l be the length and x be one side of the square base. Then our objective here is to maximize the volume of the rectangular package, given by

$$V = (\text{base area})(\text{height}) = x^2 l,$$

which involves two unknowns, x and l. The girth in this case is $4x$, and we will let length plus girth equal 108 inches in order to achieve the largest volume, so we obtain

$$4x + l = 108, \quad \text{or} \;\; l = 108 - 4x.$$

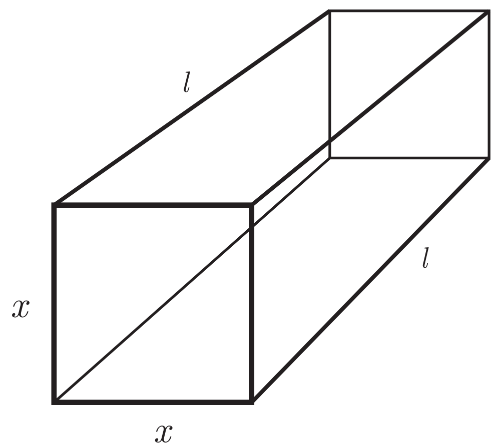

Figure 5.56: Dimensions for a rectangular package

The objective function is now given by

$$V(x) = x^2(108 - 4x) = 108x^2 - 4x^3, \tag{4.6}$$

with domain $x \in (0, 27)$ (in order for $l = 108 - 4x > 0$).

From

$$V'(x) = 216x - 12x^2 = 12x(18 - x), \quad V''(x) = 216 - 24x,$$

we know that the only critical point on the domain $(0, 27)$ is at $x = 18$. Now, the second derivative test is not easy to use, so we use the first derivative test and conclude (with a sign chart) that $V(x)$ takes the global maximum value at $x = 18$.

Since $l = 108 - 4x = 108 - (4)(18) = 108 - 72 = 36$, we conclude that the volume of the rectangular package is the largest when the dimensions are $x = 18$ and $l = 36$.

The solution indicates that the length ($l = 36$) should be twice the side of the base ($x = 18$) in order to maximize the volume. ♠

Note that the solutions for Example 5.4.4, Example 5.4.7, and Example 5.4.9 are similar: a sum is fixed, and a product is maximized.

Example 5.4.10 If you want to construct a cylindrical can so as to hold a volume of $100\,\text{ft}^3$, how should the radius and height be chosen so as to minimize the surface area of the can, including the top and bottom?

Solution. Let's use **Figure 5.57** to visualize the situation. We let r be the radius and h be the height of the can. Then our objective here is to minimize the surface area of the can. To determine this area, we note that

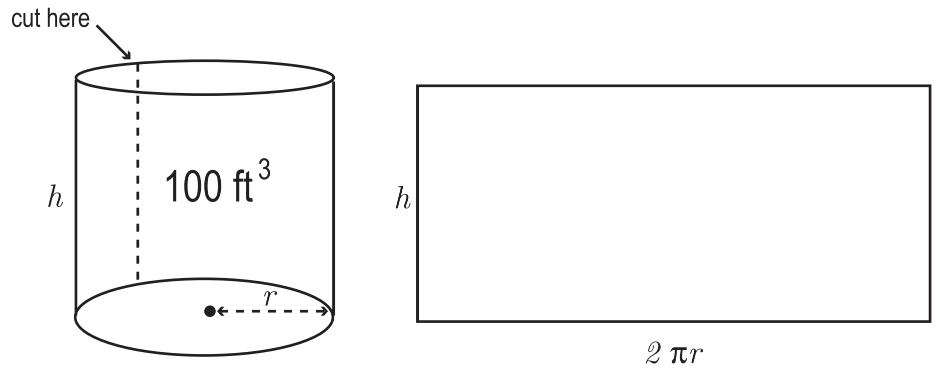

Figure 5.57: Dimensions for the cylindrical can

the area of the top or the bottom is the area of a circle, given by πr^2, and the area of the side is given by height times base, with base the same as the circumference ($2\pi r$) if you cut the side vertically and unroll it into a flat sheet, see the second graph in Figure 5.57. Thus, the surface area of the can is given by

$$S = 2\pi r^2 + 2\pi r h,$$

which involves two unknowns, r and h. To replace one unknown with the other one, we know that the volume of the can is 100, so we obtain

$$100 = (\text{base area})(\text{height}) = (\pi r^2)(h).$$

Thus we can express h in terms of r and get

$$h = \frac{100}{\pi r^2}.$$

The objective function is now given by

$$S(r) = 2\pi r^2 + 2\pi r \frac{100}{\pi r^2} = 2\pi r^2 + \frac{200}{r}, \tag{4.7}$$

with domain $r > 0$ because we only consider positive numbers.

From

$$\begin{aligned} S'(r) &= \frac{d}{dr}[2\pi r^2 + 200r^{-1}] = 4\pi r - 200r^{-2} = 4\pi r - \frac{200}{r^2}, \\ S''(r) &= 4\pi + \frac{400}{r^3} > 0, \quad r > 0, \end{aligned}$$

we get a critical point from $4\pi r = \frac{200}{r^2}$, or $r^3 = \frac{50}{\pi}$, which gives

$$r = (\frac{50}{\pi})^{1/3}.$$

Next, we know from the second derivative test that $S(r)$ takes the global minimum value at $r = (\frac{50}{\pi})^{1/3}$ (the first derivative test can also be used to derive the same conclusion). Plugging r into $h = \frac{100}{\pi r^2}$, we get

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi(\frac{50}{\pi})^{2/3}} = \frac{2 \times 50}{\pi^{1/3}(50)^{2/3}} = \frac{2 \times (50)^{1/3}}{\pi^{1/3}} = 2(\frac{50}{\pi})^{1/3} = 2r.$$

For this choice of r and h, the surface area of the can is minimized. The solution indicates that the height (h) should be the same as the diameter ($2r$) in order to minimize the surface area. ♠

Note that the solutions for Example 5.4.5, Example 5.4.8, and Example 5.4.10 are similar: a product is fixed, and a sum is minimized.

In fact, many problems we encounter here are similar to Example 5.4.4 or to Example 5.4.5: either a sum is fixed and a product is maximized, or a product is fixed and a sum is minimized. So the ideas used in the above solutions should help when solving similar optimization problems.

However, in some situations, the relationships between the two unknowns are not given by a fixed sum or a fixed product, as can be seen in the following case.

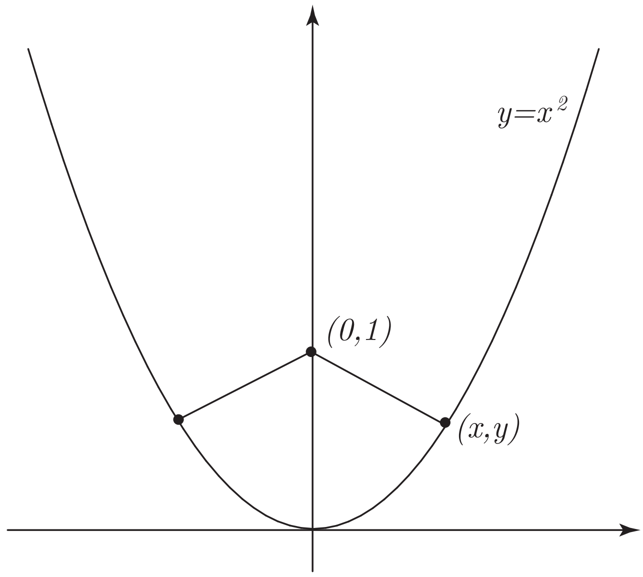

Figure 5.58: Closest points on $y = x^2$ to $(0, 1)$

Example 5.4.11 Find the points on the graph of $y = x^2$ which are closest to the point $(0, 1)$.

Solution. The situation can be visualized in **Figure 5.58**. The distance from (x_1, y_1) to (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

so we let $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (x, y)$ for (x, y) on the graph of $y = x^2$, and our objective is to minimize $d = \sqrt{(x-0)^2 + (y-1)^2} = \sqrt{x^2 + (x^2-1)^2}$ since $y = x^2$. This is equivalent to minimize

$$D(x) = x^2 + (x^2 - 1)^2 = x^4 - x^2 + 1,$$

which will be our objective function with domain $x \in (-\infty, \infty)$.

From

$$D'(x) = 4x^3 - 2x = 2x(2x^2 - 1), \quad D''(x) = 12x^2 - 2,$$

we get critical points at $x = 0, \pm\frac{\sqrt{2}}{2}$. Now, the first derivative test is easier to used, so that with the sign chart of $D'(x)$ given in **Figure 5.59**, we conclude that $D(x)$ takes its global minimum at $x = -\frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{2}$ since $D(-\frac{\sqrt{2}}{2}) = D(\frac{\sqrt{2}}{2})$. Hence, the points on the graph of $y = x^2$ which are closest to the point $(0, 1)$ are $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$ and $(\frac{\sqrt{2}}{2}, \frac{1}{2})$. ♠

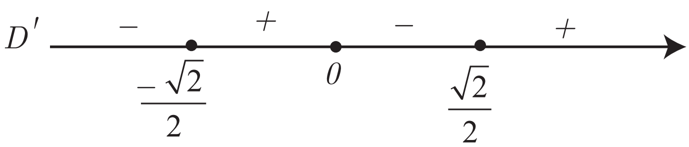

Figure 5.59: The sign chart of $D'(x) = 2x(2x^2 - 1)$

Guided Practice 5.4

1. A campus club is putting on a talent show in an attempt to raise money. In the past they have charged \$3 per person and have been able to bring

in 100 people. However it is widely believed that each increase of a dime in the cost of admission will result in one less person attending the show. If they actually would lose a person per dime increase in admission what should they set the price at to maximize their revenue?

2. You must design a drawer, which has a bottom but no top, you are told that the drawer MUST have a height of 1 foot, and a volume of 6 ft^3 . The wood on the front of the drawer is be made of Philippine teak which runs \$16 per ft^2. The remaining sides are made of a cabinet grade plywood which runs \$2 per ft^2. Your goal is to determine the width and the length of the drawer that will minimize the cost in material.

Exercises 5.4

1. The price function for a certain product is $u(x) = 8 - \frac{1}{4}x$. Find the level of production that results in the maximum revenue.

2. Suppose you are running a business for which the yearly rental fee is \$8,000, and it costs \$35 to produce a certain product, and the price function is $u(x) = 80 - 0.02x$. Find the level of production that results in the maximum yearly profit. Then find the corresponding unit price and yearly profit.

3. Suppose you are running a business for which it costs \$40 to produce a certain product, and the price function is $u(x) = 90 - 0.03x$. Find the level of production that results in the maximum profit. Then find the corresponding unit price and profit.

4. **(Fuel Economy)** The function $f(x) = 0.003x^3 - .555x^2 + 33.625x - 638.625$ comes from a cubic spline based on data representing the fuel economy $f(x)$ in miles per gallon for the average car driven at given speed x in miles per hour . The function is only valid for speeds ranging between 45 and 65 miles per hour. Based on this model what is the optimal speed at which to drive to save gas?

5. **(Optimal Pricing)** Suppose you sell a certain book on campus, and find that if you charge \$7 per book, then you can sell 1000 books per week, but if you charge \$6 per book, then you can sell 1200 books per week. Assume that the price function is a linear function, then determine the price (per book) that can maximize the weekly revenue.

6. **(Optimal Pricing)** You operate a ski-rental business at the skiing Mecca of Massanutten, Virginia. A short arrogant customer who oddly signed his receipt Napoleon Bonaparte has claimed to have rented skis all over the world and emphatically declares your current rate of 15

dollars per rental is outrageous and that you would make far more money if you dropped the price. After giving the pompous gentleman in the strange hat your shortest pair of skis you checked your records and determined that at the price of 15 dollars per rental you averaged 50 customers per day. Also you found in the wickedly popular weekly publication *Ski Rental Owners are Hot*, that typically for each dollar increase in price you can expect to lose on average one customer per day. Is the chubby little man correct? What should you charge?

7. Find two positive numbers whose sum is 64 and whose product is as large as possible.

8. Find two positive numbers whose product is 64 and whose sum is as small as possible.

9. If you want to plant a rectangular garden along one side of your house with a picket fence on the other three sides of the garden, find the dimensions of the largest garden possible if you only have 60 feet of fencing.

10. A rectangular garden of area 500 square feet is to be surrounded on three sides by a brick wall costing \$12 per foot and on one side by a fence costing \$6 per foot. Find the dimensions of the garden such that the cost of materials is minimized.

11. U.S. parcel post regulations state that packages must have length plus girth of no more than 108 inches. Find the dimensions of a cylindrical package of the largest volume that is mailable by parcel post.

12. If you want to construct a cylindrical can so as to hold a given volume V, how should the radius and height be chosen so as to minimize the surface area of the can, including the top and bottom?

13. If you want to construct an open cylindrical can so as to hold a given volume V, how should the radius and height be chosen so as to minimize the surface area of the can?

14. If you want to construct an open cylindrical can with $100\,\text{ft}^2$ of material, then how should the radius and height be chosen so as to maximize the volume of the can?

15. If you want to construct a closed cylindrical can with material $S\,\text{ft}^2$, then how should the radius and height be chosen so as to maximize the volume of the can?

16. Find the points on the graph of $y = x^2 + 1$ which are closest to the point $(0, 2)$.

17. Find the points on the graph of $y = 2 - x^2$ which are closest to the point $(0, -1)$.

Chapter 6

Integration

There are two important subjects in calculus: the differential calculus and the integral calculus. The materials covered in chapters 1 – 5 can be characterized as the differential calculus, where the derivatives are the central theme. Now, we are going to start the second important subject in calculus: the integral calculus, which can be used to find areas of plane regions bounded by functions and volumes of solids of revolution, among many other applications.

6.1 The Fundamental Theorem of Calculus

Question 1: *Do you know the formula for the area of a circle? Do you know how to derive it?*

Question 2: *If you throw a stone (horizontally) with a certain velocity, then can you locate the position of the stone?*

Probably you know the formula for the area of a circle, but don't know how to derive it. The study of these kinds of questions is related to *integration*. To explain the notion of integration, let's look at the functions shown in **Figure 6.1**.

We see that in each case it is easy to find the area bounded by the curve of the function $f(x)$ and the interval $[a, b]$ (meaning the area enclosed by the curve of $f(x)$, the vertical straight lines $x = a$ and $x = b$, and the x-axis). For example, the first function in Figure 6.1 is a constant, so the area is the

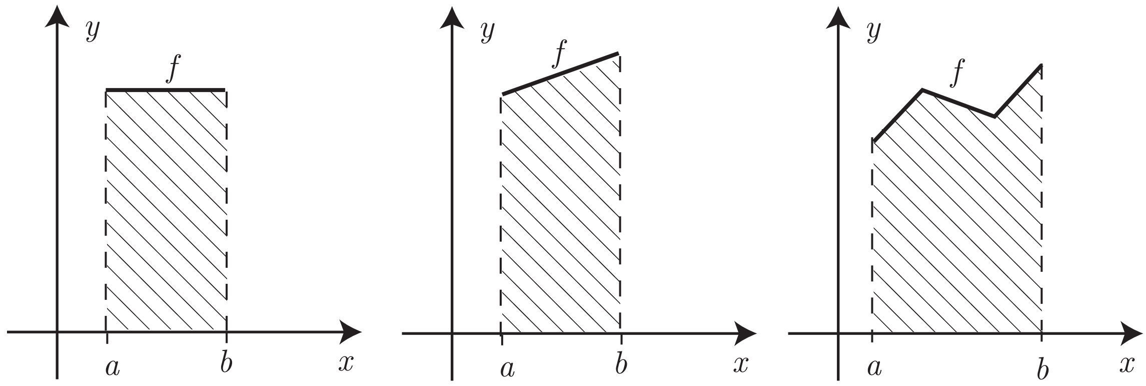

Figure 6.1: Areas under functions

area of the rectangle, given by

$$(\text{height})(\text{base}) = f(x)(b-a).$$

The area for the second function is the area of the trapezoid, which can be obtained easily by using the formula for the area of a trapezoid (or by using a rectangle and a triangle). For the area of the third function, it can be derived by dividing into several trapezoids and then finding their sum.

Question: *For a general function given in* **Figure 6.2***, how do we find the area bounded by the curve of* $f(x)$ *and the interval* $[a, b]$*?*

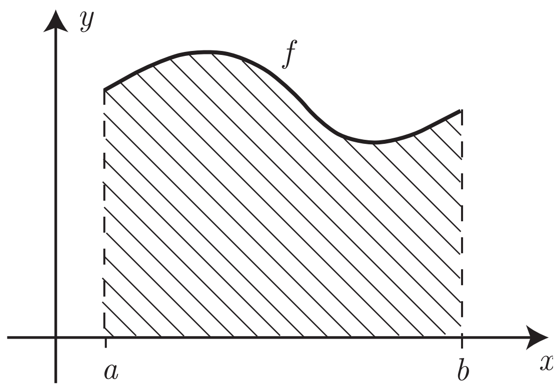

Figure 6.2: Area of a general function

The answer is that we cannot find such an area directly using what we have learned so far. Therefore, let's try to use an idea that has been very successful in differential calculus: *approximations*. For example, the slope of a tangent line couldn't be found directly, so we used the slopes of secant lines to approximate, which gave birth to differential calculus.

We will see that the following idea of approximation gives birth to the integral calculus. The idea can be seen from **Figure 6.3**.

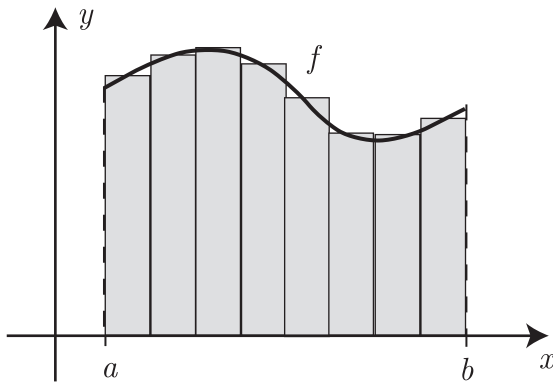

Figure 6.3: Approximate the area using areas of rectangles

That is, we cut the interval $[a, b]$ into many (say, billions of billions) smaller subintervals, and on each subinterval, we use a horizontal bar to approximate the original function. This gives very many small rectangles whose areas are easy to find (height times base). To approximate the area bounded by the function $f(x)$ and the interval $[a, b]$, we take a summation of the areas of these rectangles.

Since the size of each subinterval is tiny (say, as small as how thin a human hair is), the function $f(x)$ on a subinterval can be approximated by a horizontal bar, so that the area bounded by $f(x)$ over a subinterval and the area of the corresponding rectangle are almost the same. Therefore, the sum of the areas of these rectangles should be a very good approximation of the area bounded by the function $f(x)$ and the interval $[a, b]$.

Finally, to derive the area bounded by the function $f(x)$ and the interval $[a, b]$, we take a limit in the sense that the sizes of all subintervals go to zero, and it is plausible from geometry that this limit should be the area bounded by the function $f(x)$ and the interval $[a, b]$.

The key idea here is to use a **summation** to approximate, that explains why this study is called **integration**, meaning *putting things together*.

The following are some details for the idea just mentioned. To approximate the area bounded by the function $f(x)$ and the interval $[a, b]$ in Figure 6.2, we cut the interval $[a, b]$ into many smaller subintervals, shown in **Figure 6.4**, which is called a *partition*.

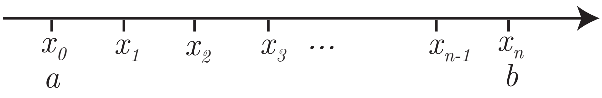

Figure 6.4: A partition of the interval $[a, b]$

We write $a = x_0$ and $b = x_n$ so that the ith subinterval is given by

$$[x_{i-1},\, x_i], \quad i = 1, 2, \cdots, n.$$

Note that the sizes of these intervals may be different. In the ith subinterval $[x_{i-1},\, x_i]$, we randomly select an x value, denoted by $x_i^\star$, and then use $f(x_i^\star)$ as the height so we can form a rectangle. See **Figure 6.5**.

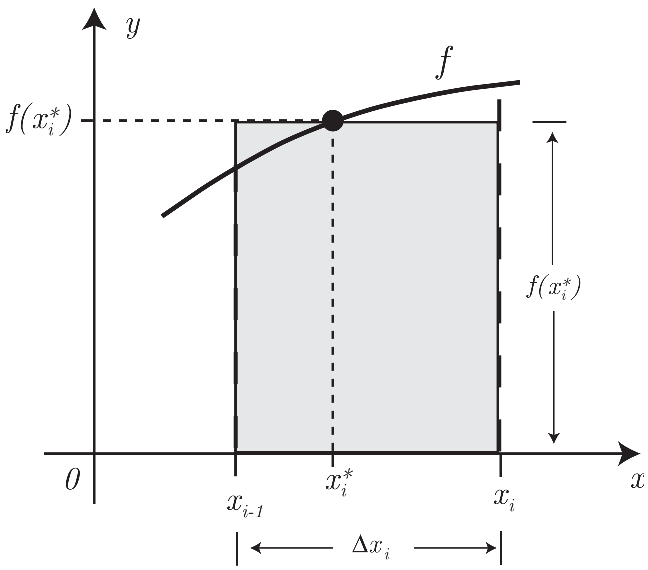

Figure 6.5: A rectangle on the ith subinterval $[x_{i-1},\, x_i]$

The area of the rectangle in Figure 6.5 is given by

$$\text{(height)(base)} = f(x_i^\star)(x_i - x_{i-1}) = f(x_i^\star)\Delta x_i,$$

where we define $\Delta x_i = x_i - x_{i-1}$, which gives the size of the ith subinterval.

Let this be done for all $i = 1, 2, \cdots, n$, so that the summation of the areas of those rectangles is given by

$$S_n = f(x_1^\star)\Delta x_1 + f(x_2^\star)\Delta x_2 + \cdots + f(x_n^\star)\Delta x_n, \tag{1.1}$$

which gives a good approximation of the area bounded by the curve of $f(x)$ and the interval $[a, b]$.

Next, we assume that as $n \to \infty$ (more and more subintervals), the sizes of all the subintervals go to zero. Then it is plausible from geometry that the following limit (if it exists)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [f(x_1^\star)\Delta x_1 + f(x_2^\star)\Delta x_2 + \cdots + f(x_n^\star)\Delta x_n] \tag{1.2}$$

should give the area bounded by the curve of $f(x)$ and the interval $[a, b]$.

To illustrate the idea and all the steps described above, and to check whether the idea works, we look at the following example.

Example 6.1.1 Use areas of rectangles to approximate the area bounded by the curve of $f(x) = 3x$ and the interval $[0, 1]$.

Solution. The geometry of $f(x) = 3x$ is a straight line $y = 3x$, given in **Figure 6.6**.

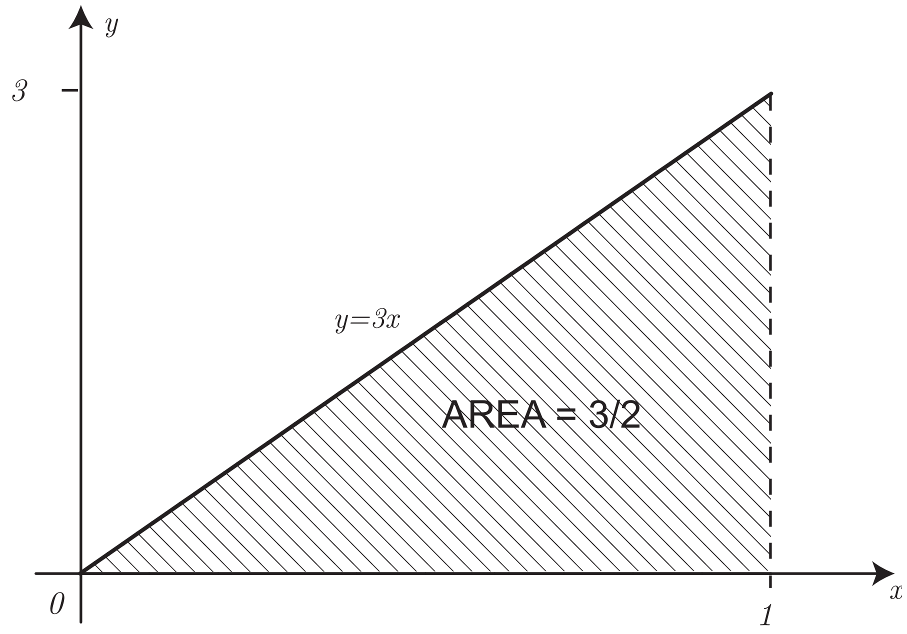

Figure 6.6: $f(x) = 3x$ forms a triangle

From Figure 6.6 we see that the area bounded by the curve of $f(x) = 3x$ on the interval $[0, 1]$ is the area of a triangle with base 1 and height 3, so the area is $\frac{(1)(3)}{2} = \frac{3}{2}$. The reason to use this simple example is that we know from geometry that the area must be $\frac{3}{2}$, so we can use this example to check whether the procedure of using summations of areas of rectangles described above gives the same answer. (Otherwise, if we use a complicated example whose area is unknown, then how can we *check* it?)

Now, to make things simple, we divide the interval $[0, 1]$ into n *equal* subintervals. See **Figure 6.7**. On each subinterval, there are many ways to select a height using the function values $f(x)$ on that subinterval, so we look at several cases.

Case 1. Select the highest height on each subinterval.

Now, on each subinterval, the height is obtained by evaluating the function $f(x) = 3x$ at the right-end point of the subinterval, that is, let $x_i^\star = x_i$. See Figure 6.7.

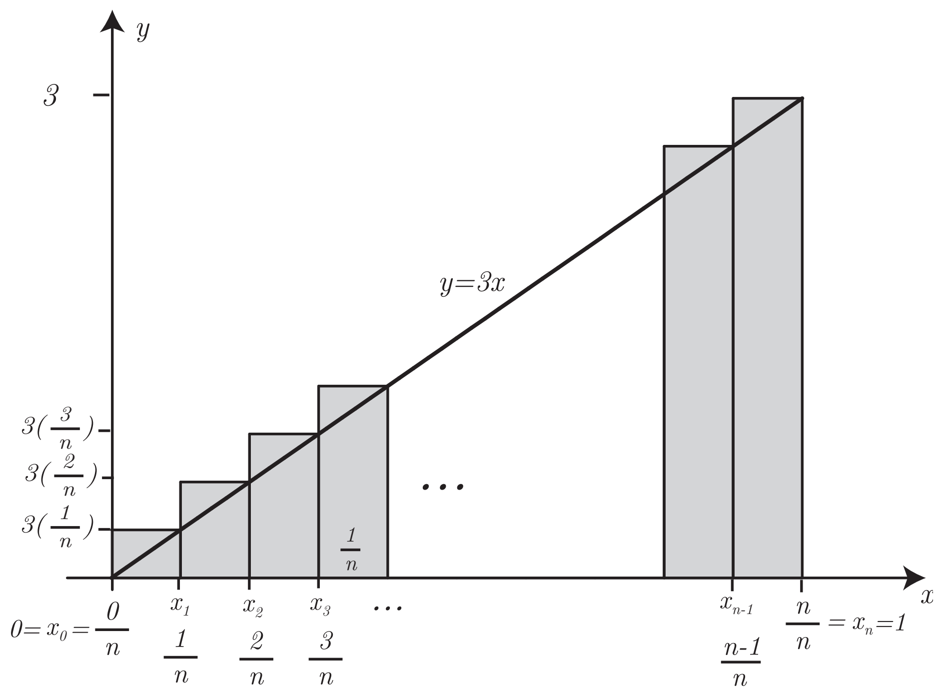

Figure 6.7: Select the highest height on each subinterval

Note that the base for each rectangle in Figure 6.7 is $\frac{1}{n}$, so the area of each rectangle in Figure 6.7 is given by:

on $[0, \frac{1}{n}]$, the height $= f(\frac{1}{n}) = 3 \cdot \frac{1}{n}$; the area $= 3 \cdot \frac{1}{n} \cdot \frac{1}{n}$,

on $[\frac{1}{n}, \frac{2}{n}]$, the height $= f(\frac{2}{n}) = 3 \cdot \frac{2}{n}$; the area $= 3 \cdot \frac{2}{n} \cdot \frac{1}{n}$,

$\cdots$

on $[\frac{n-1}{n}, \frac{n}{n}]$, the height $= f(\frac{n}{n}) = 3 \cdot \frac{n}{n}$; the area $= 3 \cdot \frac{n}{n} \cdot \frac{1}{n}$.

Following the procedure described above, we need to find the summation of the areas of these rectangles, so we have

$$\begin{aligned} S_n &= f(x_1^\star)\Delta x_1 + f(x_2^\star)\Delta x_2 + \cdots + f(x_n^\star)\Delta x_n \\ &= 3 \cdot \frac{1}{n} \cdot \frac{1}{n} + 3 \cdot \frac{2}{n} \cdot \frac{1}{n} + \cdots + 3 \cdot \frac{n}{n} \cdot \frac{1}{n} \\ &= \frac{3}{n^2}[1 + 2 + \cdots + n]. \end{aligned} \tag{1.3}$$

To find a way to sum $1 + 2 + \cdots + n$, let's rewrite $1 + 2 + \cdots + n$ in decreasing order and look at the following two rows,

$$\begin{cases} 1 & + & 2 & + & 3 & + & \cdots & + & (n-1) & + & n, \\ n & + & (n-1) & + & (n-2) & + & \cdots & + & 2 & + & 1. \end{cases} \tag{1.4}$$

Now, if we add these numbers *vertically*, then each column adds to $n+1$, so the sum of all the numbers in (1.4) must be $n \cdot (n+1)$ because there are n columns. Since the sum of the first row equals the sum of the second row (the only differences between the two rows are their orders), we see that each row adds to one half of the total. Thus,

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Therefore, (1.3) becomes

$$\begin{aligned} S_n &= \frac{3}{n^2}[1 + 2 + \cdots + n] = \frac{3}{n^2}\frac{1}{2}n(n+1) \\ &= \frac{3}{2}\frac{n+1}{n} = \frac{3}{2}(1 + \frac{1}{n}) = \frac{3}{2} + \frac{3}{2n}. \end{aligned} \tag{1.5}$$

Now, we let $n \to \infty$ (cut $[0, 1]$ into many many smaller subintervals), and obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}[\frac{3}{2} + \frac{3}{2n}] = \frac{3}{2} + \lim_{n\to\infty} \frac{3}{2n} = \frac{3}{2} + 0 = \frac{3}{2}. \tag{1.6}$$

Wow! It works! It gives exactly the same answer, $\frac{3}{2}$, that we got earlier from geometry.

The geometry explanation of $S_n = \frac{3}{2} + \frac{3}{2n}$ in (1.5) is that $\frac{3}{2}$ is the true area, and $\frac{3}{2n}$ is the extra area using rectangles, which looks like a stair. See **Figure 6.8**.

When n is increased without bound, the extra area of the stair is crushed to zero, so that the limit of S_n becomes $\frac{3}{2}$.

The above is done for Case 1. The other cases, such as selecting the lowest height on each subinterval or randomly selecting a height on each subinterval, can be done in the same way and give the same result, $\frac{3}{2}$. See exercises. ♠

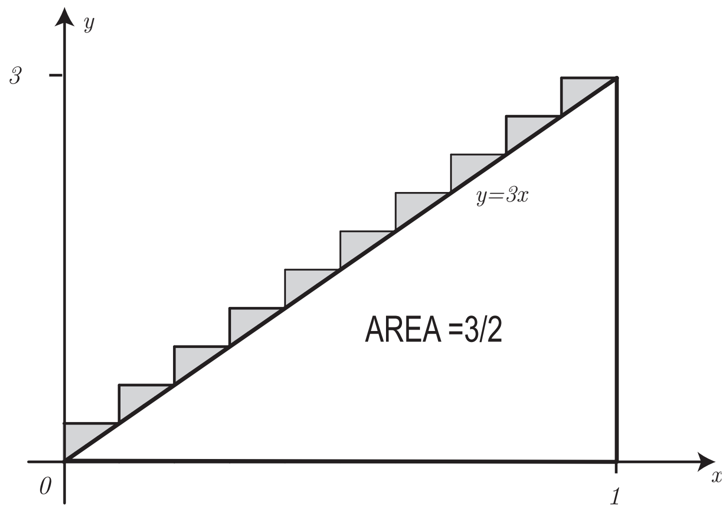

Figure 6.8: The extra area looks like a stair

Based on this example, we are going to generalize this idea and introduce the notion of an *integral*. To this end, we first introduce the $\sum$ (called "sigma") notation, which is a summation involving an index that increases by 1 every time. For example, $1+2+\cdots+200$ can be written as

$$1+2+\cdots+200=\sum_{i=1}^{200} i,$$

where the i is an index that increases by 1 every time, so that $\sum_{i=1}^{200} i$ gives the sum where the first item is 1, the second item is $1+1=2, \cdots$, and the last item is 200. Note that the index i can be replaced by any index, for example, $\sum_{k=1}^{200} k = \sum_{i=1}^{200} i$ because they give the same sum, namely, $1+2+\cdots+200$.

Example 6.1.2 Rewrite the following using $\sum$ notation.

1. $2+4+6+8+\cdots+600$.
2. $1-2+3^2-4^3+\cdots-10^9$.

Solution. 1. Now, all the numbers are even numbers, so expressions like $\sum i$ will not work because they also give odd numbers as i is increased by 1 every time. To find an index that increases by 1 every time, we note that $2=2\cdot 1$, $4=2\cdot 2$, $6=2\cdot 3$, $\cdots$, $600=2\cdot 300$, therefore, we can rewrite

$$2+4+6+8+\cdots+600=\sum_{l=1}^{300} 2l.$$

2. Note that every item is of the form of $\pm k^{k-1}$ and the $\pm$ is determined by the power $(k-1)$, so we can rewrite

$$1 - 2 + 3^2 - 4^3 + \cdots - 10^9 = \sum_{k=1}^{10} (-1)^{k-1} k^{k-1}.$$

♠

With this preparation, the summation S_n in (1.1) can be written as

$$S_n = f(x_1^\star)\Delta x_1 + f(x_2^\star)\Delta x_2 + \cdots + f(x_n^\star)\Delta x_n = \sum_{i=1}^{n} f(x_i^\star)\Delta x_i, \qquad (1.7)$$

which gives the summation of the areas of those rectangles. The summation (1.7) is called a **Riemann sum** to honor the mathematician Georg Bernhard Riemann (1826-1866), who pioneered the modern integration theory. With these notations, we are now ready to make the following definition.

Definition 6.1.3 *For a function $f(x)$ defined on an interval $[a,b]$, if the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i \qquad (1.8)$$

exists and is finite, then $f(x)$ is said to be **integrable** *on $[a,b]$ and the limit is called the* **integral** *(also called the* **definite integral***) of $f(x)$ on $[a,b]$, denoted by*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i. \qquad (1.9)$$

We require that the limit in Definition 6.1.3 be *finite* because an integral means an area in geometry, and of course we don't want an *infinite* area. The formula (1.9) indicates that the notation $\int_a^b f(x)dx$ is created *naturally* in such a way that in $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x_i$, we first replace $\lim_{n\to\infty} \sum_{i=1}^{n}$ by $\int_a^b$ to mean a summation over the interval $[a,b]$ because $\int$ is a stretched s (for summation). Then we replace $f(x_i^\star)$ by $f(x)$, and finally we replace Δx_i by dx. These can be illustrated using **Figure 6.9**.

We know that $f(x) = 3x$ is integrable on $[0,1]$, but from geometry, $f(x) = 3x$ is not integrable on $[0,\infty)$ because then the function couldn't bound a finite area. This means that domains of functions are important factors in determining whether the functions are integrable.

The solution of Example 6.1.1 indicates that the approximation procedure described above works, and this procedure is needed to develop the general theory of integrals. However, Example 6.1.1 also indicates that the approximation procedure is too tedious and too difficult to follow, even for

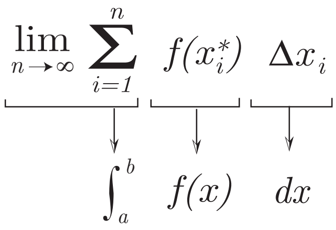

Figure 6.9: Explanation of the notation $\int_a^b f(x)dx$

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the simple linear function $f(x) = 3x$. So, you can imagine how difficult it is to carry out this approximation procedure for complicated functions.

Luckily, in this elementary calculus course, this approximation procedure is not our focus. That is, here, we will learn another *easier* way to derive integrals. But we point out again that this approximation procedure is needed in order to establish important results for the general theory of integrations, which are covered in advanced calculus courses.

Next, let's look at how to derive another *easier* method to evaluate integrals. First, note that the x in $\int_a^b f(x)\,dx$ can be replaced by other letters, for example, we have $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ because both give the same area bounded by the function f and the interval $[a,b]$, where the independent variable can be denoted by x or by t. This is similar to the $\sum$ notation where we can use any letter for the index to get the same result, for example, $\sum_{k=1}^{3} k = \sum_{j=1}^{3} j = 6$. But note that $\int_a^b f(b)\,db$ is not well-defined because b is a fixed value (the right-end point of the interval $[a,b]$) so b cannot be used also as an independent variable.

Now, let's consider a continuous function f on the interval $[a,b]$. Let $x \in [a,b]$, then $\int_a^x f(t)dt$ gives the area bounded by the function f and the interval $[a,x]$. See **Figure 6.10**.

Note that the x in $\int_a^x f(t)dt$ is a fixed number (the right-end point of the interval $[a,x]$), so we use a different letter, t, to denote the independent variable.

Accordingly,

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a,b], \tag{1.10}$$

defines a function, which gives the area bounded by the function f and the interval $[a,x]$. Next, we use some simple examples to indicate some important results.

Example 6.1.4 Find $F(x) = \int_0^x f(t)\,dt$ for $f(x) = 3x$ on $[0,1]$. Then find $F'(x)$ and $F(1) - F(0)$.

Solution. In geometry, the function f and the interval $[0,x]$ form a triangle

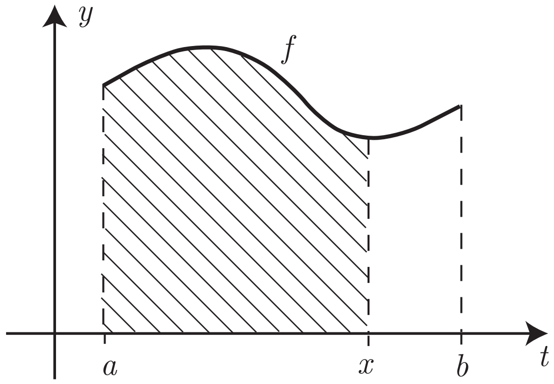

Figure 6.10: The area represented by $\int_a^x f(t)\,dt$

with base x and height $3x$, then its area is given by $\frac{(x)(3x)}{2} = \frac{3}{2}x^2$. Therefore,

$$F(x) = \int_0^x f(t)\,dt = \int_0^x 3t\,dt = \frac{3}{2}x^2.$$

Then,

$$F'(x) = \frac{d}{dx}\frac{3}{2}x^2 = 3x = f(x).$$

Next, we have $F(1) = \int_0^1 f(x)\,dx$, and $F(0) = 0$ because from geometry it is the area bounded by the function f and the interval $[0,0]$, which is just a line, and a line does not occupy any area. Therefore, we get

$$F'(x) = f(x), \quad \text{and} \quad \int_0^1 f(x)\,dx = F(1) - F(0). \tag{1.11}$$

♠

Example 6.1.5 Find $F(x) = \int_0^x f(t)\,dt$ for $f(x) = 3x + 2$ on $[0,1]$. Then find $F'(x)$ and $F(1) - F(0)$.

Solution. In geometry, the function f and the interval $[0,x]$ form a trapezoid, or compare to Example 6.1.4 the area of a rectangle with base x and height 2 is added, so the area bounded by f and the interval $[0,x]$ is given by $\frac{3}{2}x^2 + 2x$. Therefore,

$$F(x) = \int_0^x f(t)\,dt = \int_0^x (3t+2)\,dt = \frac{3}{2}x^2 + 2x.$$

Then,

$$F'(x) = \frac{d}{dx}(\frac{3}{2}x^2 + 2x) = 3x + 2 = f(x).$$

Next, we have $F(1) = \int_0^1 f(x)\,dx$ and $F(0) = 0$, so we get

$$F'(x) = f(x), \quad \text{and} \quad \int_0^1 f(x)\,dx = F(1) - F(0). \tag{1.12}$$

♠

The formula (1.11) in Example 6.1.4 (or (1.12) in Example 6.1.5) indicates a very interesting phenomenon: to find $\int_0^1 f(x)\,dx$, all we need is to find a function $F(x)$ such that $F'(x) = f(x)$, then $\int_0^1 f(x)\,dx = F(1) - F(0)$. This is much easier than going through the approximation procedure described above, so we ask

Question: *Does the formula (1.11) happen just by chance? or is it true in general?*

The good news is that it is true in general, and it was discovered by the mathematician Isaac Barrow (1630-1677), Isaac Newton's (1642-1727) teacher at Cambridge. We state the result as follows, whose verification will be given at the end of this section.

Theorem 6.1.6 (The Fundamental Theorem of Calculus) *Let $f(x)$ be continuous on $[a, b]$. If there is a function $F(x)$ such that F is continuous on $[a, b]$ and $F'(x) = f(x)$ on (a, b), then*

$$\int_a^b f(x)dx = F(b) - F(a) \overset{\text{notation}}{=} F(x)\Big|_a^b, \tag{1.13}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{1.14}$$

Also, for

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

one has

$$F'(x) = f(x), \text{ i.e., } \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad x \in (a, b). \tag{1.15}$$

♠

The notation $F(x)|_a^b$ means that you first plug b to x to get $F(b)$, then you plug a to x to get $F(a)$, and then you subtract $F(a)$ from $F(b)$.

Theorem 6.1.6 links the two most important notions in calculus: the *derivative* and the *integral*; that is why it is called the *fundamental theorem of calculus*. It indicates that integration and differentiation *cancel* each other (see (1.14)). Most importantly, it reduces the finding of $\int_a^b f(x)dx$ to the finding of a function $F(x)$ such that $F'(x) = f(x)$, which is much easier than finding $\int_a^b f(x)dx$ using the approximation procedure described above. The following are some examples.

Example 6.1.7 Check the formula (1.13) for $f(x) = 3x$ on $[0, 1]$.

Solution. As we did before that in geometry, the function $f(x) = 3x$ and the interval $[0, 1]$ form a triangle, whose area is given by $\frac{3}{2}$, thus $\int_0^1 3x\,dx = \frac{3}{2}$.

Next, to check with the formula (1.13), we need to find a function $F(x)$ such that $F'(x) = f(x) = 3x$. From our knowledge of derivatives, we know that $(x^2)' = 2x$, so that to get $3x$, we multiply by $\frac{3}{2}$ on both sides and obtain $(\frac{3}{2}x^2)' = 3x$. That is, if we let $F(x) = \frac{3}{2}x^2$, then $F'(x) = f(x)$. Now, from the formula (1.13) in the fundamental theorem of calculus, we get

$$\int_0^1 3x\,dx = F(1) - F(0) = \frac{3}{2}x^2\Big|_0^1 = \frac{3}{2} - 0 = \frac{3}{2},$$

which matches the answer from geometry. ♠

Example 6.1.1 and Example 6.1.7 deal with the same question, but the solution in Example 6.1.7 using $F'(x) = f(x)$ is *so easy* compared with Example 6.1.1 using approximations: all we need is to find $F(x)$ and then plug in values.

Example 6.1.8 Check the formula (1.13) for $f(x) = 5x$ on $[1, 3]$.

Solution. In geometry, the function $f(x) = 5x$ and the interval $[1, 3]$ form a trapezoid, whose area can be obtained to be 20 (for example, you can divide it into a triangle and a rectangle and then add the areas).

Next, similar to Example 6.1.7, we see that $(\frac{5}{2}x^2)' = 5x$. That is, if we let $F(x) = \frac{5}{2}x^2$, then $F'(x) = f(x)$. Now, from the formula (1.13) in the fundamental theorem of calculus, we get

$$\int_1^3 5x\,dx = F(3) - F(1) = \frac{5}{2}x^2\Big|_1^3 = \frac{45}{2} - \frac{5}{2} = \frac{40}{2} = 20,$$

which matches the answer from geometry. ♠

Again, Example 6.1.8 is solved easily using the fundamental theorem of calculus compared to how difficult it would be using the approximation procedure described above.

So far, we have always used a positive function so that in geometry $\int_a^b f(x)dx$ gives the area bounded by the curve of the function $f(x)$ and the interval $[a, b]$.

Question: *What does $\int_a^b f(x)dx$ mean in geometry if $f(x) < 0$?*

From **Figure 6.11**, we see that $\Delta x_i = x_i - x_{i-1}$ gives the size of the ith subinterval, so that Δx_i is always positive. Thus, the only difference in this case is that $f(x_i^\star) < 0$.

Therefore, when $f(x) < 0$, $f(x_i^\star) \cdot \Delta x_i$ gives the *negative* of the area of the ith rectangle, so that the summation $\sum_{i=1}^n f(x_i^\star) \cdot \Delta x_i$ gives the *negative*

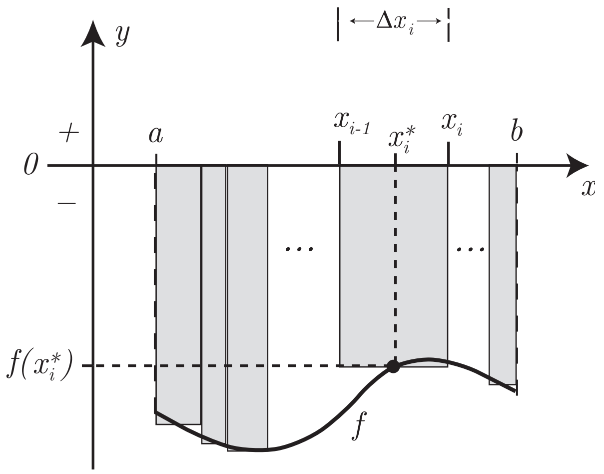

Figure 6.11: The geometry of $\int_a^b f(x)dx$ when $f(x) < 0$

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of the summation of the areas of the rectangles. Note that when we speak of "area", we mean a nonnegative value. Now, the sum of the areas of the rectangles approximate the area bounded by the curve of $f(x)$ and the interval $[a, b]$, therefore,

> When $f(x) < 0$, $\int_a^b f(x)dx$ gives the *negative* of the area bounded by the curve of $f(x)$ and the interval $[a, b]$.

The following is such an example.

Example 6.1.9 Find $\int_0^1 f(x)dx$ for $f(x) = -3x$.

Solution. Now, on the interval $[0, 1]$, we have $f(x) < 0$ (except at $x = 0$). See **Figure 6.12**.

The curve of $f(x)$ is now below the x-axis, and this curve and the interval $[0, 1]$ bound an area, given by $\frac{3}{2}$ (again, when we speak of *area*, we mean a nonnegative value).

Next, you can check that $(-\frac{3}{2}x^2)' = -3x$. That is, if we let $F(x) = -\frac{3}{2}x^2$, then we get $F'(x) = f(x)$. Now, from the fundamental theorem of calculus, we have

$$\int_0^1 (-3x)\, dx = F(1) - F(0) = -\frac{3}{2}x^2\Big|_0^1 = -\frac{3}{2} - 0 = -\frac{3}{2}.$$

Since the area bounded is $\frac{3}{2}$, we see that $\int_0^1 (-3x)\, dx = -\frac{3}{2}$ gives the negative of the area bounded by $f(x)$ and the interval $[0, 1]$. ♠

In general, a function may be positive or negative on different intervals, so we ask

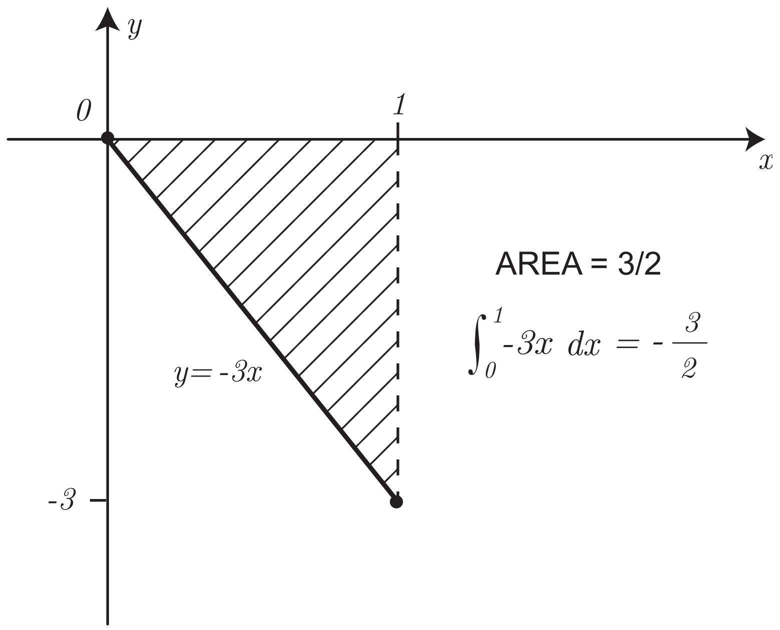

Figure 6.12: The geometry of $f(x) = -3x$ on $[0, 1]$

Question: *What does $\int_a^b f(x)dx$ mean for a general function $f(x)$?*

From **Figure 6.13**, we infer that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx,$$

because when doing approximations using the areas of the corresponding rectangles, the summations can be arranged to take place on the intervals $[a, c]$, $[c, d]$, and $[d, b]$, respectively.

Now, from the second graph in Figure 6.13, (where the numbers give the corresponding areas bounded by the function and the intervals), we obtain

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)\,dx = (3) + (-2) + (1) = 2.$$

Therefore, for a general function $f(x)$, the integral $\int_a^b f(x)dx$ means that we subtract the areas below the x-axis from the areas above the x-axis. If $\int_a^b f(x)dx > 0$, then there are more areas above the x-axis than below the x-axis; if $\int_a^b f(x)dx < 0$, then there are more areas below the x-axis than above the x-axis.

Verification of the fundamental theorem of calculus (Theorem 6.1.6).

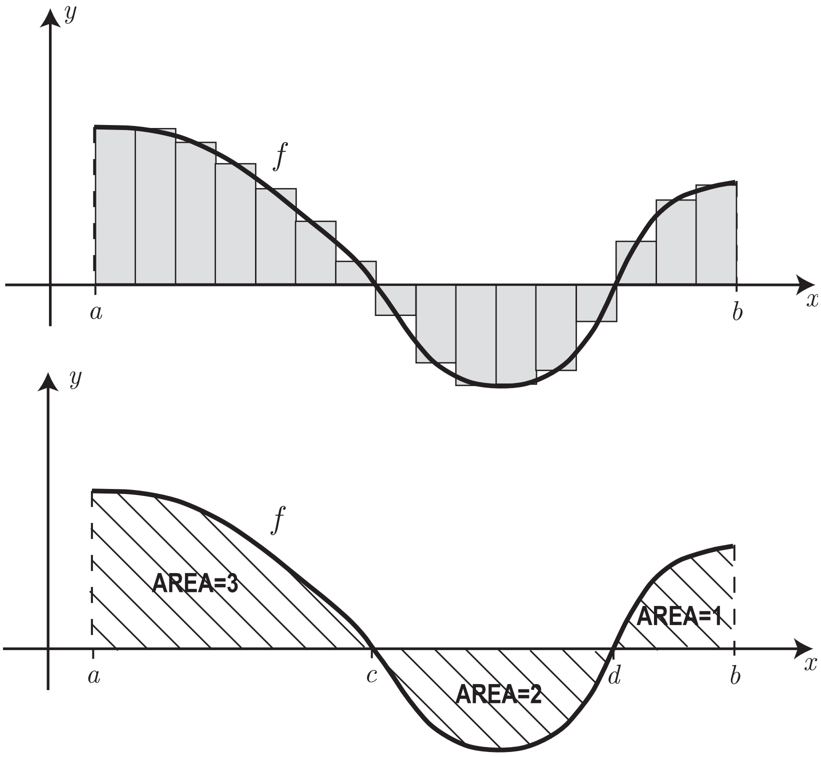

Figure 6.13: The geometry of $\int_a^b f(x)dx$ for a general $f(x)$

First, let's use the definition of derivative to find $\frac{d}{dx}\int_a^x f(t)\,dt$. So we start with

$$\lim_{h\to 0}\frac{1}{h}[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt] = \frac{1}{h}\int_x^{x+h} f(t)\,dt, \tag{1.16}$$

where, to make things easy, we assumed $h > 0$ and the geometry is used to cancel the area on the interval $[a, x]$. To find $\int_x^{x+h} f(t)\,dt$, which is the area bounded by f and the interval $[x, x+h]$, we look at a typical case, shown in **Figure 6.14**.

In Figure 6.14, if you use $f(x)$ as a height and $(x+h)-x = h$ as a base, then the area of the corresponding rectangle is less than the area given by $\int_x^{x+h} f(t)\,dt$; if you use $f(x+h)$ as a height and $(x+h)-x = h$ as a base, then the area of the corresponding rectangle is bigger than the area given by $\int_x^{x+h} f(t)\,dt$. Therefore, if you move the height from $f(x)$ to $f(x+h)$, then there must be a point $c \in (x, x+h)$ such that the area of the rectangle with height $f(c)$ and base $(x+h)-x = h$ equals the area given by $\int_x^{x+h} f(t)\,dt$. That is, there exists $c \in (x, x+h)$ such that

$$\int_x^{x+h} f(t)\,dt = f(c)h.$$

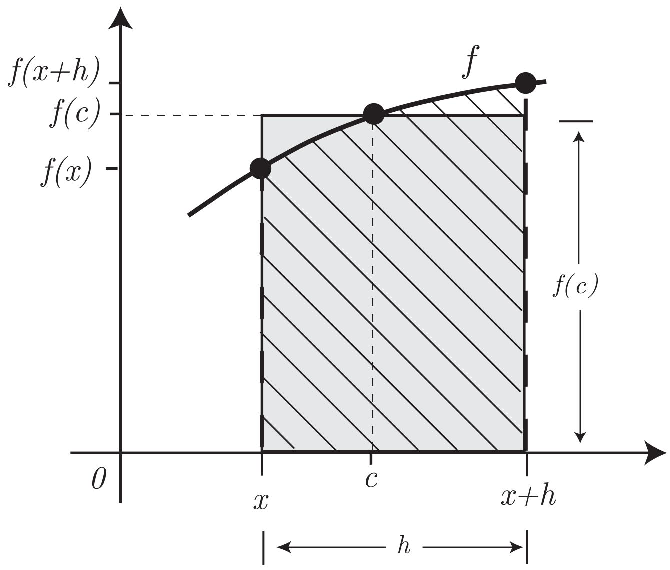

Figure 6.14: A typical case for $\int_x^{x+h} f(t)\,dt$

Then, (1.16) becomes

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0}\frac{1}{h}f(c)h = \lim_{h\to 0} f(c). \tag{1.17}$$

Now, as $h \to 0$, we have $c \to x$, thus, using the continuity of f, (1.17) becomes

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x), \tag{1.18}$$

which is (1.15).

To verify (1.13), we let $F(x)$ be any function with $F'(x) = f(x)$ and define $D(x) = \int_a^x f(t)\,dt - F(x)$, $x \in [a,b]$, which is continuous on $[a,b]$. From

$$\begin{aligned} D'(x) &= [\int_a^x f(t)\,dt - F(x)]' = \frac{d}{dx}\int_a^x f(t)\,dt - F'(x) \\ &= f(x) - f(x) = 0,\ x \in (a,b), \end{aligned}$$

and using geometry (or the mean value theorem in Chapter 5), we conclude that $D(x)$ must be a constant on $[a,b]$, that is, $D(x) = C$ for some constant C. Let $x = a$, we obtain $D(a) = C$, or $\int_a^a f(t)\,dt - F(a) = -F(a) = C$ since $\int_a^a f(t)\,dt = 0$. Thus, $\int_a^x f(t)\,dt - F(x) = C = -F(a)$, or $\int_a^x f(t)\,dt = F(x) - F(a)$. Therefore, if we let $x = b$, we get $\int_a^b f(t)\,dt = F(b) - F(a)$, that is,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This completes the verification of the fundamental theorem of calculus (Theorem 6.1.6). ♠

Guided Practice 6.1

1. For the given function $F(x)$, first set $f(x) = F'(x)$, and then for this $f(x)$ use the fundamental theorem of calculus to find $\int_1^2 f(x)\,dx$.

(a) $F(x) = \frac{1}{3}x^3$
(b) $F(x) = \frac{1}{5}x^5$
(c) $F(x) = \ln(x)$
(d) $F(x) = \frac{1}{2}e^{2x}$
(e) $F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$

2. Check the fundamental theorem of calculus for $f(x) = 2x + 1$ on $[0, 1]$.

Exercises 6.1

1. For $f(x) = x^2$ on $[0, 2]$, let the partition be such that the interval $[0, 2]$ is divided into 3 equal subintervals, and let $x_i^\star = \frac{1}{2}[x_{i-1} + x_i]$ (middle point of the subinterval). Then approximate $\int_0^2 f(x)dx$ using the corresponding Riemann sum.

2. Evaluate the following.

(a) $\sum_{i=0}^{3} 2^i$.
(b) $\sum_{i=0}^{3}(-1)^{i+1}2^i$.
(c) $\sum_{i=0}^{3}(-1)^{i+2}2^{2i}$.
(d) $\sum_{i=0}^{3}(-2)^{2i}$.

3. Rewrite the following using $\sum$ notation.

(a) $1 + 3 + 5 + 7 + \cdots + 611$.
(b) $\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \cdots + \frac{x^9}{10}$.

4. Complete Example 6.1.1 by finishing the following two cases.

(a) Select the lowest height on each subinterval.
(b) Randomly select a height on each subinterval. Then make an argument based on the other two cases (with the highest and lowest heights).

5. For the following functions, use geometry to find the areas bounded by the functions and the corresponding domains.

(a) $f(x) = 2x, \quad x \in [0, 2]$.
(b) $f(x) = 4x + 7, \quad x \in [1, 5]$.
(c) $f(x) = 4x + 7, \quad x \in [-1, 3]$.

6. For the following functions, use the fundamental theorem of calculus to find the areas bounded by the functions and the corresponding domains. Then compare the results with geometry.

(a) $f(x) = 2x, \quad x \in [0, 2]$.
(b) $f(x) = 4x + 7, \quad x \in [1, 5]$.
(c) $f(x) = 4x + 7, \quad x \in [-1, 3]$.

7. For $f(x) = 6x + 5$ on $[2, 4]$, find $\int_2^4 f(x)dx$ using geometry. Next, find $F(x) = \int_0^x f(t)dt$ (using geometry) and $F'(x)$. Then find $F(4) - F(2)$ and compare with $\int_2^4 f(x)dx$.

8. Can you make an argument using geometry that $F(x) = \int_0^x f(t)dt$ is a continuous function when f is a continuous function? Can you verify the result?

6.2 Antiderivatives

The fundamental theorem of calculus and the examples from the previous section indicate that to find $\int_a^b f(x)dx$, we can just find a function $F(x)$ such that $F'(x) = f(x)$. That is, if we can find such a function $F(x)$, then the fundamental theorem of calculus provides an easier way to find $\int_a^b f(x)dx$, as the procedure of using summations of areas of rectangles is avoided.

This gives a *reason* to raise the following

Question: *For a given function $f(x)$, how do we find a function $F(x)$ such that $F'(x) = f(x)$?*

Note that this is in the *reverse* direction of taking derivatives. That is, for example, to take the derivative of x^2, we have

$$(x^2)' = 2x.$$

But now, for a given function, such as $2x$, we do not take the derivative of $2x$, instead, we look for a function, denoted by "?", such that

$$(?)' = 2x. \tag{2.1}$$

Note that in (2.1) we can take "?" to be x^2 or $x^2 + C$ for any constant C. For example, we have $(x^2 + 1)' = 2x$, $(x^2 - 3.78)' = 2x$, etc. In this sense, we call this reverse direction of taking derivatives as taking *anti*-derivatives, and call $x^2 + C$ *antiderivatives* of $2x$. Accordingly, we have the following definition.

Definition 6.2.1 *If $F'(x) = f(x)$, then we use*

$$\int f(x)\,dx = F(x) + C \tag{2.2}$$

to denote **antiderivatives** *(also called* **indefinite integrals***) of $f(x)$, where C is any constant. This is the family of all functions whose derivatives equal $f(x)$.*

Our order of presentation in this chapter is to present the fundamental theorem of calculus first, and then present the notion of antiderivatives. This way, it is very *natural* to see *why* the symbol $\int f(x)\,dx$ is used for antiderivatives, because it reminds us that what we are doing is related to finding $\int_a^b f(x)\,dx$. According to the fundamental theorem of calculus, if we can find $\int f(x)\,dx$, then we just evaluate $\int f(x)\,dx$ at a and b, that is, $(\int f(x)\,dx)\big|_a^b$, which gives $\int_a^b f(x)\,dx$. In formula, this means

$$\int_a^b f(x)\,dx = \Big(\int f(x)\,dx\Big)\Big|_a^b.$$

If we plug $F'(x) = f(x)$ into (2.2), then we get

$$\int F'(x)\,dx = F(x) + C, \tag{2.3}$$

which means derivative and antiderivative cancel, that is, antiderivative does *anti* derivative, explaining why this subject is called *antiderivatives*. Note that (2.2) or (2.3) indicates that we can find a function if we know its derivative (provided the antiderivative can be carried out).

Now we look at some examples.

Example 6.2.2 Find $\int f(x)\,dx$ for

1. $f(x) = 0$.
2. $f(x) = 1$.
3. $f(x) = x$.
4. $f(x) = x^2$.
5. $f(x) = \sqrt{x} = x^{1/2}$.
6. $f(x) = \frac{1}{x} = x^{-1}$.
7. $f(x) = e^{3x}$.

Solution. 1. We know that the derivative of a constant is zero, so we get

$$\int 0\,dx = C, \quad (C \text{ denotes any constant}).$$

2. We know that $x' = 1$, so we get

$$\int dx = x + C.$$

3. We know that $(x^2)' = 2x$, so that to get x, we divide by 2 on both sides and obtain $(\frac{x^2}{2})' = x$. So we get

$$\int x\,dx = \frac{x^2}{2} + C.$$

4. Similar to the previous one, we know that $(\frac{x^3}{3})' = x^2$, so we get

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

5. Based on the previous cases, it seems all we need is to raise the power of x by 1 and then divide by the resulting power of x. This also works here because we can check that $(\frac{x^{3/2}}{(3/2)})' = x^{1/2}$, so we get

$$\int x^{1/2}\,dx = \frac{x^{3/2}}{(3/2)} + C = \frac{2}{3}x^{3/2} + C.$$

6. What we just mentioned above doesn't work for x^{-1} because after raising the power of x^{-1} by 1 the resulting power would be zero, and we cannot divide by zero. In this case, we know that

$$(\ln|x|)' = \frac{1}{x},$$

so we obtain

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

7. We know that $(e^{3x})' = 3e^{3x}$, so that to get e^{3x}, we divide by 3 on both sides and obtain $(\frac{e^{3x}}{3})' = e^{3x}$. So we get

$$\int e^{3x}\,dx = \frac{e^{3x}}{3} + C.$$

♠

Based on these examples, we obtain the following property using the power rule.

Property 6.2.3 *The following are true.*

1.

$$\int x^r\,dx = \begin{cases} \frac{x^{r+1}}{r+1} + C, & r \neq -1, \\ \ln|x| + C, & r = -1. \end{cases} \tag{2.4}$$

2. $\displaystyle\int e^{kx}\,dx = \frac{e^{kx}}{k} + C,$ (k is a nonzero constant). ♠

Note that to check the formula $\int f(x)\,dx = F(x) + C$, all we need is to take the derivative of $F(x) + C$ and see if it equals $f(x)$. For example, $\int x^2\,dx = \frac{x^3}{3} + C$ is true because the derivative of $\frac{x^3}{3} + C$ equals x^2. Using this idea, we have (see exercises) the following property.

Property 6.2.4 *The following are true.*

1. $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx,$ k is a constant.

2. $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$

3. If f has derivatives, then $\displaystyle\int f'(x)\,dx = f(x) + C$ *for some constant C.*

♠

Example 6.2.5 Find $\int (x^{-3} + 7e^{5x} + \frac{4}{x})\,dx$.

Solution. Using properties 6.2.3 – 6.2.4, we get

$$\begin{aligned} \int (x^{-3} + 7e^{5x} + \frac{4}{x})\,dx &= \frac{x^{-2}}{-2} + 7\frac{e^{5x}}{5} + 4\ln|x| + C \\ &= -\frac{1}{2}x^{-2} + \frac{7}{5}e^{5x} + 4\ln|x| + C. \end{aligned}$$

♠

Remark 6.2.6 The ideas used in Example 6.2.2 (such as how $\int \sqrt{x}\,dx = \frac{x^{3/2}}{(3/2)} + C$ was derived) can also be used in other cases. For example, to find $\int \sqrt{5x+2}\,dx$, we can regard $5x + 2$ as "one variable" such as z and use $\int \sqrt{z}\,dz = \frac{z^{3/2}}{(3/2)} + C$ to try a derivative on $\frac{z^{3/2}}{(3/2)} = \frac{(5x+2)^{3/2}}{(3/2)}$. Then we find that we get an extra 5, so we divide by 5 and obtain

$$\int \sqrt{5x+2}\,dx = \frac{(5x+2)^{3/2}}{(3/2)5} + C = \frac{2}{15}(5x+2)^{3/2} + C.$$

Also, for example, you can use the same idea to obtain

$$\int \frac{1}{3x+5} = \frac{\ln(3x+5)}{3} + C.$$

♠

For linear functions (straight lines), the fundamental theorem of calculus can be checked using geometry as the areas bounded by the straight lines and their corresponding domains. Next, we will use the fundamental theorem of calculus to find areas or integrals for more complicated functions, where the answers cannot be checked using geometry.

Example 6.2.7 Find the area bounded by $f(x) = x^2 + x$ and the interval $[0, 2]$.

Solution. The function $f(x) = x^2 + x$ on the interval $[0, 2]$ is positive, so it bounds an area. Now, the function $f(x) = x^2 + x$ is nonlinear (not a straight line), so the geometry will not help. Thus, we need to use the fundamental theorem of calculus and properties 6.2.3 – 6.2.4, which give

$$\int_0^2 (x^2 + x)\,dx = (\frac{x^3}{3} + \frac{x^2}{2})\Big|_0^2 = (\frac{8}{3} + 2) - 0 = \frac{14}{3}.$$

♠

Note that in Example 6.2.7 we do not add a constant C because if we do, it just cancels out:

$$(\frac{x^3}{3} + \frac{x^2}{2} + C)\Big|_0^2 = (\frac{8}{3} + 2 + C) - (0 + C) = \frac{14}{3},$$

which gives the same answer. Therefore, the constant C is needed for antiderivatives (also called indefinite integrals) to denote a family of functions, but not needed for integrals (also called definite integrals).

Example 6.2.8 Find the area bounded by $f(x) = \frac{1}{x}$ and the interval $[1, e]$.

Solution. We have

$$\int_1^e \frac{1}{x}\,dx = \ln|x|\Big|_1^e = \ln e - \ln 1 = 1 - 0 = 1.$$

♠

In general, a function may be positive somewhere and negative somewhere, so the integration gives how much is left after cancel the positive and negative areas.

Example 6.2.9 Find the following.

1. $\int_0^1 (7x^{-2/3} + \sqrt{x} - 32)\,dx$.
2. $\int_0^1 (x^2 + e^{3x})\,dx$.
3. $\int_1^e (x^2 - 9 + \frac{3}{x})\,dx$.

Solution. 1. We have

$$\begin{aligned}\int_0^1 (7x^{-2/3} + \sqrt{x} - 32)\,dx &= \Big(7\frac{x^{1/3}}{(1/3)} + \frac{x^{3/2}}{(3/2)} - 32x\Big)\Big|_0^1 \\ &= (21x^{1/3} + \frac{2}{3}x^{3/2} - 32x)\Big|_0^1 \\ &= (21 + \frac{2}{3} - 32) - 0 = -\frac{31}{3}.\end{aligned}$$

The geometry explanation is that for the function $7x^{-2/3}+\sqrt{x}-32$ on the interval $[0,1]$, there are more areas below the x-axis than above the x-axis.

2. We have

$$\begin{aligned}\int_0^1 (x^2 + e^{3x})\,dx &= \Big(\frac{x^3}{3} + \frac{e^{3x}}{3}\Big)\Big|_0^1 \\ &= (\frac{1}{3} + \frac{e^3}{3}) - (\frac{1}{3}) = \frac{e^3}{3}.\end{aligned}$$

3. We have

$$\begin{aligned}\int_1^e (x^2 - 9 + \frac{3}{x})\,dx &= (\frac{x^3}{3} - 9x + 3\ln|x|)\Big|_1^e \\ &= (\frac{e^3}{3} - 9e + 3\ln e) - (\frac{1}{3} - 9 + 3\ln 1) \\ &= \frac{e^3}{3} - 9e + 3 - \frac{1}{3} + 9 = \frac{e^3}{3} - 9e + \frac{35}{3}.\end{aligned}$$

♠

The following result gives another way to understand $\ln x$.

Example 6.2.10 Find the area bounded by $f(t) = \frac{1}{t}$ and the interval $[1, x]$.

Solution. We have

$$\int_1^x \frac{1}{t}\,dt = \ln|t|\Big|_1^x = \ln x - \ln 1 = \ln x - 0 = \ln x.$$

That is, in geometry, $\ln x$ means the area bounded by $f(t) = \frac{1}{t}$ and the interval $[1, x]$. ♠

Remark 6.2.11 Referring to Remark 1.3.3 in Chapter 1, to present a rigorous treatment for exponential and logarithmic functions, we should study integration first and then use the result of Example 6.2.10 to define the natural logarithmic function as

$$\ln x = \int_1^x \frac{1}{t}\,dt,$$

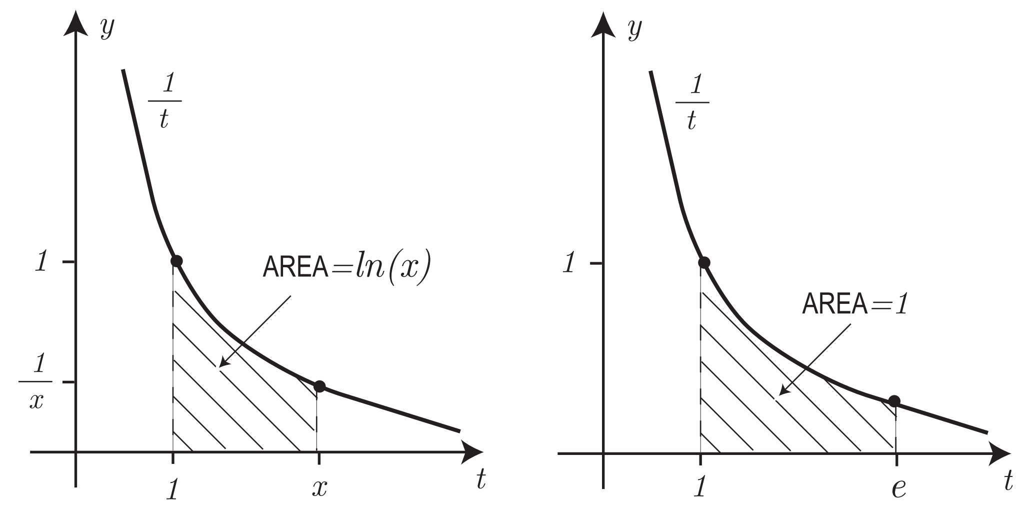

Figure 6.15: Definition of $\ln x$ and e

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and then use the result of Example 6.2.8 to define e to be the value such that

$$\ln e = \int_1^e \frac{1}{t}\, dt = 1.$$

See **Figure 6.15**. Then the exponential function with base e is defined such that $y = e^x$ if and only if $\ln y = x$. Based on these, all the identities we have seen before concerning e^x and $\ln x$ can be proved *using integral calculus*. Then, numbers such as $1.32^{0.53}$ is understood to be $e^{0.53 \ln 1.32}$, which can now be explained using integrals. But again, presenting a simple treatment for exponential and logarithmic functions in Chapter 1 is appropriate for this elementary calculus course. ♠

Next, let's look at the role the constant C plays in antiderivatives. For example, we have

$$\int 2x\, dx = x^2 + C.$$

In geometry, the curve of x^2 is a parabola passing through $(0, 0)$, shown in **Figure 6.16**.

If we let $C = 1$, then the curve of $x^2 + 1$ is also a parabola that can be obtained by moving the curve of x^2 *up* by 1 unit. Similarly, $x^2 - 2.54$ can be obtained by moving the curve of x^2 *down* by 2.54 units. That is, if we let C take *all* real values, then in geometry, the family of antiderivatives $x^2 + C$ give many (in fact, uncountably many) disjoint curves (two curves are said to be *disjoint* if they do not cross each other), and these disjoint curves *cover the entire xy-plane.*

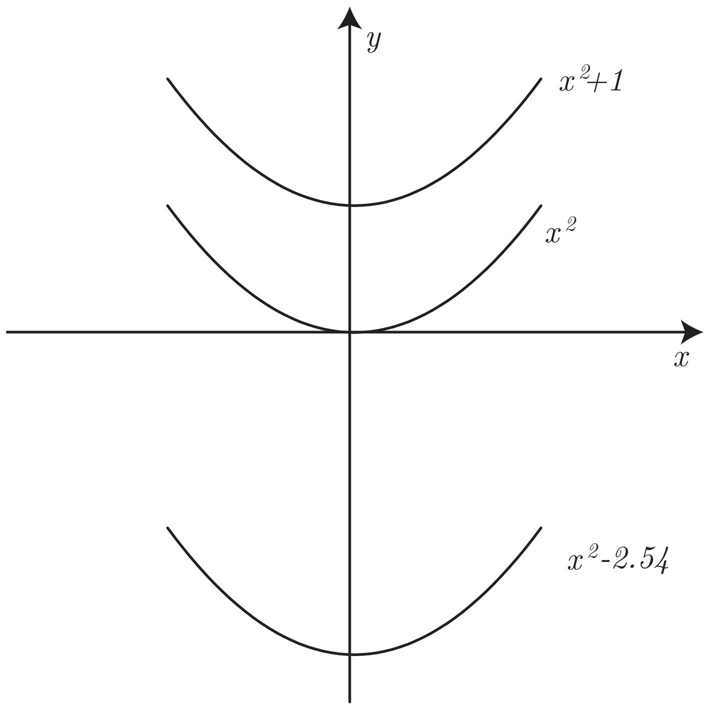

Figure 6.16: The geometry of $\int 2x\,dx = x^2 + C$

Consequently, if we specify any point $(x_0,\, y_0)$ in the xy-plane, then from geometry there must be a unique curve from the family of curves of antiderivatives $x^2 + C$ that passes through the point $(x_0,\, y_0)$. To find such a curve, all we need is to solve for a particular constant C such that when $x = x_0$, the function value is y_0. That is, we need to solve C from

$$(x_0)^2 + C = y_0.$$

Example 6.2.12 Find the function $f(x)$ such that $f'(x) = 2x$ and $f(x)$ passes through the point $(1, 5)$.

Solution. The condition $f'(x) = 2x$ means that $f(x)$ is from the antiderivatives of $f'(x) = 2x$. Next, the condition that $f(x)$ passes through the point $(1, 5)$ means that $f(1) = 5$. Thus, from

$$\int 2x\,dx = x^2 + C,$$

we know that there is a special C such that $f(x) = x^2 + C$ satisfies $f(1) = 5$. That is,

$$(1)^2 + C = 5, \quad \text{or } \ C = 5 - 1 = 4.$$

Thus, the function $f(x)$ we want to find is given by

$$f(x) = x^2 + 4.$$

♠

Example 6.2.13 Find the function $f(x)$ such that $f'(x) = 2x^3 - 5$ and $f(-1) = -4$.

Solution. We first have

$$\int (2x^3 - 5)\, dx = \frac{1}{2}x^4 - 5x + C,$$

then we solve C such that

$$\frac{1}{2}(-1)^4 - 5(-1) + C = -4, \quad \text{or } C = -\frac{19}{2}.$$

Thus, the function $f(x)$ we want to find is given by

$$f(x) = \frac{1}{2}x^4 - 5x - \frac{19}{2}.$$

♠

In a similar way, we can carry out the following applications, such as finding a cost function if given the marginal cost function; finding position of a moving object if given the velocity of the moving object, since they all fall in the same structure: find a function if we know its derivative, which is exactly what the antiderivative will do.

Example 6.2.14 Given a marginal cost function $C'(x) = 3x^2 - 60x + 500$, find

1. the cost function $C(x)$.
2. the increase in cost if the production is raised from 20 to 30 units.

Solution. 1. We know that $C(x)$ is from the antiderivatives of $C'(x) = 3x^2 - 60x + 500$. Thus, from

$$\int 3x^2 - 60x + 500\, dx = x^3 - 30x^2 + 500x + C,$$

we know that there is a special C such that

$$C(x) = x^3 - 30x^2 + 500x + C.$$

Evaluate at $x = 0$, we obtain $C = C(0)$, which is the cost of producing no products, such as the rental fee for runing a certain business. $C(0)$ is called the *fixed cost*, and can be obtained in a particular application. Therefore, the cost function is given by

$$C(x) = x^3 - 30x^2 + 500x + C(0).$$

2. The increase in cost if the production is raised from 20 to 30 units is given by $C(30) - C(20)$. From $C(x) = x^3 - 30x^2 + 500x + C(0)$, we get

$$C(30) - C(20) = (30^3 - 30 \cdot 30^2 + 500 \cdot 30) - (20^3 - 30 \cdot 20^2 + 500 \cdot 20) = 9000.$$

♠

Using the same idea, we can find revenue and profit functions if we are given their marginal functions. Next, recall that if $p(t)$ denotes the position of a moving object at time t, then its velocity is given by $p'(t)$ and its acceleration is given by $p''(t)$. So that we can find $p(t)$ if given $p'(t)$ and find $p'(t)$ if given $p''(t)$.

Example 6.2.15 Assume that you throw a stone (horizontally) with a velocity of $8 - t$ (ft/sec), where $0 \leq t \leq 8$ is measured in seconds. Then find the position of the stone at time t (that is, how far it is away from you).

Solution. Let $p(t)$ be the position of the stone at time t, then $p'(t) = 8 - t$ and $p(0) = 0$. We know that $p(t)$ is from the antiderivatives of $p'(t) = 8 - t$, thus, from

$$\int (8 - t)\, dt = 8t - \frac{t^2}{2} + C,$$

we know that there is a special C such that $p(t) = 8t - \frac{t^2}{2} + C$.

Evaluate at $t = 0$, we get $C = p(0) = 0$. Thus, the position of the stone at time t is

$$p(t) = 8t - \frac{t^2}{2}.$$

♠

Example 6.2.16 Assume that an object is moving (horizontally) at a velocity of $\frac{2}{1+t}$ (ft/min), where $t \geq 0$ is measured in minutes. Then find the position of the object at time t.

Solution. Let $p(t)$ be the position of the object at time t, then $p'(t) = \frac{2}{1+t}$. Thus, from

$$\int \frac{2}{1+t}\, dt = 2\ln(1 + t) + C,$$

we know that there is a special C such that $p(t) = 2\ln(1 + t) + C$.

Evaluate at $t = 0$, we get $C = p(0)$ which denotes the initial position. Thus,

$$p(t) = 2\ln(1 + t) + p(0).$$

♠

Example 6.2.17 Assume that when testing a new car, you apply brakes when the car is traveling at 30 mi/h (44 ft/sec) and also assume that the brakes produce a constant deceleration of 22 ft/sec^2, then how far will the car travel before coming to a complete stop?

Solution. Let $p(t)$ be the distance (in feet) the car traveled t seconds after the brakes are applied. Since the brakes produce a constant deceleration of 22 ft/sec^2, the acceleration is given by $p''(t) = -22$ ft/sec^2. Thus, the velocity $p'(t)$ is from

$$\int p''(t)\,dt = \int -22\,dt = -22t + C,$$

so that $p'(t) = -22t + 44$ because $p'(0) = 44$ ft/sec (the initial velocity when you apply the brakes is 44 ft/sec). Therefore, the position function $p(t)$ is from

$$\int p'(t)\,dt = \int (-22t + 44)\,dt = -11t^2 + 44t + C,$$

so that $p(t) = -11t^2 + 44t$ because $p(0) = 0$ (the initial distance when you apply the brakes is zero). Next, the car comes to a complete stop when its velocity is zero, so we have

$$-22t + 44 = 0, \quad \text{or} \quad t = 2.$$

This means that after you apply the brakes, the car travels $t = 2$ seconds before coming to a complete stop. During 2 seconds, or after you apply the brakes, the car travels

$$p(2) = -11(2)^2 + 44(2) = 44 \text{ (ft)}$$

before coming to a complete stop. ♠

Exercises 6.2

1. Find $\int f(x)\,dx$ for

(a) $f(x) = x^{2/5} + 8x^2 - 8$.

(b) $f(x) = x^{-2/3} - x^2 + 8$.

(c) $f(x) = x^{1/2} - \frac{4}{5x}$.
(d) $f(x) = e^4 - e^{-4x} + \frac{3}{x^3}$.
(e) $f(x) = e^4 - \frac{e^{-4x}}{e^4} + \frac{3}{x}$.
(f) $f(x) = e^4 - \frac{e^{-4x}}{4} + \frac{3}{xe^4}$.
(g) $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{x}$.
(h) $f(x) = \frac{5x^5+6x^3-6x^{1/6}+9}{\sqrt{x}}$.

2. Find the area bounded by $f(x) = \sqrt{x} + \frac{8}{3x} + 2 + e^{3x}$ and the interval $[1, 2]$.

3. Find the area bounded by $f(x) = x^{3/2} + \frac{5}{2x} + 5 + e^{5x}$ and the interval $[1, 2]$.

4. Find the area bounded by $f(x) = \sqrt{x+5} + \frac{8}{3x} + 2$ and the interval $[1, 2]$.

5. Find the area bounded by $f(x) = \sqrt{2x+5} + \frac{8}{7+4x} + 2$ and the interval $[1, 2]$.

6. Find the following.

(a) $\int_0^1 (x^{2/5} + 8x^2 - 8)\, dx$.
(b) $\int_0^1 (x^{-2/3} - x^2 + 8)\, dx$.
(c) $\int_1^e (x^{1/2} - \frac{4}{5x})\, dx$.
(d) $\int_1^e (e^4 - e^{-4x} + \frac{3}{x^3})\, dx$.
(e) $\int_1^e (e^4 - \frac{e^{-4x}}{e^4} + \frac{3}{x})\, dx$.
(f) $\int_1^e (e^4 - \frac{e^{-4x}}{4} + \frac{3}{xe^4})\, dx$.
(g) $\int_1^2 (\frac{5x^5+6x^3-6x^{1/6}+9}{x})\, dx$.
(h) $\int_0^1 (\frac{5x^5+6x^3-6x^{1/6}+9}{\sqrt{x}})\, dx$.

7. Assume that you throw a stone (horizontally) with a velocity of $\frac{2}{5+3t}$ (ft/min), where $t \geq 0$ is measured in minutes. Then find the position of the stone at time t (that is, how far it is away from you).

8. A boy runs away from a lamppost along a straight road with a velocity of $10t - t^2$ (ft/min), $t \in [0, 10]$, where t is measured in minutes. Find the position of the boy at time t (that is, how far he is away from the lamppost) if he was 4 feet away from the lamppost initially.

9. Given a marginal cost function $C'(x) = x^2 - 22x + 160$, find the cost function $C(x)$.

10. Given a marginal cost function $C'(x) = x^2 - 22x + 160$, find the increase in cost if the production is raised from 10 to 20 units.

11. Given a marginal revenue function $R'(x) = 22 - 2x$, find the revenue function $R(x)$.

12. Given a marginal revenue function $R'(x) = 22 - 2x$, find the increase in revenue if the production is raised from 10 to 14 units.

13. Assume that when testing a new car, you apply brakes when the car is traveling at 40 mi/h and also assume that the brakes produce a constant deceleration of 22 ft/sec^2, then how far will the car travel before coming to a complete stop?

14. Assume that when testing a new car, you apply brakes when the car is traveling at 30 mi/h and also assume that the brakes produce a constant deceleration of 21 ft/sec^2, then how far will the car travel before coming to a complete stop?

15. Find the function $f(x)$ such that $f'(x) = e^{2x} - x^{-2/3} + 3$ and $f(0) = -4$.

16. Find the function $f(x)$ such that $f'(x) = 4x^{-7/4} + \frac{3}{2x} + x - 2$ and $f(1) = 3$.

17. Consider

$$\int 2x\,dx = x^2 + C.$$

Verify that the curves $x^2 + C$ are disjoint curves. Then verify that these disjoint curves cover the entire xy-plane.

18. Verify Property 6.2.3.

19. Verify Property 6.2.4.