Math 238

Linear Algebra with Differential Equations

Chapter 1. Matrices and Determinants

§1.1. Systems of Linear Equations

Ex. (Page 1) A boat traveling at a constant speed in a river with a constant current speed can travel 48 miles downstream in 4 hours. The same trip upstream takes 6 hours. What is the speed of the boat in still water and what is the speed of the current?

Sol: Let x = speed of boat in still water,

y = speed of current.

$$\left(\frac{d}{t} = s\right) \quad \frac{48}{4} = x + y, \quad \frac{48}{6} = x - y$$

$$\begin{cases} x + y = 12 & ① \\ x - y = 8 & ② \end{cases} \qquad ① + ②: \quad \begin{array}{r} x + y = 12 \\ +)\ x - y = 8 \\ \hline 2x = 20 \end{array} \quad \Rightarrow \begin{cases} x = 10 \\ y = 2 \end{cases}$$

$\Rightarrow \{x = 10, y = 2\}$ is the solution.

In general,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad \text{(SOLE)}$$

is a <u>system of linear equations</u>, which is equivalent to

$$\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \qquad \text{(AM)}$$

called the <u>augmented matrix</u>.

To find solutions of (SOLE), use the Gauss-method:
Do the following <u>elementary row operations</u> on (AM):

① interchange two rows,
② multiply a row by a nonzero number,
③ replace a row by itself plus a multiple of another row.

These operations will NOT change solutions.

Ex. 0.

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 4 & 5 & 8 \\ 0 & 0 & 6 & 9 \end{array}\right]$$

Sol: Now, solve x_3 first: $6x_3 = 9 \Rightarrow x_3 = 3/2$.
Then from $4x_2 + 5x_3 = 8$ to solve x_2 and then from $x_1 + 2x_2 + 3x_3$ to solve x_1.

Ex. 1. (p. 6)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \begin{array}{l} \\ R_2 + (-2R_1) \\ R_3 + 2R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3 + (-3R_2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} R_1 + (-R_2) \\ \\ \\ \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} 3R_1 \\ 3R_2 \\ \\ \end{array}$$

(Can solve from here but keep go on)

$$\left[\begin{array}{ccc|c} 3 & 0 & -3 & 6 \\ 0 & -3 & 6 & -6 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} R_1 + (-R_3) \\ R_2 + 2R_3 \\ \\ \end{array} \quad \left[\begin{array}{ccc|c} 3 & 0 & 0 & -7 \\ 0 & -3 & 0 & 20 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{3}R_2 \\ -\frac{1}{3}R_3 \end{array}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{array}\right] \Rightarrow \begin{cases} x_1 = -7/3 \\ x_2 = -20/3 \\ x_3 = -13/3 \end{cases}$$

is the <u>unique solution</u>.

Ex. 2. (P. 7)

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & -4 & 0 & 1 \\ 2 & 1 & 2 & 5 & 1 \end{array}\right] \begin{array}{l} \\ R_2+(-R_1) \\ R_3+(-R_1) \\ R_4+(-2R_1) \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right] R_2 \Leftrightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right] R_4+R_2 \quad \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right] R_4+(-R_3)$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

No solutions.

$$\stackrel{\dots}{\Longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -1 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Ex. 3 (P. 8)

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right] \stackrel{\dots}{\Longrightarrow} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\Rightarrow \begin{cases} x_1 - 8x_3 = -3 \\ x_2 + 5x_3 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = -3 + 8x_3 \\ x_2 = 3 - 5x_3 \\ x_3 = (?)\ x_3 \ (\underline{\text{free}}) \end{cases}$$

Infinitely many solutions.

Ex. 4. (P. 9)

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - 2x_2 + x_5 = 0 \\ x_3 = 0 \\ x_4 - x_5 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 2x_2 - x_5 \\ x_2 = (?)\ x_2 \ (\text{free}) \\ x_3 = 0 \\ x_4 = x_5 \\ x_5 = (?)\ x_5 \ (\text{free}) \end{cases}$$

Now, x_2 and x_5 are free. That is, ANY variable can be free, NOT always the last ones.

Definition. (P. 7) Reduced row-echelon form (RRE-Form)

① Any rows of zeros (called zero rows) appear at the bottom.

② The first nonzero entry of a nonzero row is 1 (called a leading 1.)

③ The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

④ All the other entries of a column containing a leading 1 are zero.

Definition: (SOLE) is homogeneous if all $b_i = 0$.

Zero solution: all $x_j = 0$. otherwise, nonzero.

Ex. 6. (P. 13) Determine if

$$\begin{cases} 2x_1 + x_2 + x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ 3x_1 - x_2 = 0 \\ 4x_1 - 3x_2 - x_3 = 0 \end{cases}$$

has nonzero solutions.

Sol:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & -3 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{cases} x_1 = -\frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_2 = -\frac{3}{5}x_3 \\ x_3 = x_3 \text{ (free)} \end{cases}$$

Yes. It has nonzero solutions.

Thm 1.1. (P. 11) A homogeneous system of m equations in n variables with $m < n$ has infinitely many nonzero solutions.

Pf: Go to RRE-form: $m\,\overset{n}{\boxed{\diagdown\qquad\qquad}}$. Some variables have no equations to control. So they are free variables. $\blacksquare$

§1.2. Matrices and Matrix Operations

Def:

$$A_{m\times n} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a <u>matrix</u> with m rows and n columns.

<u>Notations</u>: $a_{ij} = \text{ent}_{ij}(A)$.

$M_{m\times n}(\mathbb{R})$ = set of all $m\times n$ matrices with entries from real numbers.

<u>Ex.</u> (Dot product)

$$[1, 2, 3]\cdot\begin{bmatrix}4\\5\\6\end{bmatrix} = (1)(4)+(2)(5)+(3)(6) = 32.$$

For $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m\times n}(\mathbb{R})$, and $c \in \mathbb{R}$, define $A+B = [a_{ij}+b_{ij}]$, $cA = [c\,a_{ij}]$.

For $A_{m\times n} = [a_{ij}]_{m\times n}$ and $B_{n\times \ell} = [b_{ij}]_{n\times \ell}$, define $A_{m\times n}B_{n\times \ell} = [P_{ij}]_{m\times \ell}$, where

$$P_{ij} = \underset{(\text{row } i \text{ from } A)}{[a_{i1}, a_{i2}, \cdots, a_{in}]}\cdot\begin{bmatrix}b_{1j}\\b_{2j}\\\vdots\\b_{nj}\end{bmatrix} (\text{column } j \text{ from } B)$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

<u>Ex.</u> $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $B = \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}$.

$$\Rightarrow A+B = \begin{bmatrix}1+5 & 2+6\\3+7 & 4+8\end{bmatrix} = \begin{bmatrix}6 & 8\\10 & 12\end{bmatrix},\quad 5\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\15 & 20\end{bmatrix},$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}.$$

In general, $AB \neq BA$. If $AB = BA$, then we say A and B commute.

<u>Thm 1.2.</u> (P20) For matrices A, B, C, and scalars $\bar{c}, \bar{d}$,

1. $A + B = B + A$.
2. $A + (B+C) = (A+B) + C$.
3. $\bar{c}(\bar{d}A) = (\bar{c}\bar{d})A$.
4. $\bar{c}(A+B) = \bar{c}A + \bar{c}B$.
5. $(\bar{c}+\bar{d})A = \bar{c}A + \bar{d}A$.

<u>Thm 1.3.</u> (P.23) For matrices A, B, C, and scalar d,

1. $A(BC) = (AB)C$.
2. $A(B+C) = AB + AC$.
3. $(A+B)C = AC + BC$.
4. $d(AB) = (dA)B = A(dB)$.

<u>Pf</u>: ②. $LHS = \mathrm{ent}_{ij}[A(B+C)] = \sum_{k=1}^{n} a_{ik}[\mathrm{ent}_{kj}(B+C)] = \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj})$.

$RHS = \mathrm{ent}_{ij}(AB+AC) = \mathrm{ent}_{ij}(AB) + \mathrm{ent}_{ij}(AC) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$

$$= \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = LHS. \quad \blacksquare$$

If A is a square matrix, define $A^2 = AA$, $A^3 = AAA, \cdots$.

Define $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} = I_n = I$: <u>the identity</u> matrix of size n.

Now, (SOLE) can be written as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}.$$

<u>Thm L1.</u> The RRE-Form of $A_{n\times n}$ must be given by

$$\begin{bmatrix} * & & ? \\ & * & \vdots \\ 0 & \ddots & \\ & & * \end{bmatrix}, \quad \text{where } * = 0 \text{ or } 1.$$

<u>Pf:</u> If the first column is zero, ok. Otherwise, we get

$$\begin{bmatrix} 1 & \\ 0 & ? \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}, \quad \text{so we get} \quad \begin{bmatrix} * & \\ 0 & \\ 0 & ? \\ \vdots & \\ 0 & \end{bmatrix}, \quad \text{where } * = 0 \text{ or } 1.$$

Now, cut the first row and first column, and then repeat. $\blacksquare$

Thm 1.2. For any $B_{n\times 1}$, $A_{n\times n} X_{n\times 1} = B_{n\times 1}$ has unique solution

$\Leftrightarrow$ The RRE-Form of $A_{n\times n}$ is I_n.

Pf: ($\Leftarrow$) $[A|B] \Rightarrow \left[\begin{array}{ccc|c} 1 & & 0 & b_1 \\ & \ddots & & b_2 \\ 0 & & 1 & \vdots \\ & & & b_n \end{array}\right] \Rightarrow$ unique solution.

($\Rightarrow$) Let $B_{n\times 1} = \vec{0} \Rightarrow X = \vec{0}$ is the only solution of $AX = \vec{0}$. Then use Thm 1.1 and check $* = 0$ or 1, and then apply Thm 1.1:

$$\begin{bmatrix} * & & ? \\ & * & \vdots \\ 0 & \ddots & \\ & & * \end{bmatrix} \Rightarrow \begin{bmatrix} * & & 0 \\ & * & \vdots \\ 0 & * & 0 \\ & & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}. \quad \blacksquare$$

§1.3. Inverses of Matrices

Ex. $3^{-1} = \frac{1}{3} \Leftrightarrow (3)(3^{-1}) = (3^{-1})(3) = 1$.

Def. A is called invertible if there is B (called the inverse of A) s.t. $AB = BA = I$.

Thm 1.4. (P.28) If A is invertible, then the inverse of A is unique.

Pf: Suppose B and C are inverses of A, then,

$$\begin{cases} BAC = (BA)C = IC = C \\ BAC = B(AC) = BI = B \end{cases} \Rightarrow B = C. \quad \blacksquare$$

Notation: A^{-1} = the inverse of A.

Thm 1.5' (P. 32) A is invertible $\Rightarrow$ $AX = B$ has a unique solution for any B.

Pf: $A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$. ▨

Def. Elementary Matrices:

$E[i,j] \overset{def.}{=} I(R_i \leftrightarrow R_j)$; $E[c(i)] = I(cR_i)$; $E[i, c(j)] = I(R_i + cR_j)$.

Ex. For $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E[1,2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E[3(1)] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$,

$E[2, 4(1)] = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$.

Thm 1.6' (P. 33)

① $(R_i \leftrightarrow R_j)$ in $A = E[i,j]A$.

② (cR_i) in $A = E[c(i)]A$.

③ $(R_i + cR_j)$ in $A = E[i, c(j)]A$.

④ For any A, there are elementary matrices $E_1, E_2, \cdots, E_m$, s.t. $E_1 E_2 \cdots E_m A$ = RRE-Form of A.

"Pf": For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

①: LHS = $(R_1 \leftrightarrow R_2)$ in $A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

RHS = $E[1,2]A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$. $\Rightarrow$ LHS = RHS.

▨

Thm 1.8 (P. 34) Any elementary matrix is invertible. And

① $E[i,j]^{-1} = E[i,j]$.

② $E[c(i)]^{-1} = E[\frac{1}{c}(i)]$.

③ $E[i, c(j)]^{-1} = E[i, (-c)(j)]$.

Pf: ① $E[i,j]E[i,j] = E[i,j]E[i,j]I = E[i,j](R_i \leftrightarrow R_j \text{ in } I)$
$= (R_i \leftrightarrow R_j) \text{ in } (R_i \leftrightarrow R_j \text{ in } I) = I$. ▨

Thm 1.7 (P. 34) If A, B are invertible, then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Pf: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.
$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

Thm 1.10′ (P. 35) The following are equivalent.

① A is invertible.

② The RRE-Form of $[A \mid I]$ is $[I \mid A^{-1}]$.

③ A is a product of elementary matrices.

Pf: ① ⇒ ②. A is invertible $\Rightarrow$ For any B, $AX = B$ has a unique solution ($X = A^{-1}B$). $\xRightarrow{\text{Thm } 1.2}$ RRE-Form of A is I $\xRightarrow{\text{Thm } 1.6}$ there are elementary matrices $E_1, E_2, \cdots E_m$ s.t. $E_1E_2 \cdots E_m A = I$. Then,

$[A \mid I] \to [E_m A \mid E_m I] \to [E_{m-1} E_m A \mid E_{m-1} E_m I]$

$\longrightarrow [E_1 E_2 \cdots E_m A \mid E_1 E_2 \cdots E_m I] = [I \mid \bar{B}]$,

where $\bar{B} = E_1 E_2 \cdots E_m I$. $\Rightarrow$

$\bar{B} A = E_1 E_2 \cdots E_m A = I$. And $\bar{B} A A^{-1} = A^{-1}$

$\Rightarrow \bar{B} = A^{-1}$. Thus the RRE-Form of

$[A \mid I]$ is $[I \mid \bar{B}] = [I \mid A^{-1}]$.

<u>② ⇒ ③</u>. Now, there are elementary matrices $E_1, \cdots, E_m$ s.t. $E_1 E_2 \cdots E_m A = I$. $\Rightarrow A = E_m^{-1} \cdots E_2^{-1} E_1^{-1}$ is a product of elementary matrices by using Thm 1.8.

<u>③ ⇒ ①</u>. Use Thm 1.7. ▨

<u>Thm 1.9</u>. (P35) If $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

<u>Pf</u>: Only do the case $AB = I$. If A is not invertible, then from Thm L1 and Thm 1.10', the RRE-Form of A contains a zero row.

$$\begin{bmatrix} * & & ? \\ & * & \\ 0 & \ddots & \\ & & * \end{bmatrix}, \quad * = 0 \text{ or } 1.$$

Now, RRE-Form of $A = E_1 E_2 \cdots E_m A$ for some elementary matrices $E_1, \cdots, E_m$. $\Rightarrow$

$[E_1E_2\cdots E_mA]B = E_1E_2\cdots E_m(AB) = E_1E_2\cdots E_m$ also contains a zero row. This is contradiction since $E_1E_2\cdots E_m$ is invertible. $\blacksquare$

Ex. (P.30) Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Sol: $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$. $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

Ex. 1. (P.30)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

A^{-1}

Ex. 2. (P. 31)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

A has No inverse.

Thm 1.5. (P.32) A is invertible $\Leftrightarrow$ $AX = B$ has a unique solution for any B.

Pf: Use Thm 1.1 and Thm 1.10'. $\blacksquare$

§ 1.4. Special Matrices

Diagonal matrix : $\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \overset{\text{def.}}{=\!=} \operatorname{diag}(d_1, d_2, \cdots, d_n)$.

<u>Thm 1.11</u> (P. 38) For $A = \operatorname{diag}(a_1, a_2, \cdots, a_n)$, $B = \operatorname{diag}(b_1, \cdots, b_n)$,

① $A + B = \operatorname{diag}(a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$.

② $AB = \operatorname{diag}(a_1 b_1, a_2 b_2, \cdots, a_n b_n)$.

③ A is invertible $\Leftrightarrow$ each $a_i \neq 0$ $\Leftrightarrow$ $A^{-1} = \operatorname{diag}(\frac{1}{a_1}, \cdots, \frac{1}{a_n})$.

$\begin{bmatrix} * & & & * \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix}$ upper triangular $\qquad$ $\begin{bmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ * & & & * \end{bmatrix}$ lower triangular

<u>Thm 1.12</u> (P. 38)

① If A and B are both upper (or lower) triangular, then so are $A+B$ and AB.

② An upper (or lower) triangular matrix A is invertible $\Leftrightarrow$ each $a_{ii} \neq 0$.

<u>Definition</u> : A^T = transpose of A s.t. $\operatorname{ent}_{ij}(A^T) = a_{ji}$.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Thm 1.13. (P. 39) For matrices A, B, and scalar $\bar{c}$,

① $(A^T)^T = A$.

② $(A+B)^T = A^T + B^T$.

③ $(\bar{c}A)^T = \bar{c}A^T$.

④ $(AB)^T = B^T A^T$.

⑤ $(A^T)^{-1} = (A^{-1})^T$.

Pf: ④. LHS: $ent_{ij}(AB)^T = ent_{ji}(AB) = \sum_k a_{jk} b_{ki}$

RHS: $ent_{ij}(B^T A^T) = \sum_k ent_{ik}(B^T)\, ent_{kj}(A^T) = \sum_k b_{ki} a_{jk}$. ∎

Definition: A is symmetric if $A^T = A$.

Thm 1.14. (P. 40)

① AA^T and $A^T A$ are symmetric.

② If A, B are symmetric, then so are $A+B$ and $\bar{c}A$, $\bar{c} \in \mathbb{R}$.

③ If A is invertible and symmetric, then so is A^{-1}.

§ 1.5. Determinants

Definition: The minor $M_{ij} \overset{\text{def.}}{=\!=}$ the matrix from A after deleting row i and column j.

Determinant of $A = [a_{ij}] \overset{\text{def.}}{=\!=} \det(A) \overset{\text{def.}}{=\!=} |A|$:

For $n=1$: $\det([a_{11}]) = a_{11}$.

For $n \geq 2$: $|A| = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(M_{1n})$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(M_{1j})$$

$$= \sum_{j=1}^{n} a_{1j} [(-1)^{1+j} \det(M_{1j})]$$

<u>Definition</u> : The <u>cofactor</u> $C_{ij} = (-1)^{i+j} \det(M_{ij})$.

<u>Fact</u> : $|A| = \sum_{j=1}^{n} a_{1j} C_{1j}$.

<u>Thm 1.16</u> (P. 46).

$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$ (cofactor expansion about the ith row)

$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ (cofactor expansion about the jth column)

<u>Ex.</u> $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\det(M_{11}) - a_{12}\det(M_{12})$

$$= a_{11}\det([a_{22}]) - a_{12}\det([a_{21}])$$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

Ex.

$$\begin{vmatrix} 2 & 3 & -2 \\ -1 & 6 & 3 \\ 4 & -2 & 1 \end{vmatrix} \xrightarrow{\text{3rd row}} 4(-1)^{3+1}\begin{vmatrix} 3 & -2 \\ 6 & 3 \end{vmatrix} +$$

$$+ (-2)(-1)^{3+2}\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ -1 & 6 \end{vmatrix}$$

$$= 4(21) + 2(4) + 15 = 107.$$

$$\xrightarrow{\text{2nd column}} 3(-1)^{1+2}\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 6(-1)^{2+2}\begin{vmatrix} 2 & -2 \\ 4 & 1 \end{vmatrix} + (-2)(-1)^{3+2}\begin{vmatrix} 2 & -2 \\ -1 & 3 \end{vmatrix}$$

$$= -3(-13) + 6(10) + 2(4) = 107.$$

Ex. 1. (P 47)

$$\begin{vmatrix} 7 & -3 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & -2 & -5 \\ 0 & 4 & 0 & 6 \end{vmatrix} \xlongequal{\text{3rd column}} (-2)\begin{vmatrix} 7 & -3 & 4 \\ 0 & 1 & 3 \\ 0 & 4 & 6 \end{vmatrix}$$

$$\xlongequal{\text{1st col.}} (-2)(7)\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -14(-6) = 84.$$

Thm 1.17 – 1.18 (P. 47) If A has a zero row or zero column, then $|A| = 0$. If A is triangular (upper or lower), then $|A|$ = product of its diagonal entries.

Thm 1.19. (P. 48) $|A^T| = |A|$.

"pf": Do 2×2 and then use induction.

Use row operations to find $|A|$:

Thm 1.20' (p. 48)

① $|A| = -|E[i,j]A|$, $|E[i,j]| = -1$, $|E[i,j]A| = |E[i,j]||A|$.

② $|A| = \frac{1}{c}|E[c(i)]A|$, $|E[c(i)]| = c$, $|E[c(i)]A| = |E[c(i)]||A|$.

③ $|A| = |E[i, c(j)]A|$, $|E[i, c(j)]| = 1$, $|E[i, c(j)]A| = |E[i, c(j)]||A|$.

④ If E is elementary, then $|EA| = |E||A|$.

"Pf": Do 2×2 and then use induction.

Ex. 2. (p. 49)

$$\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & 1 & 4 \end{vmatrix} \begin{matrix} \\ R_2+(-2R_1) \\ R_3+(-R_1) \\ R_4+(-R_1) \end{matrix} = \begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & -1 & 1 \end{vmatrix} \begin{matrix} \\ \\ 3R_3 \\ \\ \end{matrix}$$

$$= \frac{1}{3}\begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 6 & -9 & -15 \\ 0 & 0 & -1 & 1 \end{vmatrix} \begin{matrix} \\ \\ R_3+(-2R_2) \\ \\ \end{matrix} = \frac{1}{3}\begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -1 & 1 \end{vmatrix} \begin{matrix} \\ \\ R_3 \Leftrightarrow R_4 \\ \\ \end{matrix}$$

$$= -\frac{1}{3}\begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -5 & -5 \end{vmatrix} \begin{matrix} \\ \\ \\ R_4+(-5R_3) \end{matrix} = -\frac{1}{3}\begin{vmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -2 & -5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -10 \end{vmatrix}$$

$$= -\frac{1}{3}(1)(3)(-1)(-10) = -10.$$

§1.6. More on Determinants

<u>Thm 1.23'</u> (p.52) For elementary matrices $E_1, E_2, \cdots E_m$,

$$|E_1E_2\cdots E_mA| = |E_1||E_2|\cdots|E_m||A| = |E_1E_2\cdots E_m||A|.$$

<u>Pf</u>: From Thm 1.20',

$$|E_1E_2\cdots E_mA| = |E_1||E_2E_3\cdots E_mA| = |E_1||E_2||E_3\cdots E_mA|$$

$$= \cdots = |E_1||E_2|\cdots|E_m||A|.$$

Next, let $A = I \Rightarrow |E_1E_2E_3\cdots E_m| = |E_1||E_2|\cdots|E_m|$. ▨

<u>Thm 1.21.</u> (p.51) A is invertible $\Leftrightarrow |A| \neq 0$.

<u>pf</u>: ($\Rightarrow$). Now, using Thm 1.10', there are elementary matrices $E_1, E_2, \cdots, E_m$ s.t. $E_1E_2\cdots E_mA = I$.

$\xrightarrow{\text{Thm 1.23'}} |E_1E_2\cdots E_m||A| = |I| = 1. \Rightarrow |A| \neq 0.$

($\Leftarrow$). Let B = RRE-Form of $A = E_1E_2\cdots E_mA$ for some elementary matrices. $\xrightarrow[\text{Thm 1.20'}]{\text{Thm 1.23'}} |B| = |E_1||E_2|\cdots|E_m||A| \neq 0.$

$$\xrightarrow{\text{Thm 1.1}} B = \begin{bmatrix} * & & ? \\ & * & \\ 0 & & \ddots \\ & & & * \end{bmatrix}, \quad * = 0 \text{ or } 1. \Rightarrow B = I.$$

$\xrightarrow{\text{Thm 1.10'}}$ A is invertible. ▨

Thm 1.24. (P.52) $|AB| = |A||B|$.

Pf: Case 1: $|A| \neq 0$. Then from Thm 1.21, A is invertible

$\xRightarrow{\text{Thm 1.10'}}$ $A = E_1 E_2 \cdots E_m$ for some elementary matrices

$\Rightarrow |AB| = |E_1 E_2 \cdots E_m B| \overset{\text{Thm 1.23'}}{=\!=} |E_1 E_2 \cdots E_m||B| = |A||B|$.

Case 2: $|A| = 0$. Now need to show $|AB| = 0$.

From Thm 1.21, A is not invertible $\Rightarrow$ there are elementary matrices $E_1, E_2, \cdots, E_m$ s.t. $E_1 E_2 \cdots E_m A$ (RRE) has a zero row $\Rightarrow (E_1 E_2 \cdots E_m A)B$ has a zero row.

$\Rightarrow 0 = |E_1 E_2 \cdots E_m A B| \overset{\text{Thm 1.23'}}{=\!=} |E_1||E_2| \cdots |E_m||AB|$

$\Rightarrow |AB| = 0$. ▨

Thm 1.25. (P.53) If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$.

Pf: $|A^{-1}||A| \overset{\text{Thm 1.24}}{=\!=} |A^{-1}A| = |I| = 1$. ▨

For $A_{n \times n}$ and cofactor $C_{ij} = (-1)^{i+j}|M_{ij}|$,

$$\text{cofactor matrix of } A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ & \cdots & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix},$$

$$\text{adjoint of } A = \text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ & \cdots & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

Thm 1.26. (P.54) $A(\text{adj}(A)) = (\text{adj}(A))A = |A| I$.

Thm 1.27. (P.54) If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

Pf: From Thm 1.26. $\blacksquare$

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $ad-bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

To solve systems, consider the following.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \quad \vdots \; a_{22}R_1 + (-a_{12}R_2)$$

$\Rightarrow (a_{11}a_{22} - a_{12}a_{21})x_1 = b_1 a_{22} - b_2 a_{12}$.

If $a_{11}a_{22} - a_{12}a_{21} \neq 0 \Rightarrow$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}.$$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$, $A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$,

$$\Rightarrow x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}.$$

Thm 1.28. (P.55, Cramer's Rule) For $A_{n\times n} X = B$ with $|A| \neq 0$, let A_i = replace ith column of A by B. Then

$$X_i = \frac{|A_i|}{|A|}, \quad i = 1, 2, 3, \ldots, n.$$

Ex. 2. (P.56) solve

$$\begin{cases} x_1 e^{2t}\sin t - x_2 e^{2t}\cos t = 1, \\ 2x_1 e^{2t}\cos t + 2x_2 e^{2t}\sin t = t. \end{cases}$$

Sol: $$|A| = \begin{vmatrix} e^{2t}\sin t & -e^{2t}\cos t \\ 2e^{2t}\cos t & 2e^{2t}\sin t \end{vmatrix} = 2e^{4t} > 0.$$

$$|A_1| = \begin{vmatrix} 1 & -e^{2t}\cos t \\ t & 2e^{2t}\sin t \end{vmatrix} = 2e^{2t}\sin t + te^{2t}\cos t.$$

$$|A_2| = \begin{vmatrix} e^{2t}\sin t & 1 \\ 2e^{2t}\cos t & t \end{vmatrix} = te^{2t}\sin t - 2e^{2t}\cos t$$

$$x_1 = e^{-2t}\sin t + \tfrac{1}{2}te^{-2t}\cos t, \quad x_2 = \tfrac{1}{2}te^{-2t}\sin t - e^{-2t}\cos t.$$

Chapter 2. Vector Spaces

§2.1. Vector spaces

Definition: $$R^n = M_{n\times 1}(R) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in R \right\}.$$

<u>Definition</u>: (P. 66) A nonempty set V is a <u>vector space</u> if there are <u>addition</u> and <u>scalar multiplication</u> s.t.

① If $u, v \in V$, then $u+v \in V$ and $u+v = v+u$.

② For $u, v, w \in V$, $u+(v+w) = (u+v)+w$.

③ There is a zero vector $\vec{0}$ in V s.t. for $u \in V$, $u+\vec{0} = u$.

④ For each $u \in V$, there is $-u \in V$ s.t. $u+(-u) = \vec{0}$.

⑤ For $u, v \in V$, $c \in R$, $cu \in V$ and $c(u+v) = cu+cv$.

⑥ For $u \in V$, $c, d \in R$, $(c+d)u = cu+du$.

⑦ For $u \in V$, $c, d \in R$, $c(du) = (cd)u$.

⑧ For $u \in V$, $1 \cdot u = u$.

Elements of a vector space are called <u>vectors</u>.

<u>Ex.</u> R^n, $M_{m \times n}(R)$ are vector spaces.

<u>Ex. 3.</u> (P. 68) Let $F(a,b)$ = set of all real-valued functions on (a,b). Define $(f+g)(x) = f(x) + g(x)$, $(cf)(x) = cf(x)$, then $F(a,b)$ is a vector space.

<u>Ex. 4.</u> (P. 70) For $V = \{(x,y) : x, y \in R\}$. Define

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1+x_2+1, y_1+y_2),$$

$$c \odot (x,y) = (cx, cy).$$

Will it form a vector space?

Sol: ⑤ is not true:

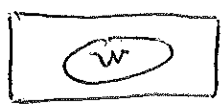
$$LHS = c \odot ((x_1, y_2) \oplus (x_2, y_2)) = c \odot (x_1 + x_2 + 1,\ y_1 + y_2)$$
$$= (cx_1 + cx_2 + c,\ cy_1 + cy_2).$$

$$RHS = c \odot (x_1, y_1) \oplus c \odot (x_2, y_2) = (cx_1, cy_1) \oplus (cx_2, cy_2)$$
$$= (cx_1 + cx_2 + 1,\ cy_1 + cy_2) \neq LHS.$$

$\Rightarrow$ V is not a vector space.

§2.2. Subspaces and Spanning Sets

Definition (p.74) A subset W of a vector space V is a subspace of V if W is itself a vector space under the same operations of V restricted to W.



V

Ex 1. (p.74)

$$W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\} \text{ is a subspace of } \mathbb{R}^3.$$

Thm 2.3 (p.76) Let W be a nonempty subset of a vector space V. W is a subspace $\Leftrightarrow$ for all $u, v \in W$ and $c \in \mathbb{R}$, $u + v \in W$ and $cu \in W$.
(Closed under the operations)

Ex. 2. (P.76) Is $W = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} : x \in R \right\}$ a subspace of $\mathbb{R}^2$?

Sol: $\begin{bmatrix} x_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2 \end{bmatrix} \notin W. \Rightarrow$ Not a subspace.

Ex. 3. (P.76) Is $W = \left\{ \begin{bmatrix} x \\ y \\ x-2y \end{bmatrix} : x, y \in R \right\}$ a subspace of R^3?

Sol: $\begin{bmatrix} x_1 \\ y_1 \\ x_1 - 2y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ x_2 - 2y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) - 2(y_1 + y_2) \end{bmatrix} \in W.$

$c \begin{bmatrix} x \\ y \\ x-2y \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cx - 2cy \end{bmatrix} \in W.$ Yes, a subspace.

Thm 2.4 (P.77) The solutions of $A_{m \times n} \mathbf{X} = \vec{0}$ is a subspace of R^n.

Pf: Let $W = \{ \mathbf{X} \in R^n : A\mathbf{X} = \vec{0} \}$. If $\mathbf{X}_1, \mathbf{X}_2 \in W$

$\Rightarrow A\mathbf{X}_1 = \vec{0}$ and $A\mathbf{X}_2 = \vec{0}$. Then

$A(\mathbf{X}_1 + \mathbf{X}_2) = A\mathbf{X}_1 + A\mathbf{X}_2 = \vec{0} + \vec{0} = \vec{0} \Rightarrow \mathbf{X}_1 + \mathbf{X}_2 \in W.$

$A(c\mathbf{X}_1) = cA\mathbf{X}_1 = c\vec{0} = \vec{0} \Rightarrow c\mathbf{X}_1 \in W.$ $\blacksquare$

Ex. 9 (P.78)

$\mathbb{P} = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : n \geq 0$ is an integer, $a_i \in \mathbb{R}, i = 0, 1, \ldots, n \}$ is a vector space.

($\blacksquare$ $\mathbb{P} \subseteq F(a, b)$ and is closed under the operations)

Ex. 10 (P.78) For an integer $k \geq 0$,

$$P_k = \{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 : n \leq k,\ a_i \in \mathbb{R} \}$$

is a vector space.

$$P_0 = \{ a_0 : a_0 \in \mathbb{R} \} = \mathbb{R}.$$

$$P_1 = \{ mx + b : m, b \in \mathbb{R} \}.$$

$$P_2 = \{ ax^2 + bx + c : a, b, c \in \mathbb{R} \}.$$

$$P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \quad \subseteq P.$$

Ex. Define

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad \ldots,\quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Then for any $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$.

Definition. (P.78) For $v_1, v_2, \ldots, v_n \in V$ and $c_1, c_2, \ldots c_n \in \mathbb{R}$, $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ is called a linear combination of $v_1, v_2, \ldots, v_n$.

Thm 2.5. (P.78) For $v_1, v_2, \ldots v_n \in$ a vector space V, the set W of all linear combinations of $v_1, v_2, \ldots, v_n$ is a subspace of V.

Pf: $[c_1v_1 + c_2v_2 + \cdots + c_nv_n] + [c_1'v_1 + c_2'v_2 + \cdots + c_n'v_n]$

$= (c_1 + c_1')v_1 + (c_2 + c_2')v_2 + \cdots + (c_n + c_n')v_n \in W.$

$c[c_1v_1 + c_2v_2 + \cdots + c_nv_n] = cc_1v_1 + cc_2v_2 + \cdots + cc_nv_n \in W.$ ∎

Definition (P.79-80) The subspace of V from Thm 2.5 is called the span of $v_1, v_2, \cdots, v_n$, denoted by span$\{v_1, \cdots, v_n\}$. And the vectors $v_1, v_2, \cdots, v_n$ span V if span$\{v_1, \cdots, v_n\} = V$.

Ex. 11 (P.79)

$$\text{Is } \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix} \text{ in span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} ?$$

Sol: Try to find c_1, c_2, c_3 s.t.

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 10 \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ -1 & -2 & 0 & -5 \\ 2 & -1 & 1 & 1 \\ 3 & 2 & 3 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

It has a sol.

Yes. In the span.

Ex. 12. (P. 80) Is $2x^2 + x + 1$ in span$\{x^2 + x, x^2 - 1, x + 1\}$?

Sol: Try to find c_1, c_2, c_3 s.t.

$$c_1(x^2 + x) + c_2(x^2 - 1) + c_3(x + 1) = 2x^2 + x + 1.$$

$$\Rightarrow (c_1+c_2)x^2 + (c_1+c_3)x + (-c_2+c_3) = 2x^2+x+1$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 2 \\ c_1 + c_3 = 1 \\ -c_2 + c_3 = 1 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

No solutions. Not in span.

Ex. 13. (p. 80) Do $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ span $\mathbb{R}^2$?

Sol. For any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, try to find c_1, c_2 s.t.

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 2 & a \\ -2 & -4 & b \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 0 & b+2a \end{array}\right].$$

If $b+2a \neq 0$, then $\therefore$ no solutions. $\Rightarrow$ No.

Ex. 14. (p. 81) Do x^2+x-3, $x-5$, 3 span P_2 ?

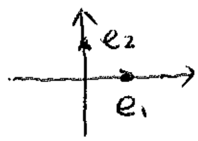
Sol: For any $ax^2+bx+c \in P_2$ (any a, b, c), try to find c_1, c_2, c_3 s.t.

$$c_1(x^2+x-3) + c_2(x-5) + c_3(3) = ax^2+bx+c.$$

$$\Rightarrow \begin{cases} c_1 = a \\ c_1 + c_2 = b \\ -3c_1 - 5c_2 + 3c_3 = c. \end{cases}$$

It has a solution (solve c_1 first, ...), so yes, they span P_2.

§2.3. Linear Independence and Bases

Ex. 

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ point to different directions. They have No linear relationship:

$$a e_1 + b e_2 = \vec{0} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow a = b = 0.$$

Definition (P. 83) Let $v_1, v_2, \cdots, v_n \in V$, and solve

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0} :$$

If the only solution is the zero solution (all $c_i \equiv 0$), then $v_1, v_2, \cdots, v_n$ are linearly independent. Otherwise, if there is a nonzero solution (some $c_i \neq 0$), then $v_1, v_2, \cdots, v_n$ are linearly dependent.

Ex. 1. (p. 84) Are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ linearly dependent?

Sol: Solve

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 1 & 5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Has a nonzero solution. Yes, linearly dependent.

Ex. 2. (P. 84) Are $x^2+1,\ x^2-x+1,\ x+2$ linearly dependent?

Sol: Solve $c_1(x^2+1)+c_2(x^2-x+1)+c_3(x+2)=\vec{0}=0x^2+0x+0$

$$\Rightarrow \begin{cases} c_1+c_2 = 0 \\ -c_2+c_3 = 0 \\ c_1+c_2+2c_3=0 \end{cases}$$

$\Rightarrow 0+2c_3=0 \Rightarrow c_3=0 \Rightarrow c_2=0 \Rightarrow c_1=0. \Rightarrow$ linearly independent.

Thm 2.6. (P. 85) $v_1, v_2, \cdots, v_n$ are linearly dependent

$\Leftrightarrow$ one of $v_1, v_2, \cdots, v_n$ is a linear combination of the others.

Pf: ($\Rightarrow$) Now, there are $c_1, c_2, \cdots, c_n$, not all zero, s.t.

$$c_1v_1+c_2v_2+\cdots+c_nv_n=\vec{0}.$$

Assume $c_1 \neq 0 \Rightarrow v_1=(-\frac{c_2}{c_1})v_2+(\frac{-c_3}{c_1})v_3+\cdots+(-\frac{c_n}{c_1})v_n$.

($\Leftarrow$) Assume $v_1=d_2v_2+d_3v_3+\cdots+d_nv_n,\ d_i \in R$.

$\Rightarrow (1)v_1+(-d_2)v_2+\cdots+(-d_n)v_n=\vec{0} \Rightarrow$ linearly dep. ▨

Note: For v_1 and v_2: v_1 and v_2 are linearly dependent

$\Leftrightarrow v_1=cv_2$ or $v_2=cv_1$ for some $c \in R$.

Definition (P. 86) $v_1, v_2, \cdots, v_n$ form a basis for a vector space V if

① $v_1, v_2, \cdots, v_n$ are linearly independent.

② $v_1, v_2, \cdots, v_n$ span V.

Ex. In R^n,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{form a basis.}$$

Ex. 7. (P. 87) For integer $n \geq 0$,

$$x^n, \; x^{n-1}, \; \cdots, \; x, \; 1$$

form a basis for P_n.

These bases are called standard bases. Other bases are possible.

Ex. 8. (p. 87) Show that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for R^3.

Sol: Independency:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \Rightarrow \text{only zero solution.}$$

$\Rightarrow$ linearly independent.

Span: $c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ (any a, b, c)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & 1 & 1 & b \\ 1 & 1 & 1 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & c-a \end{array}\right].$$ Has a solution. $\Rightarrow$ span.

Ex. 9. (p. 88) Show that x^2+x-3, $x-5$, 3 form a basis for P_2.

Sol: Independency: $c_1(x^2+x-3)+c_2(x-5)+c_3(3)=\vec{0}$

$$\Rightarrow \begin{cases} c_1 = 0 \\ c_1+c_2=0 \\ -3c_1-5c_2+3c_3=0 \end{cases}$$

$\Rightarrow c_1=0 \Rightarrow c_2=0 \Rightarrow c_3=0$.

$\Rightarrow$ linearly independent.

span: See Ex. 14. (p. 81) $\Rightarrow$ span.

Ex. 10. (p. 88) Do x^2+x-1, x^2-x+1 form a basis for P_2?

Sol: $x^2+x-1 \neq c(x^2-x+1) \Rightarrow$ linearly independent.

Span: $c_1(x^2+x-1)+c_2(x^2-x+1) = ax^2+bx+c$ (any a.b.c)

$$\Rightarrow \begin{cases} c_1+c_2 = a \\ c_1-c_2=b \\ -c_1+c_2=c \end{cases}$$

$\Rightarrow b+c=0 \Rightarrow$ Not span $(b=c=1)$.

$\Rightarrow$ Not a basis.

Ex. 11. (p. 89) Do $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ form a basis for $\mathbb{R}^2$?

Sol: Indep: $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} c_1 - c_2 + 2c_3 = 0 \\ c_1 + c_2 + 3c_3 = 0 \end{cases} \xRightarrow{\text{Thm 1.1}} \text{many nonzero solutions.}$$

$\Rightarrow$ linearly dependent. $\Rightarrow$ Not a basis.

§2.4. Dimension, Null space, Row space, Column space

Ex. In R^3, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are linearly dependent: $v = e_1 + 2e_2 + 3e_3$.

Thm 2.8 (P.95) If $v_1, v_2, \cdots, v_n$ are a basis, then every set $\{w_1, w_2, \cdots, w_m\}$ with $m > n$ is linearly dependent.

Thm 2.9. (P.96) If $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_m$ both form bases, then $n = m$.

Pf: Apply Thm 2.8 to $v_1, \cdots, v_n \Rightarrow m \le n$.
Then apply Thm 2.8 to $w_1, \cdots, w_m \Rightarrow n \le m$. ▨

Definition. (P.96) If a vector space V has a basis of n vectors, then the dimension of V is n. ($\dim(V) = n$).

Ex. $\dim(R^n) = n$, $\dim(P_n) = n+1$, $\dim(M_{m\times n}(R)) = m\times n$.
$\dim\{\vec{0}\} = 0$.

Thm 2.10. (P.97) $\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{w_1, \cdots, w_m\}$
$\Leftrightarrow$ each v_i (w_i) is a linear combination of $w_1, \cdots, w_m$ ($v_1, \cdots, v_n$).

Thm 2.11–2.12. (P. 97–98) Let $\dim(\mathcal{V}) = n > 0$.

① If $v_1, v_2, \cdots, v_n$ are linearly independent, then they form a basis.

② If $\text{span}\{v_1, v_2, \cdots, v_n\} = \mathcal{V}$, then $v_1, \cdots, v_n$ form a basis.

③ If $v_1, v_2, \cdots, v_k$ $(k < n)$ are linearly independent, then there are $v_{k+1}, \cdots, v_n$ s.t. $v_1, v_2, \cdots, v_n$ form a basis.

④ If $\text{span}\{v_1, \cdots, v_k\} = \mathcal{V}$ with $k > n$, then there is a subset of $v_1, \cdots, v_k$ that forms a basis.

Pf: ①. Need to show $\text{span}\{v_1, \cdots, v_n\} = \mathcal{V}$.

If $\mathcal{W} = \text{span}\{v_1, \cdots, v_n\} \neq \mathcal{V}$, $\Rightarrow$ there is $u \in \mathcal{V}$ and $u \notin \mathcal{W}$. As $\dim(\mathcal{V}) = n$, $\mathcal{V}$ has a basis with n vectors. $\xRightarrow{\text{Thm 2.8}}$ $v_1, v_2, \cdots, v_n, u$ are linearly dependent.

But next we prove that $v_1, v_2, \cdots, v_n, u$ are linearly independent, which leads to a contradiction.

If $c_1, c_2, \cdots, c_n, c_{n+1}$ solve

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n + c_{n+1} u = \vec{0},$$

then $c_{n+1} = 0$ $(u \notin \text{span}\{v_1, \cdots, v_n\})$.

$$\Rightarrow \quad c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}.$$

Now, $v_1, \cdots, v_n$ are linearly independent $\Rightarrow c_1 = c_2 \cdots = c_n = 0$.

$\blacksquare$

Ex 1. (P. 99) Show that $x^2-1,\ x^2+1,\ x+1$ form a basis for P_2.

Sol: $\dim(P_2)=3$. So only check dependence (or spanning).

$$c_1(x^2-1)+c_2(x^2+1)+c_3(x+1)=\vec{0}.$$

$$\Rightarrow \begin{cases} c_1+c_2 = 0 \\ c_3 = 0 \\ -c_1+c_2+c_3=0 \end{cases} \Rightarrow c_3=0 \Rightarrow 2c_2=0 \Rightarrow c_2=0 \Rightarrow c_1=0.$$

$\Rightarrow$ linearly independent $\Rightarrow$ form a basis.

Definition. (P. 99) The <u>Null Space</u> (or Kernel) of $A_{m\times n}$ is

$$NS(A_{m\times n}) = \{ \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \vec{0} \}$$ (a subspace by Thm 2.4).

Ex. 2. (P. 99) Find a basis for $NS(A)$ if

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 1 & 1 & 0 & 4 & 1 \\ 1 & 4 & -3 & 1 & -2 \end{bmatrix}$$

Sol:

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 4 & 1 & 0 \\ 1 & 4 & -3 & 1 & -2 & 0 \end{array}\right] \xrightarrow{RRE} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 5 & 2 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$NS(A): \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3-5x_4-2x_5 \\ x_3+x_4+x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_4 \underbrace{\begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + x_5 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

$\Rightarrow \text{span}\{v_1, v_2, v_3\}$.

And v_1, v_2, v_3 are linearly independent $\Rightarrow$ v_1, v_2, v_3 form a basis for $NS(A)$.

<u>Definition</u>. (P. 100)

$$RS(A) = \text{row space of } A = \text{span}\{\text{rows of } A\}.$$

<u>Thm 2.13.</u> (P. 101) If A and B are row equivalent (some elementary row operations on A gives B), then

$$RS(A) = RS(B).$$

<u>Pf</u>: Use Thm 2.10 (same span). $\blacksquare$

<u>Ex. 3.</u> (P. 101) In Ex 2 (P. 99), find a basis for $RS(A)$.

<u>Sol</u>: From Ex. 2,

$$A \xrightarrow[\text{equiv.}]{\text{row}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 5 & 2 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} v_1 \\ v_2 \\ \end{array}$$

v_1 and v_2 are linearly independent, and $\text{span}\{v_1, v_2\} = RS(A)$, so v_1 and v_2 form a basis for $RS(A)$.

<u>Thm 2.14.</u> (P. 102). For $A_{m\times n}$,

$$\dim[RS(A)] + \dim[NS(A)] = n.$$

<u>"Pf:"</u> Solve $A\underline{X} = \vec{0}$: $A \Rightarrow$ RRE-Form,

Free variables $\Rightarrow \dim[NS(A)]$.

Nonfree variables $\Rightarrow$ nonzero rows $= \dim[RS(A)]$. $\blacksquare$

Definition (P. 102)

$CS(A)$ = column space of A = span{columns of A}.

Thm. $CS(A) = [RS(A^T)]^T$.

Ex. 4. (P. 102) In Ex. 2 (P. 99), find a basis for $CS(A)$.

Sol:

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -3 \\ 3 & 4 & 1 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$CS(A) = [RS(A^T)]^T$: $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ form a basis for $CS(A)$.

§2.5. Wronskians

For the dependence of n functions $f_1(x), \cdots, f_n(x)$ on (a,b),

solve $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = \vec{0}$ (zero function on (a,b)).

$$\Rightarrow \begin{cases} c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \\ c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0 \\ \cdots \\ c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots + c_n f_n^{(n-1)} = 0 \end{cases} \qquad Eq(\heartsuit)$$

Definition (P. 107)

$$W(x) = \text{Wronskian } W(f_1(x), \cdots, f_n(x)) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ & \cdots & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

<u>Thm 2.15.</u> (p. 108) If $W(f_1(x), \cdots, f_n(x))$ is nonzero for some x in (a,b), then $f_1(x), \cdots, f_n(x)$ are linearly independent.

<u>Pf:</u> $W(x) \neq 0 \Rightarrow W^{-1}(x)$ exists. Now, Eq $(\otimes) \Rightarrow W(x)C = \vec{0}$

$\Rightarrow C = W^{-1}(x)\vec{0} = \vec{0}. \Rightarrow c_i = 0, \ i = 1, 2, \cdots, n.$ ▨

<u>Ex. 1.</u> (p. 108) Are e^x, $\cos x$, $\sin x$ linearly independent?

<u>Sol:</u>

$$W(x) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = 2e^x > 0. \text{ Yes.}$$

Chapter 3. First Order Differential Equations

§ 3.1. Introduction

Around 1800: For some quantity $P(t)$ of time t, the rate of change of $P(t)$ is proportional to $P(t)$, with some proportional constant k.

$\Rightarrow P'(t) = kP(t)$ (Differential Equation).

<u>Ex.</u> $P'(t) = 3P(t)$. Now $P(t) = ce^{3t}$ (c is a constant) is a <u>general solution</u>.

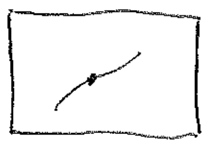
Check: $(ce^{3t})' = c3e^{3t} = 3(ce^{3t})$, i.e. $P'(t) = 3P(t)$.

Ex. Initial Value Problem.

$$p'(t) = 3p(t), \quad p(1) = 2.$$

Now, $p(t) = \frac{2}{e^3} e^{3t}$ is the unique solution.

Thm 3.1. (P. 114) Existence and Uniqueness.



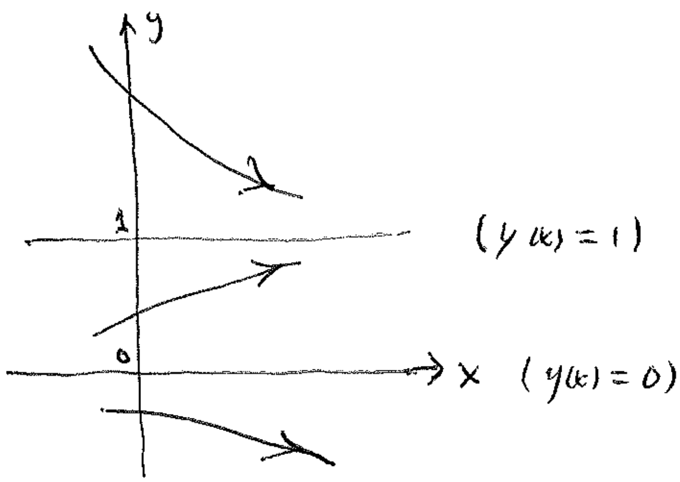
Definition. Graphs of possible constant solutions and other solutions with increasing/decreasing are called phase portraits.

Ex. 1 – Ex. 3 (P. 116–118) Sketch a phase portrait of

ⓐ $y'(x) = y(x) - y^2(x)$. ⓑ $y' = xy$. ⓒ $y' = x^2 + y^2$.

Sol: ⓐ $y' = y(1-y)$, Set $y' = 0 \Rightarrow y(1-y) = 0 \Rightarrow$

$y(x) = 1$, $y(x) = 1$ are constant solutions.

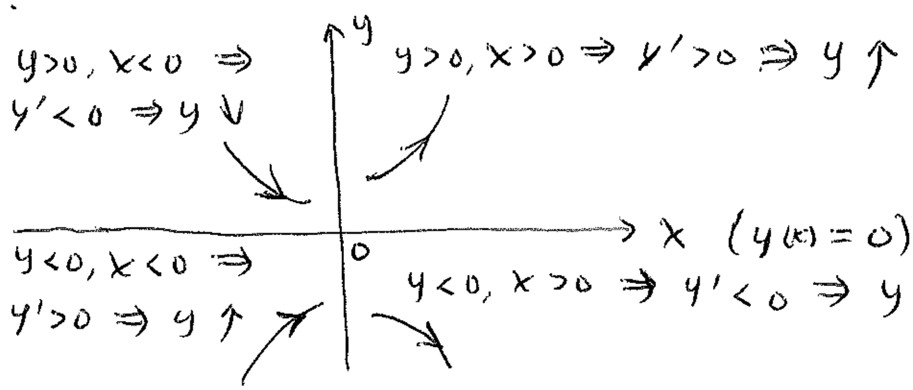


$y > 1 \Rightarrow y' = y(1-y) < 0$
$\Rightarrow y(x)$ decreasing

$0 < y < 1 \Rightarrow y' = y(1-y) > 0$
$\Rightarrow y(x)$ increasing

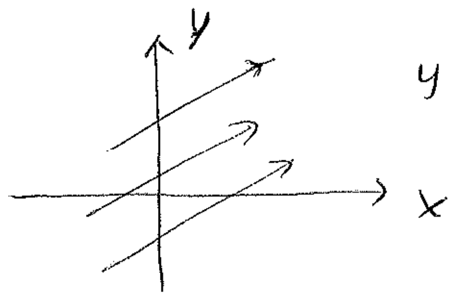
$y < 0 \Rightarrow y' = y(1-y) < 0$
$\Rightarrow y(x)$ decreasing

(b) $y' = xy$. set $y' = 0 \Rightarrow xy = 0 \Rightarrow y(x) = 0$ is a constant solution.



(c) $y' = x^2 + y^2$. set $y' = 0 \Rightarrow x^2 + y^2 = 0 \Rightarrow$ No solutions

$\Rightarrow$ No constant solutions.



$y' = x^2 + y^2 > 0$.

§3.2. Separable Differential Equations

$$\frac{dy}{dx} = M(x)N(y(x)) = M(x)N(y) \text{ : separable}$$

Ex. $y'(x) = x + y$: Not separable.

Ex. 1. (p. 121) Solve $y'(x) = a\,y(x)$.

Sol: $y(x) = 0$ is a constant solution. Next,

$$\frac{dy}{dx} = ay \xrightarrow{y \neq 0} \frac{dy}{y} = a\,dx \Rightarrow \int \frac{dy}{y} = \int a\,dx$$

$\Rightarrow \ln|y| = ax + k \Rightarrow |y| = e^{ax+k} = e^k e^{ax} \Rightarrow$

$y(x) = \pm e^k e^{ax} = ce^{ax}$, a general solution.

($y(x) = 0$ is included in $y(x) = ce^{ax}$, $c \in \mathbb{R}$)

Ex. 2. Solve $y' = xy - 4x$.

Sol: $y' = x(y-4)$, $y(x) = 4$ is a constant solution.

Next, $\frac{dy}{y-4} = x\,dx \Rightarrow \int \frac{dy}{y-4} = \int x\,dx \Rightarrow \ln|y-4| = \frac{x^2}{2} + k$

$\Rightarrow |y-4| = e^k e^{\frac{x^2}{2}} \Rightarrow y(x) = 4 + ce^{\frac{x^2}{2}}$.

Ex. 3. (P. 122) Solve $x^2 y\,dx + (y^2-1)\,dy = 0$.

Sol: $y(x) = 0$ is a constant solution. Next,

$$\frac{y^2-1}{y}\,dy = -x^2\,dx \Rightarrow \int \left(y - \frac{1}{y}\right)dy = \int -x^2\,dx$$

$$\Rightarrow \frac{y^2}{2} - \ln|y| = -\frac{x^3}{3} + k.$$

Ex. 4. Solve $y' = y\cos x - xy$, $y(0) = 1$.

Sol: $\frac{dy}{y} = (\cos x - x)\,dx \Rightarrow \ln|y| = \sin x - \frac{x^2}{2} + k$.

$\Rightarrow |y| = e^k e^{\sin x - \frac{x^2}{2}}$, $y(x) = Ce^{\sin x - \frac{x^2}{2}}$

$y(0) = 1 \Rightarrow 1 = ce^0 = c \Rightarrow y(x) = e^{\sin x - \frac{x^2}{2}}$.

§3.4. Linear Differential Equations

$$y'(x) + p(x)y(x) = q(x). \qquad \text{(LDE)}$$

Homogeneous if $q(x) = 0$.

<u>The method of variation of constants:</u>

First, solve $y'(x) + p(x)y(x) = 0$ (homogeneous)

$\Rightarrow \dfrac{dy}{y} = -p(x)dx \Rightarrow \ln|y| = -\int p(x)dx \Rightarrow$

$y(x) = c_1 e^{-\int p(x)dx}$ for some constant c_1.

<u>Vary this constant c_1 to a function $C(x)$.</u>

<u>Idea:</u> If $y(x) = C(x) e^{-\int p(x)dx}$ is a solution of (LDE),

then find conditions on $C(x)$.

Now, $y'(x) = C'(x) e^{-\int p(x)dx} + C(x) e^{-\int p(x)dx}(-p(x))$

(LDE) $\Rightarrow C'(x)e^{-\int p(x)dx} + C(x)e^{-\int p(x)dx}(-p(x)) + p(x)C(x)e^{-\int p(x)dx} = q(x).$

$\Rightarrow \quad C'(x) = e^{\int p(x)dx} q(x) \Rightarrow C(x) = \int e^{\int p(x)dx} q(x)dx + C$

$\Rightarrow y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} q(x)\,dx + C\right].$

<u>Check:</u> It is a solution of (LDE).

Ex. 1. (P. 132) Solve $y' = 2y + x$.

Sol: $y' - 2y = x$, $p(x) = -2$, $q(x) = x$. $\int p(x)dx = \int -2\,dx = -2x$,

$$\int e^{\int p(x)dx} q(x)dx = \int e^{-2x} x\,dx \overset{\text{by parts}}{=\!=\!=} -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}.$$

$$\Rightarrow y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} q(x)dx + c\right]$$

$$= e^{2x}\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c\right]$$

$$= -\frac{1}{2}x - \frac{1}{4} + ce^{2x}.$$

Ex. 3. (P. 133) Solve $y' = xy - x$, $y(1) = 2$.

Sol: $p(x) = -x$, $q(x) = -x$, $\int p(x)dx = \int -x\,dx = -\frac{x^2}{2}$.

$$\int e^{\int p(x)dx} q(x)dx = \int e^{-\frac{x^2}{2}}(-x)\,dx \overset{\text{by sub}}{=\!=\!=} e^{-\frac{x^2}{2}}.$$

$$y(x) = e^{\frac{x^2}{2}}\left[e^{-\frac{x^2}{2}} + c\right] = 1 + ce^{\frac{x^2}{2}}.$$

Next, $y(1) = 2 \Rightarrow 2 = 1 + ce^{\frac{1}{2}} \Rightarrow c = e^{-\frac{1}{2}}$.

$$\Rightarrow y(x) = 1 + e^{-\frac{1}{2}}e^{\frac{x^2}{2}}.$$

Ex. 4. (P. 133) Solve $xy' + 2y = 3x$, $y(1) = 0$.

Sol: $y' + \frac{2}{x}y = 3$. $p(x) = \frac{2}{x}$, $q(x) = 3$. $\int p(x)dx = \int \frac{2}{x}dx = \ln(x^2)$

$$\int e^{\int p(x)dx} q(x)dx = \int e^{\ln x^2}(3)\,dx = \int 3x^2 dx = x^3$$

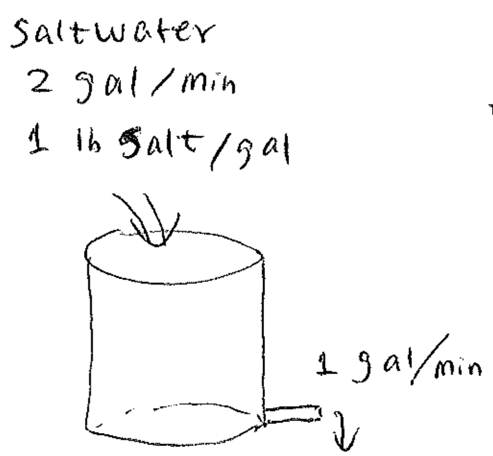
$$y(x) = e^{-\ln x^2}\left[x^3 + c\right] = \frac{1}{x^2}(x^3 + c) = x + \frac{c}{x^2}.$$

$y(1) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1 \Rightarrow y(x) = x - \frac{1}{x^2}$.

§3.6. Modeling with Differential Equations

Ex. 2. (P. 147) (Mixture problem) 10 pounds of salt is dissolved in a 200-gallon tank containing 100 gallons of water. A saltwater solution containing 1 lb salt/gal is then poured into the tank at a rate of 2 gal/min. The tank is continuously well-stirred and drained at a rate of 1 gal/min. How much salt is in the tank after a half hour? How much salt is in the tank when the tank overfills?

Sol.



The volume of water in the tank is increasing at a rate of 1 gal/min $\Rightarrow$ after t min, the volume $= (100+t)$ gal.

The rate of salt (in pounds) entering the tank:

$$\left(2\,\frac{\text{gal}}{\text{min}}\right)\left(1\,\frac{\text{lb}}{\text{gal}}\right) = 2\,\frac{\text{lb}}{\text{min}}.$$

Let $s(t)$ = salt (in pounds) in the tank at time t.

$\Rightarrow$ concentration of salt in the tank $= \dfrac{S(t)}{100+t} \left(\dfrac{lb}{gal}\right)$

$\Rightarrow$ rate of salt leaving the tank :

$$\left(1 \ \frac{gal}{min}\right)\left(\frac{S(t)}{100+t} \ \frac{lb}{gal}\right) = \frac{S(t)}{100+t} \ \frac{lb}{min}.$$

$\Rightarrow$ rate of change of salt in the tank :

$$S'(t) = 2 - \frac{S(t)}{100+t} \qquad \left(y'(x) + p(x)y(x) = q(x)\right)$$

$$\Rightarrow \quad S(t) = 100 + t + \frac{c}{100+t}.$$

Now, $S(0) = 10 \Rightarrow c = -9000$

$$\Rightarrow \quad S(t) = 100 + t - \frac{9000}{100+t}.$$

<u>after a half hour</u> : $S(30) = 130 - \dfrac{9000}{130} \approx 60.77$ lb
salt in the tank.

<u>Overfills</u> : $t = 100$ min. $\Rightarrow S(100) = 155$ lb salt in the tank.

<u>Newton's Law of Cooling</u> :

$$\theta'(t) = K[\theta(t) - H],$$

$\theta(t)$: temperature an object at time t,

H : constant surrounding temperature,

K : a constant.

Ex. 3 (p. 148) A turkey of $40^\circ F$ is placed in an oven of $325^\circ F$. After 20 min, the temperature of the turkey is checked to be $60^\circ F$. How long (total time) will it take to get $185^\circ F$?

Sol: $\theta(t)$: temperature of the turkey at time t. $H = 325$

$\Rightarrow \theta'(t) = k[\theta(t) - 325] \Rightarrow [\theta(t) - 325]' = k[\theta(t) - 325]$

$(y'(x) = a y(x)) \Rightarrow \theta(t) - 325 = c e^{kt} \Rightarrow \theta(t) = 325 + c e^{kt}$.

Now, $\theta(0) = 40 \Rightarrow c = -285 \Rightarrow \theta(t) = 325 - 285 e^{kt}$.

Next, $\theta(20) = 60 \Rightarrow 60 = 325 - 285 e^{k(20)} \Rightarrow$

$$k = \frac{\ln(53/57)}{20}$$

$$\Rightarrow \quad \theta(t) = 325 - 285 e^{\frac{\ln(53/57)}{20} t}.$$

Find t such that $\theta(t) = 185$:

$$325 - 285 e^{\frac{\ln(53/57)}{20} t} = 185 \Rightarrow t \approx 185 \text{ min.} \approx 3 \text{ hours}.$$

Chapter 4. Linear Differential Equations

§4.1. The Theory of High Order Linear Diff. Equ.

(nLDE) $\quad q_n(x) y^{(n)}(x) + q_{n-1}(x) y^{(n-1)}(x) + \cdots + q_1(x) y'(x) + q_0(x) y(x) = g(x)$,

$(q_n(x) \neq 0$ for $x \in (a,b))$

(nLDE)(0): Homogeneous: $g(x) = 0$.

Ex. $y''(x) = 1 \Rightarrow \int_0^x y''(t)dt = \int_0^x dt \Rightarrow y'(x) - y'(0) = x \Rightarrow$

$\int_0^x y'(t)\,dt = \int_0^x (y'(0) + t)\,dt \Rightarrow y(x) - y(0) = y'(0)x + \frac{x^2}{2}$

$\Rightarrow \quad y(x) = y(0) + y'(0)x + \frac{x^2}{2}$.

(Need $y(0)$ and $y'(0)$ to solve $y(x)$ uniquely)

<u>Initial Value Problem</u>: $y(x_0) = k_0,\ y'(x_0) = k_1,\ \cdots,\ y^{(n-1)}(x_0) = k_{n-1}$.

<u>Thm 4.1</u> (P. 181) Existence and Uniqueness for initial value Problems.

<u>Thm 4.2</u> (P. 181) The solutions to an (nLDE)(0) on (a,b) form a vector space of dimension n.

<u>Pf</u>: The solutions of (nLDE)(0) form a subset of $F(a,b)$.

If y_1 and y_2 are solutions of (nLDE)(0) $\Rightarrow$

$y_1 + y_2$ and $c\,y_1$ are also solutions of (nLDE)(0).

$\Rightarrow$ Solutions of (nLDE)(0) form a vector space.

Next, pick $x_0 \in (a,b)$. Let $y_1(x)$ be the solution of (nLDE)(0) with

$y_1(x_0) = 1,\ y_1'(x_0) = 0,\ \cdots,\ y_1^{(n-1)}(x_0) = 0.$ (i.e. e_1)

Let $y_2(x)$: $y_2(x_0) = 0,\ y_2'(x_0) = 1,\ y_2''(x_0) = 0,\ \cdots,\ y_2^{(n-1)}(x_0) = 0.$ (i.e. e_2)

$\vdots$

Let $y_n(x)$: $y_n(x_0) = 0,\ \cdots\ \ y_n^{(n-2)}(x_0) = 0,\ y_n^{(n-1)}(x_0) = 1,$ (i.e. e_n)

We will show $y_1(x), \cdots, y_n(x)$ form a basis.

Indep: The Wronskian at x_0 is

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots, & y_n'(x_0) \\ & \vdots & & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{vmatrix} = 1$$

$\Rightarrow y_1(x), \cdots, y_n(x)$ are linearly independent.

Span: Let $f(x)$ be any solution of (HLDE)(0). Denote

$$f(x_0) = C_1,\ f'(x_0) = C_2,\ \cdots,\ f^{(n-1)}(x_0) = C_n.$$

Let $F(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$. Then $F(x)$ is a solution of (HLDE)(0) (these solutions form a vector space). And

$$F(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) + \cdots + C_n y_n(x_0) = C_1 = f(x_0),$$

$$F'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) + \cdots + C_n y_n'(x_0) = C_2 = f'(x_0),$$

$$\vdots$$

$$F^{(n-1)}(x_0) = C_1 y_1^{(n-1)}(x_0) + \cdots + C_n y_n^{(n-1)}(x_0) = C_n = f^{(n-1)}(x_0)$$

$\Rightarrow f(x)$ and $F(x)$ are solutions of (HLDE)(0) with the same initial values. $\xrightarrow[\text{uniqueness}]{\text{Thm 4.1}}$ $f(x) = F(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$.

$\Rightarrow y_1(x), \cdots, y_n(x)$ span the vector space. $\Rightarrow \dim = n$.

$\boxtimes$

Def. (P. 183) A set of n linearly independent solutions of (nLDE)(0) is called a fundamental set of solutions. Using Thm 2.11 – 2.12 and Thm 4.2, they form a basis.

If $y_1(x), \cdots, y_n(x)$ is a fundamental set of solutions of (nLDE)(0), then the general solution of (nLDE)(0) is given by $y_H(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ for some c_i.

Thm 4.3. (P. 183) If $y_1, \cdots, y_n$ is a fundamental set and if y_p is a solution of (nLDE)(0) (called a particular solution), then the general solution of (nLDE) is

$$y(x) = y_H(x) + y_p(x) = c_1 y_1(x) + \cdots + c_n y_n(x) + y_p(x)$$

for some c_i.

Pf: If $y(x)$ is a solution of (nLDE), then $y(x) - y_p(x)$ is a solution of (nLDE)(0). $\Rightarrow$ $y(x) - y_p(x) = y_H(x)$. ∎

§4.2. Homogeneous constant coefficient Linear Differential Equations

(nLDE)(0)(c) $\quad a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots a_1 y'(x) + a_0 y(x) = 0.$

Ex. $(e^{\lambda x})' = \lambda e^{\lambda x}$, $(e^{\lambda x})'' = \lambda^2 e^{\lambda x}$, $(e^{\lambda x})''' = \lambda^3 e^{\lambda x}$, ...

Thm $y(x) = e^{\lambda x}$ is a solution of (nLDE)(0)(C) $\Leftrightarrow$
λ is a solution of
$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \quad (CE)$$
(called the characteristic equation)

§4.2.1. Characteristic Equations with Real Distinct Roots

Ex. 1. (p. 190) Find the general solution to
$$y'' + 4y' + 3y = 0.$$

Sol: The characteristic equation is
$$\lambda^2 + 4\lambda + 3 = 0 \Rightarrow (\lambda+1)(\lambda+3) = 0$$
$\Rightarrow \lambda_1 = -1,\ \lambda_2 = -3 \Rightarrow y_1(x) = e^{-x},\ y_2(x) = e^{-3x}.$

They are linearly independent ($e^{-x} \neq C e^{-3x}$).

$\Rightarrow$ fundamental set. $\Rightarrow$ the ~~general~~ general solution
$$y(x) = c_1 e^{-x} + c_2 e^{-3x}.$$

Ex. 2. (P. 191) Find the general solution to
$$y''' - 4y' = 0.$$

Sol: $\lambda^3 - 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = \lambda(\lambda-2)(\lambda+2) = 0$

$\Rightarrow \quad y_1(x) = e^{0x} = 1, \quad y_2(x) = e^{2x}, \quad y_3(x) = e^{-2x}.$

Wronskian $\begin{vmatrix} 1 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 0 & 4e^{2x} & 4e^{-2x} \end{vmatrix} = 16 \Rightarrow$ fundamental

$\Rightarrow \quad y(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}.$

~~Ex 3 (p. 191)~~

Thm 4.6 – 4.7 (P. 192 – 193) If $\lambda_1, \cdots, \lambda_k$ are distinct real roots of (CE), then $e^{\lambda_1 x}, \cdots, e^{\lambda_k x}$ are linearly independent solutions of (hLDE)(0)(C). If $k = n$, then $e^{\lambda_1 x}, \cdots, e^{\lambda_n x}$ form a fundamental set of solutions.

§4.2.2. Characteristic Equations with Real Repeated Roots

Thm 4.8. (P. 195) If λ_0 is a root of multiplicity m of the characteristic equation, then

$$e^{\lambda_0 x}, \; x e^{\lambda_0 x}, \; \cdots, \; x^{m-1} e^{\lambda_0 x}$$

are linearly independent solutions of (hLDE)(0)(C).

(Indep: $1, x, x^2, \cdots, x^{m-1}$ are linearly indep.)

Ex. 5. (P. 195) Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Sol: $\lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0 \Rightarrow m = 2, \lambda_0 = -2.$

$\Rightarrow$ general solution $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$

Ex. 6. (p. 195) Find the general solution to

$$y''' - 9y'' + 27y' - 27y = 0.$$

Sol: $\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0 = (\lambda - 3)^3.$

$\Rightarrow y(x) = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}.$

§ 4.2.3. Characteristic Equations with Complex Roots

Ex. For $z = 2 + 3i$ $(i^2 = -1)$, $\Rightarrow \mathrm{Re}(z) = 2$, $\mathrm{Im}(z) = 3$ (imaginary part), $\bar{z} = 2 - 3i$ (conjugate).

Math 236 $\Rightarrow e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

Def: $e^{i\theta} \stackrel{\theta \in \mathbb{R}}{=\!=} \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \cdots$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$\Rightarrow \quad e^{i\theta} = \cos\theta + i\sin\theta, \ \theta \in \mathbb{R}$ (Euler's formula)

<u>Def:</u> $e^{a+bi} = e^{a}e^{bi} = e^{a}\cos b + i e^{a}\sin b, \ a, b \in \mathbb{R}$.

If $\lambda = a+bi$ is a complex root of the characteristic equation, then

$$e^{\lambda x} = e^{(a+bi)x} = e^{ax+bxi} = e^{ax}\cos bx + i e^{ax}\sin bx$$

is a complex-valued solution to (nLDE)(0)(C).

To get real solutions:

<u>Thm 4.9</u> (P. 198) If $u(x) + i v(x)$ is a complex solution to (nLDE)(0)(C) ($a_i \in \mathbb{R}$), then $u(x)$ and $v(x)$ are real solutions to (nLDE)(0)(C).

<u>Pf:</u> Now, $a_n[u(x)+iv(x)]^{(n)} + a_{n-1}[u(x)+iv(x)]^{(n-1)} + \cdots = 0$

$\Rightarrow [a_n u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots] + i[a_n v^{(n)}(x) + a_{n-1}v^{(n-1)}(x) + \cdots] = 0 + i0$

$\Rightarrow a_n u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots = 0$ and

$a_n v^{(n)}(x) + a_{n-1}v^{(n-1)}(x) + \cdots = 0$. $\blacksquare$

<u>Thm 4.10</u> (P. 199) If $a+bi$ is a complex root of the characteristic equation, then

$$e^{ax}\cos bx \quad \text{and} \quad e^{ax}\sin bx$$

are linearly independent solutions to (nLDE)(0)(C).

Pf: From Thm 4.9, they are solutions. Next, $e^{ax}\cos bx \neq C e^{ax}\sin bx$,

$\Rightarrow$ independent. $\boxtimes$

Ex. (P.199) Find the general solution to $y'' + 4y = 0$.

Sol: $\lambda^2 + 4 = 0$, $\lambda = \pm 2i$, use $2i$ (using $-2i$ gives the same general solution) $\Rightarrow$ $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ are fundamental set $\Rightarrow$ general solution $y(x) = C_1 \cos 2x + C_2 \sin 2x$.

Ex. 9. (P. 199) Find the general solution to

$$2y'' + 8y' + 26y = 0.$$

Sol: $\lambda^2 + 4\lambda + 13 = 0 \Rightarrow \lambda = -2 \pm 3i$

$$\Rightarrow y(x) = C_1 e^{-2x}\cos 3x + C_2 e^{-2x}\sin 3x.$$

§4.4. The Method of Variation of Constants

To get particular solutions, consider

(2LDE) $q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = g(x)$.

Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to (2LDE)(0). Then $y_H(x) = C_1 y_1(x) + C_2 y_2(x)$ is the general solution to (2LDE)(0) for some constants C_1 and C_2.

Now, vary the constants C_1 and C_2 to functions $u_1(x)$ and $u_2(x)$.

Idea: If $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ is a solution to (2LODE), then find conditions on $u_1(x)$ and $u_2(x)$.

Now, $y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$.

Impose: $u_1'y_1 + u_2'y_2 = 0 \Rightarrow y_p' = u_1y_1' + u_2y_2'$.

$\Rightarrow y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$.

$$\text{(2LODE)} \Longrightarrow q_2y'' + q_1y' + q_0y$$

$$\begin{pmatrix}\text{If } y_p \text{ is} \\ \text{a solution}\end{pmatrix} \quad = q_2(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + q_1(u_1y_1' + u_2y_2') + q_0(u_1y_1 + u_2y_2) = g.$$

$$\Rightarrow u_1[q_2y_1'' + q_1y_1' + q_0y_1] + u_2[q_2y_2'' + q_1y_2' + q_0y_2] + q_2[u_1'y_1' + u_2'y_2'] = g.$$

$$\xrightarrow[\text{of (2LODE)(0)}]{y_1 \,\&\, y_2 \text{ sols}} u_1[0] + u_2[0] + q_2[u_1'y_1' + u_2'y_2'] = g.$$

$$\Rightarrow \begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(x)/q_2(x) \end{cases}$$ and solve for u_1' and u_2'.

Now, $\left|\begin{matrix}\text{coefficient} \\ \text{matrix}\end{matrix}\right| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \overset{W}{=} \text{wronskian} \neq 0$ (y_1 & y_2 indep.)

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ \frac{g}{q_2} & y_2' \end{vmatrix}$. $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & \frac{g}{q_2} \end{vmatrix}$

Cramer's rule $\Rightarrow$ $u_1' = \frac{W_1}{W}$, $u_2' = \frac{W_2}{W}$

$\Rightarrow$ $u_1 = \int \frac{W_1}{W} dx$, $u_2 = \int \frac{W_2}{W} dx$

$$\Rightarrow \quad y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$= \left(\int \frac{W_1(x)}{W(x)} dx \right) y_1(x) + \left(\int \frac{W_2(x)}{W(x)} dx \right) y_2(x).$$

<u>Check</u>: It is indeed a solution to (2LOE).

<u>Ex. 1.</u> (P. 214) Find a particular solution to

$$y'' + 4y = \tan 2x.$$

<u>Sol</u>: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$

are fundamental.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$W_1 = \begin{vmatrix} 0 & \sin 2x \\ \tan 2x & 2\cos 2x \end{vmatrix} = -\tan 2x \sin 2x$$

$$W_2 = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \tan 2x \end{vmatrix} = \cos 2x \tan 2x = \sin 2x$$

$$\Rightarrow u_1 = \int \frac{w_1}{w}\,dx = \int \frac{-\tan 2x \sin 2x}{2}\,dx = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = \frac{1}{2}\int (\cos 2x - \sec 2x)\,dx$$

$$= \frac{1}{4}\sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x|.$$

$$u_2 = \int \frac{w_2}{w}\,dx = \int \frac{\sin 2x}{2}\,dx = -\frac{1}{4}\cos 2x.$$

$$\Rightarrow y_p(x) = u_1 y_1 + u_2 y_2$$

$$= \left[\frac{1}{4}\sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x|\right]\cos 2x - \frac{1}{4}\cos 2x \sin 2x.$$

Thm 4.12 (p. 215) If $y_1, y_2, \cdots, y_n$ form a fundamental set of solutions to (nLDE)(0), then let $w = \mathrm{Wron}(y_1, \cdots, y_n)$ and let w_i be the determinant obtained from w by replacing the ith column by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ g/a_n \end{bmatrix}$. Then

$$y_p = u_1 y_1 + \cdots + u_n y_n \quad \text{where} \quad u_i = \int \frac{w_i}{w}\,dx$$

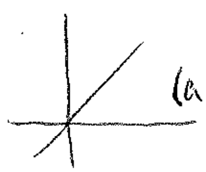
is a particular solution to (nLDE).

Chapter 5. Linear Transformations, Eigenvalues and Eigenvectors

§ 5.1. Linear Transformations

<u>Ex.</u> $f(x) = ax$, $\Rightarrow f(x_1+x_2) = a(x_1+x_2) = ax_1 + ax_2$
$$= f(x_1) + f(x_2).$$

and $f(cx) = a(cx) = c(ax) = cf(x)$.



$(a>0)$ f is a straight line, <u>a linear function</u>.

<u>Def.</u> (P. 232) For vector spaces V and W, a function $T: V \to W$ is called a <u>linear transformation</u> if

① $T(u+v) = T(u) + T(v)$, $u, v \in V$.

② $T(cu) = cT(u)$, $u \in V$, c a scalar.

<u>Ex. 1.</u> (P. 232) Determine if

$$T: R^3 \to R^2 \; ; \; T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix} \text{ is linear.}$$

<u>Sol:</u> For $T(u+v) = T(u) + T(v)$:

$$LHS = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)+(y_1+y_2)-(z_1+z_2) \\ (x_1+x_2)+2(y_1+y_2)+(z_1+z_2) \end{bmatrix}$$

$$RHS = T\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1-z_1 \\ x_1+2y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2+y_2-z_2 \\ x_2+2y_2+z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2+y_1+y_2-z_1-z_2 \\ x_1+x_2+2y_1+2y_2+z_1+z_2 \end{bmatrix} = LHS.$$

For $T(cu) = cT(u)$:

$$LHS = T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} = \begin{bmatrix} cx+cy-cz \\ cx+2cy+cz \end{bmatrix}.$$

$$RHS = cT\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} x+y-z \\ x+2y+z \end{bmatrix} = \begin{bmatrix} cx+cy-cz \\ cx+c2y+cz \end{bmatrix} = LHS.$$

$\Longrightarrow$ Linear transformation.

<u>Ex. 2.</u> (p. 233) Determine if

$$T: \mathbb{R}^2 \to \mathbb{R}^2;\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ x+y+1 \end{bmatrix} \text{ is linear.}$$

<u>Sol:</u> For $T(u+v) = T(u) + T(v)$:

$$LHS = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix} = \begin{bmatrix} (x_1+x_2)^2 \\ (x_1+x_2)+(y_1+y_2)+1 \end{bmatrix}$$

$$RHS = T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1+y_1+1 \end{bmatrix} + \begin{bmatrix} x_2^2 \\ x_2+y_2+1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^2+x_2^2 \\ x_1+x_2+y_1+y_2+2 \end{bmatrix} \neq LHS. \Rightarrow \text{Not linear.}$$

(Also, $T(cu) \neq cT(u)$)

Thm 5.1. (p. 234) $T : R^n \to R^m$ defined by

$$T(\mathbf{x}) = A_{m \times n}\, \mathbf{x}$$

is a linear transformation (called a matrix transformation).

Thm 5.2 (p. 236) If T is a linear transformation, then

$$T(c_1 u_1 + c_2 u_2 + \cdots + c_n u_n) = c_1 T(u_1) + \cdots + c_n T(u_n)$$

for $u_i \in V$ and scalars c_i.

Def. (p. 239) Kernel of $T : V \to W$ = ker(T) $= \{ u \in V : T(u) = \vec{0} \}$.

Ex. For $T\mathbf{x} = A\mathbf{x}$, $\Rightarrow$ ker(T) = NS(A) (Nullspace).

Ex. 5. (p. 239) Find a basis for ker(T) where

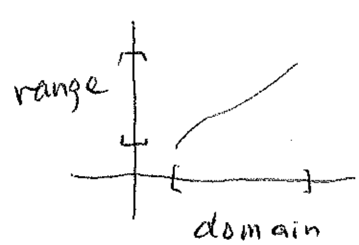
$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Sol: Solve $A\mathbf{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{RRE} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -2z \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \text{ is a basis.}$$

Def. (p240) For $T : V \to W$, range(T) $= \{ T(u) : u \in V \}$



Thm: If $T\mathbf{x} = A\mathbf{x}$, then range $(T) = CS(A)$ (column space).

Pf: range (T): $T(u) = T\mathbf{x} = A\mathbf{x} = [v_1, v_2, \cdots, v_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$: $CS(A)$. ▨

Ex. 6. (p240) Find a basis of range (T),

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ z \\ z \end{bmatrix}.$$

Sol: Same as $CS(A)$:

$$A^T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{row space}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $CS(A) = $ range (T).

Thm 5.3. (P.241) If $T: V \to W$ is a linear transformation, then $\ker(T)$ is a subspace of V and range (T) is a subspace of W.

Pf: $\ker(T)$: Similar to nullspace $A\mathbf{x} = \vec{0}$ (Thm 2.4).
range (T): closed under $+$ and $\cdot$. ▨

<u>Thm 5-4.</u> (p. 242) If $T: V \to W$ is a linear transformation with $\dim(V) = n$ $(< \infty)$, then

$$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(V). \quad (\heartsuit)$$

<u>pf:</u> Only do the case $0 < \dim(\ker(T)) < \dim(V)$.

Let $\dim(\ker(T)) = k$. $\Rightarrow 0 < k < n$. Choose a basis $v_1, v_2, \dots, v_k$ for $\ker(T)$. By Thm 2.11 – 2.12, we can get a basis $v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n$ for V.

<u>Claim</u>: $T(v_{k+1}), T(v_{k+2}), \dots, T(v_n)$ form a basis for $\text{range}(T)$. So that $\dim(\text{range}(T)) = n - k \Rightarrow (\heartsuit)$.

<u>To prove the claim</u>:

<u>Span</u>: For any $y \in \text{range}(T)$, $y = T(u)$ for some $u \in V$.

$\Rightarrow u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k + c_{k+1} v_{k+1} + \cdots + c_n v_n$

for some c_i ($v_1, v_2, \dots v_k, \dots v_n$ form a basis).

$$\Rightarrow y = T(u) = T[c_1 v_1 + \cdots + c_n v_n]$$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_k T(v_k) + c_{k+1} T(v_{k+1}) + \cdots c_n T(v_n)$$

$$= c_{k+1} T(v_{k+1}) + \cdots + c_n T(v_n). \Rightarrow \text{Span}.$$

<u>Indep</u>: Solve $c_{k+1} T(v_{k+1}) + \cdots + c_n T(v_n) = \vec{0}$

$$\Rightarrow T[c_{k+1} v_{k+1} + \cdots + c_n v_n] = \vec{0}$$

$$\Rightarrow c_{k+1} v_{k+1} + \cdots + c_n v_n \in \ker(T)$$

$\Rightarrow c_{k+1}v_{k+1} + \cdots + c_n v_n = c_1 v_1 + \cdots + c_k v_k$ for some $c_1, \cdots, c_k$ ($v_1, v_2, \cdots, v_k$ form a basis for $\ker(T)$).

$\Rightarrow c_1 v_1 + c_2 v_2 + \cdots + c_k v_k - c_{k+1}v_{k+1} \cdots - c_n v_n = \vec{0}$.

Note: $v_1, v_2, \cdots, v_k, v_{k+1}, \cdots v_n$ are linearly independent

$\Rightarrow c_i = 0,\ i = 1, 2, \cdots n$. $\Rightarrow$ Indep. $\square$

Thm 5.5. (P. 242) For a matrix $A_{m\times n}$

$$\dim(RS(A)) = \dim(CS(A)) \quad (\text{called } \underline{\text{rank}(A)}).$$

pf: From Thm 2.14,

$$\dim(RS(A)) = n - \dim(NS(A)). \quad (\Delta)$$

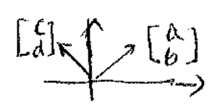
Let $T\mathbf{x} = A_{m\times n}\mathbf{x} : R^n \to R^m \Rightarrow$

$$\dim(RS(A)) \overset{(\Delta)}{=\!=} \dim(R^n) - \dim(\ker(T))$$

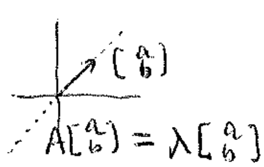
$$\overset{\text{Thm 5.4}}{=\!=} \dim(\text{range}(T)) = \dim(CS(A)). \ \square$$

§5.4. Eigenvalues and Eigenvectors of Matrices.

Ex. In $\mathbb{R}^2$, $A_{2\times 2}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$.



If $A\begin{bmatrix} a \\ b \end{bmatrix} = \lambda\begin{bmatrix} a \\ b \end{bmatrix}$, then $A\begin{bmatrix} a \\ b \end{bmatrix}$ is "along" the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.



Def. (P. 269) If $A_{n\times n} v = \lambda v$ for a nonzero $v \in \mathbb{R}^n$ and a scalar λ, then λ is called an eigenvalue of A, and v is called an eigenvector of A for eigenvalue λ.

Note: $Av = \lambda v \Leftrightarrow Av - \lambda v = \vec{0} \Leftrightarrow (A - \lambda I) v = \vec{0}$.

Thm 5.16. (P. 270) λ is an eigenvalue of $A_{n\times n} \Leftrightarrow$

$$\det(A - \lambda I) = 0. \quad \text{(Characteristic equation)}$$

Pf: ($\Rightarrow$) If $\det(A - \lambda I) \neq 0 \Rightarrow A - \lambda I$ is invertible.
Then $(A - \lambda I) v = \vec{0} \Rightarrow v = (A - \lambda I)^{-1} \vec{0} = \vec{0}$. $\rightarrow\leftarrow$.

($\Leftarrow$) Now, from Thm L1, $A - \lambda I$ has a zero row in its RRE-form $\Rightarrow (A - \lambda I) v = \vec{0}$ has nonzero solutions v. $\blacksquare$

Ex. 1. (P. 270) Find the eigenvalues for $A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$.

Sol: The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

$$= (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1.$$

Def. (P. 271)

$$\text{Eigenspace of } \lambda = E_\lambda = NS(A - \lambda I) = \left\{ \begin{matrix} \text{all eigenvectors of } \lambda \\ \text{and the } \vec{0} \text{ vector} \end{matrix} \right\}.$$

Ex. 2. (P. 271) Find the eigenvectors of A in Ex. 1.

Sol: From Ex 1. $\lambda = 4, -1$.

For $\lambda = 4$: $(A - \lambda I)U = \vec{0} \Rightarrow$

$$[A - \lambda I \mid \vec{0}] = \left[\begin{array}{cc|c} 1-\lambda & -3 & 0 \\ -2 & 2-\lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & -3 & 0 \\ -2 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}.$$

eigenvectors for $\lambda = 4$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $y \neq 0$.

For $\lambda = -1$:

$$\left[\begin{array}{cc|c} 1-\lambda & -3 & 0 \\ -2 & 2-\lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

eigenvectors for $\lambda = -1$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}y \\ y \end{bmatrix} = y = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$, $y \neq 0$.

Ex. 2' Find E_λ of A in Ex. 1.

Sol: From Ex. 2, $E_4 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $E_{-1} = \text{span}\left\{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right\}$.

($\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for E_4; $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ forms a basis for E_{-1})

Ex. 3. (P. 272) Find E_λ of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Sol: $$\begin{vmatrix} 2-\lambda & -1 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(1+\lambda)^2 = 0 \Rightarrow \lambda = 2, -1.$$

For $\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{RRE} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{pmatrix} z = 0 \\ y = 0 \\ 0x = 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ forms a basis for } E_2.$$

$$\Rightarrow E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda = -1$:

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{RRE} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & -\frac{1}{3} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for E_{-1}.

$$\Rightarrow E_{-1} = \text{span}\left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

<u>Ex. 5.</u> (p. 275) Find E_λ for

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Sol: $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2+1 = \lambda^2-2\lambda+2=0$

$\lambda = 1 \pm i$.

For $\lambda = 1+i$:

$$A-\lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \overset{RRE}{\Longrightarrow} \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

$\Rightarrow E_{1+i} = \text{span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 1-i$:

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \overset{RRE}{\Longrightarrow} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -iy \\ y \end{bmatrix} = y\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

$\Rightarrow E_{1-i} = \text{span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$.

§5.5. Similar Matrices, Diagonalization

Def: (p. 278-279) $A_{n\times n}$ and $B_{n\times n}$ are similar if there is $P_{n\times n}$ s.t. $B = P^{-1}AP$. A is said to be diagonalizable if it is similar to a diagonal matrix.

Thm 5.18. (p. 279) $A_{n\times n}$ is diagonalizable $\Longleftrightarrow$ there are n linearly independent eigenvectors in $\mathbb{R}^n$.

Pf: ($\Rightarrow$) Let $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$, $P = [p_1, p_2, \cdots, p_n]$.

$$\Rightarrow \quad AP = P\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} = [P_1, \cdots P_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

$$= [\lambda_1 P_1, \lambda_2 P_2, \cdots, \lambda_n P_n].$$

And $AP = A[P_1, \cdots, P_n] = [AP_1, AP_2, \cdots AP_n]$

$\Rightarrow \quad [AP_1, AP_2, \cdots, AP_n] = [\lambda_1 P_1, \lambda_2 P_2, \cdots, \lambda_n P_n]$

$\Rightarrow \quad AP_i = \lambda_i P_i, \ i = 1, \cdots, n. \Rightarrow P_i$ are eigenvectors.

Next. $|P| \neq 0 \quad (P\mathbf{x} = \vec{0} \Rightarrow \mathbf{x} = P^{-1}\vec{0} = 0) \Rightarrow$

P_i are linearly independent eigenvectors.

($\Leftarrow$) Use the above proof and going back:

From $AP_i = \lambda_i P_i$, define $P = [P_1, P_2, \cdots, P_n]$.

$\Rightarrow \cdots\cdots \Rightarrow \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$. ∎

<u>Thm 5.18'</u> $AP_i = \lambda_i P_i$ with $P_1, P_2 \cdots P_n$ linearly independent eigenvectors, then $P = [P_1, P_2, \cdots, P_n]$ and $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$.

<u>Thm 5.19</u> (P. 279) If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of $A_{n\times n}$ with $\{V_{11}, V_{12}, \cdots, V_{1\ell_1}\}$ a basis for E_{λ_1}, $\{V_{21}, V_{22}, \cdots, V_{2\ell_2}\}$ a basis for E_{λ_2}, $\cdots$, and $\{V_{k1}, V_{k2}, \cdots, V_{k\ell_k}\}$ a basis for E_{λ_k}, then

$V_{11}, \cdots, V_{1\ell_1}, V_{21}, \cdots, V_{2\ell_2}, \cdots, V_{k1}, \cdots, V_{k\ell_k}$ are linearly independent in $\mathbb{R}^n$.

Thm 5.20. (p. 280) If $\lambda_1, \lambda_2, \cdots, \lambda_K$ are all distinct eigenvalues of $A_{n\times n}$, then A is diagonalizable $\Longleftrightarrow$

$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_K}) = n.$$

Pf: ($\Longrightarrow$) From Thm 5.19. we get

$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_K}) \leq n.$$

Next, from Thm 5.18. A has n linearly independent eigenvectors $v_1, v_2, \cdots, v_n$. Now, distribute them to all eigenspaces:

$$\left\{ \begin{matrix} v_1, & v_2, & v_3, & \cdots, & v_n \\ E_{\lambda_1}, & E_{\lambda_2}, & \cdots & & , E_{\lambda_K} \end{matrix} \right\}$$

(Ex. $\left\{ \begin{matrix} v_1, v_2, v_3, v_4, v_5, v_6 \\ E_{\lambda_1}, \quad E_{\lambda_2} \end{matrix} \right\}$ $\{v_1, v_2, v_3, v_4\} \subset E_{\lambda_1}$. $\{v_5, v_6\} \subset E_{\lambda_2}$.

Then $\dim(E_{\lambda_1}) \geq 4$ (Thm 2.8, p. 95) and $\dim(E_{\lambda_2}) \geq 2$.

$\Rightarrow \dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) \geq 4 + 2 = 6$)

Every v_i must belong to some E_λ and $\dim(E_\lambda) \geq$ how many belong to E_λ. Thus

$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_K}) \geq n.$$

($\Longleftarrow$) Now, we have n eigenvectors. From Thm 5.19, they are independent. Then apply Thm 5.18. $\blacksquare$

Ex. 1 (P.280) If A is diagonalizable, then find P s.t. $P^{-1}AP$ is diagonal. $A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$.

Sol: From Ex. 1 and Ex. 2 in § 5.4, $\lambda = 4, -1$ and

$$E_4 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}, \quad E_{-1} = \text{span}\left\{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right\}.$$

$\Rightarrow$ $\dim(E_4) + \dim(E_{-1}) = 1 + 1 = 2 \Rightarrow$ diagonalizable.

From Thm 5.19 and Thm 5.18' $\Rightarrow$

$$P = \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix}, \quad \text{s.t. } P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}.$$

Ex. 2. (P.281) Same as Ex. 1 for $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Sol: From Ex. 3 in § 5.4, $\lambda = 2, -1$, and

$$E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \quad E_{-1} = \text{span}\left\{\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

$\Rightarrow$ $\dim(E_2) + \dim(E_{-1}) = 1 + 2 = 3 \Rightarrow$ diagonalizable.

$$P = \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Ex. 3 (P.282) $\sum \dim(E_{\lambda_i}) \neq n \Rightarrow$ not diagonalizable.

Chapter 6. Systems of Differential Equations

§6.1. Some Basic Results

Let $A(x) = [a_{ij}(x)]_{n\times n}$, $G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{bmatrix}$, $Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$.

Define $Y'(x) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}$.

(SLDEn) : $Y'(x) = A(x)Y(x) + G(x)$.

(SLDEn)(0) : $G(x) = 0$.

Ex. $$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}' = \begin{bmatrix} \sin x & \cos x \\ e^x & \ln x \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} \tan x \\ \cot x \end{bmatrix}.$$

$$\Leftrightarrow \begin{cases} y_1'(x) = \sin x \, y_1(x) + \cos x \, y_2(x) + \tan x \\ y_2'(x) = e^x \, y_1(x) + \ln x \, y_2(x) + \cot x. \end{cases}$$

Thm 6.1. (p. 296) Existence and Uniqueness.

Def. (p. 298) For any $S_1(x), S_2(x), \cdots, S_n(x) \in \mathbb{R}^n(x)$, define Wronskian $W(S_1(x), \cdots, S_n(x)) = |S_1(x), S_2(x), \cdots, S_n(x)|$.

Thm 6.4. (p. 298) If $W(S_1(x), \cdots, S_n(x)) \neq 0$ for some $x \in (a, b)$, then $S_1(x), \cdots, S_n(x)$ are linearly independent.

Pf: Similar to Thm 2.15 (p.108). $(WC = \vec{0} \Rightarrow C = W^{-1}\vec{0} = \vec{0})$.

Thm 6.2. Solutions to (SLDEn)(0) form a vector space of dimension n.

Pf: Similar to Thm 4.2, (p.181).

($Y_i(k_0) = e_i \Rightarrow$ form a basis).

Similar to Chapter 4, we call a set of n linearly independent solutions $Y_1(k), \cdots, Y_n(k)$ to a (SLDE)(0) a fundamental set of solutions.

We call $M(k) = [Y_1(k), \cdots, Y_n(k)]$ a matrix of fundamental solutions. And the general solution to (SLDE)(0) is

$$Y_H(k) = C_1 Y_1(k) + C_2 Y_2(k) + \cdots + C_n Y_n(k) = M(k)C, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}.$$

Thm 6.3. (p.297) If $M(k) = [Y_1(k), \cdots, Y_n(k)]$ is fundamental and $Y_p(k)$ is a solution to (SLDEn) (called a particular solution), then the general solution to (SLDEn) is given by

$$Y(k) = Y_H(k) + Y_p(k) = C_1 Y_1(k) + \cdots + C_n Y_n(k) + Y_p(k) = M(k)C + Y_p(k).$$

Pf: Similar to Thm 4.3 (p.183).

(If $Y(k)$ is a solution to (SLOEn), then $Y - Y_p$ is a solution to (SLDEn)(0) $\Rightarrow Y - Y_p = Y_H$.)

§6.2. Homogeneous Systems with Constant Coefficients : The Diagonalizable Case

$Y'(x) = AY(x)$, A = constant matrix.

Ex. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Leftrightarrow \begin{cases} y_1' = y_1 + 2y_2 \\ y_2' = 3y_1 + 4y_2 \end{cases}$

very difficult to solve directly.

Ex. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Leftrightarrow \begin{cases} y_1' = \lambda_1 y_1, \\ y_2' = \lambda_2 y_2. \end{cases}$

$\Rightarrow \quad y_1(x) = c_1 e^{\lambda_1 x}, \quad y_2(x) = c_2 e^{\lambda_2 x}.$

Ex. $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}' = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{bmatrix}$

$\Rightarrow$ the general solution is $\begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{bmatrix}$ for some c_i.

Thm 6.7. (P. 303) If $P^{-1}AP = B$ (A and B similar) and if Z is a solution of $Z' = BZ$, then PZ is a solution of $Y' = AY$. Also, if $Z_1, Z_2, \cdots, Z_n$ are fundamental for $Z' = BZ$, then $PZ_1, PZ_2, \cdots, PZ_n$

are fundamental for $Y' = AY$.

$$\left(\begin{tikzcd}
Y' = AY \arrow[rr, "P^{-1}AP = B"] & & Z' = BZ \arrow[d] \\
PZ \arrow[u] & & Z \arrow[ll, "P"'] \arrow[d] \\
PZ_1, PZ_2, \cdots, PZ_n \arrow[u] & & Z_1, Z_2, \cdots, Z_n \arrow[ll, "P"']
\end{tikzcd}\right)$$

Pf: Z is a solution $\Rightarrow Z' = BZ = P^{-1}APZ \Rightarrow$

$PZ' = APZ \Rightarrow (PZ)' = A(PZ) \Rightarrow PZ$ is a solution to $Y' = AY$.

Next, if $Z_1, Z_2, \cdots, Z_n$ are fundamental, then

Wronskian $W(PZ_1, PZ_2, \cdots, PZ_n) = |PZ_1, PZ_2, \cdots, PZ_n|$

$= |P[Z_1, Z_2, \cdots, Z_n]| = |P||Z_1, Z_2, \cdots, Z_n| \neq 0$.

$\Rightarrow PZ_1, PZ_2, \cdots, PZ_n$ are linearly independent. $\blacksquare$

Thm LL. If $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$, then

① $Z_1(x) = \begin{bmatrix} e^{\lambda_1 x} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $Z_2(x) = \begin{bmatrix} 0 \\ e^{\lambda_2 x} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\cdots$, $Z_n(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e^{\lambda_n x} \end{bmatrix}$

are fundamental to $Z' = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} Z$.

② $M(x) = P[Z_1, Z_2, \cdots Z_n] = P\begin{bmatrix} e^{\lambda_1 x} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 x} & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots 0 & e^{\lambda_n x} \end{bmatrix}$

is a fundamental matrix for $Y' = AY$.

③ $P\begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{bmatrix}$ is the general solution to $Y' = AY$.

<u>Pf</u>: ① $Z_1(x), \cdots, Z_n(x)$ are linearly independent solutions.

② From Thm 6.7, $PZ_1, \cdots, PZ_n$ are fundamental for $Y' = AY$. $\Rightarrow M(x) = [PZ_1, \cdots, PZ_n] = P[Z_1, \cdots, Z_n]$.

③ The general solution to $Y' = AY$ is

$$c_1PZ_1 + c_2PZ_2 + \cdots + c_nPZ_n = P[c_1Z_1 + \cdots + c_nZ_n]$$

$$= P\begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{bmatrix}. \quad \blacksquare$$

<u>Ex. 1</u>. (P. 304) Find the general solution to

$$Y' = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} Y.$$

<u>Sol</u>: (Very difficult to solve directly).

From Ex. 1 in § 5.5, $\lambda = 4, -1$.

$$P = \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Use Thm LL③ $\Rightarrow$ general solution $= P \begin{bmatrix} c_1 e^{4x} \\ c_2 e^{-x} \end{bmatrix}$

$$= \begin{bmatrix} -1 & 3/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{4x} \\ c_2 e^{-x} \end{bmatrix} = \begin{bmatrix} -c_1 e^{4x} + \frac{3}{2} c_2 e^{-x} \\ c_1 e^{4x} + c_2 e^{-x} \end{bmatrix}. \quad \text{(check it!)}$$

(Linear Algebra $\xrightarrow{\text{help}}$ Differential Equations)

Thm 6.8. (P.306) If $u(x) + iv(x)$ is a complex solution to $Y' = AY$ (A real), then $u(x)$ and $v(x)$ are real solutions to $Y' = AY$.

Pf: $[u + iv]' = A[u + iv] \Rightarrow u' = Au, \; iv' = iAv.$ $\boxtimes$

Ex. 3. (P.307) Find the general solution to

$$Y' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} Y.$$

Sol: From Ex. 5 in §5.4, $\lambda = 1 \pm i$ and

$$E_{1+i} = \text{span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}, \quad E_{1-i} = \text{span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}.$$

$$\Rightarrow P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

From Thm LL③, one solution of $Y' = AY$ is given by

$$P\begin{bmatrix} e^{(1+i)x} \\ 0 \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^x\cos x + ie^x \sin x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -e^x \sin x \\ e^x \cos x \end{bmatrix} + i\begin{bmatrix} e^x \cos x \\ e^x \sin x \end{bmatrix} = u(x) + iv(x).$$

By Thm 6.8 $\Rightarrow$ $u(x)$ and $v(x)$ are real solutions to $Y' = AY$. And $u(x) \neq c\,v(x)$, $c \in \mathbb{R}$, $\Rightarrow$ $u(x)$ and $v(x)$ are linearly independent. $\Rightarrow$

The general solution to $Y' = AY$ is

$$c_1 u(x) + c_2 v(x) = c_1 \begin{bmatrix} -e^x \sin x \\ e^x \cos x \end{bmatrix} + c_2 \begin{bmatrix} e^x \cos x \\ e^x \sin x \end{bmatrix}.$$

§6.4. Nonhomogeneous Linear Systems

To find a particular solution to (SLDEn):

$$Y'(x) = A(x)Y(x) + B(x).$$

Let $M(x)$ be a fundamental matrix for (SLDEn)(0) which has the general solution $M(x)C$ for a constant vector C.

<u>Variation of Constants:</u> Vary this constant vector C to a vector function $V(x)$.

<u>Idea:</u> If $Y_p(x) = M(x)V(x)$ is a solution to (SLDEn), then find conditions on $V(x)$.

Now, $Y_P' = M'U + MU' \overset{(SLOEA)}{=\!=\!=} A(MU) + G$.

We have $M' = AM$ $\left(M' = [Y_1', \cdots, Y_n'] = [AY_1, \cdots AY_n] = AM \right)$.

$\Rightarrow \quad MU' = G \Rightarrow U' = M^{-1}G \Rightarrow U(x) = \int M^{-1}(x)G(x)dx$

$\Rightarrow \quad Y_P(x) = M(x)U(x) = M(x)\int M^{-1}(x)G(x)dx$.

<u>Check</u>: It's indeed a solution to (SLOEA)

(check for $n=1$: $y' = a(x)y(x) + g(x)$).

<u>Ex. 1</u> (P. 317) Find the general solution to

$$Y' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} Y + \begin{bmatrix} 2 \\ x \end{bmatrix}.$$

<u>Sol:</u> The general solution $Y = Y_H + Y_P$.

<u>For Y_H</u>: $\lambda = 2, 3$, $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\Rightarrow \quad Y_H(x) \overset{\text{Thm LL③}}{=\!=\!=} P\begin{bmatrix} c_1e^{2x} \\ c_2e^{3x} \end{bmatrix} = \begin{bmatrix} 2c_1e^{2x} + c_2e^{3x} \\ c_1e^{2x} + c_2e^{3x} \end{bmatrix}.$$

<u>For Y_P</u>: $M(x) \overset{\text{Thm LL②}}{=\!=\!=} P\begin{bmatrix} e^{2x} & 0 \\ 0 & e^{3x} \end{bmatrix} = \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix}$.

$$\left(\text{If } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \neq 0 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$\Rightarrow \quad M^{-1}(x) = \frac{1}{e^{5x}}\begin{bmatrix} e^{3x} & -e^{3x} \\ -e^{2x} & 2e^{2x} \end{bmatrix} = \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix}.$$

$$\Rightarrow \int M^{-1}(x)G(x)\,dx = \int \begin{bmatrix} e^{-2x} & -e^{-2x} \\ -e^{-3x} & 2e^{-3x} \end{bmatrix} \begin{bmatrix} 2 \\ x \end{bmatrix} dx$$

$$= \int \begin{bmatrix} 2e^{-2x} - xe^{-2x} \\ -2e^{-3x} + 2xe^{-3x} \end{bmatrix} dx = \begin{bmatrix} (-\frac{3}{4} + \frac{x}{2})e^{-2x} \\ (\frac{4}{9} - \frac{2}{3}x)e^{-3x} \end{bmatrix}.$$

$$\Rightarrow Y_p(x) = M(x)\int M^{-1}(x)G(x)\,dx$$

$$= \begin{bmatrix} 2e^{2x} & e^{3x} \\ e^{2x} & e^{3x} \end{bmatrix} \begin{bmatrix} (-\frac{3}{4} + \frac{x}{2})e^{-2x} \\ (\frac{4}{9} - \frac{2x}{3})e^{-3x} \end{bmatrix} = \begin{bmatrix} -\frac{19}{18} + \frac{x}{3} \\ -\frac{11}{36} - \frac{x}{6} \end{bmatrix}.$$

§6.5. Converting Differential Equations to First Order Systems

For $q_n(x)y^{(n)}(x) + q_{n-1}(x)y^{(n-1)} + \cdots + q_1(x)y' + q_0(x)y = 0$, $(\otimes)$

let $U_1 = y$, $U_2 = y'$, $U_3 = y''$, $\ldots$, $U_n = y^{(n-1)}(x)$,

$$\Rightarrow \begin{cases} U_1' = y' = U_2 \\ U_2' = y'' = U_3 \\ \quad\vdots \\ U_{n-1}' = y^{(n-1)} = U_n \\ U_n' = y^{(n)} = -\frac{q_0(x)}{q_n(x)}U_1 - \frac{q_1}{q_n}U_2 \cdots - \frac{q_{n-1}}{q_n}U_n \end{cases}$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & & \vdots \\ & & \vdots & & & 1 \\ -\frac{q_0}{q_n} & -\frac{q_1}{q_n} & \cdots & & & -\frac{q_{n-1}}{q_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad (\star\star)$$

<u>Thm 6.10</u> (P. 320) Equations $(\star)$ and $(\star\star)$ are equivalent. The first entry of $(\star\star)$, v_1, is a solution of $(\star)$. Also, when $q_i(x)$ are constants, their characteristic equations are the same.

<u>"pf":</u> For $n = 2$:

$$(\star\star) \Rightarrow \text{The (CE) (§5.4)} = \begin{vmatrix} 0-\lambda & 1 \\ -\frac{q_0}{q_2} & -\frac{q_1}{q_2} - \lambda \end{vmatrix} = \lambda\left(\frac{q_1}{q_2} + \lambda\right) + \frac{q_0}{q_2} = 0$$

$$\Rightarrow \lambda(q_1 + q_2\lambda) + q_0 = q_2\lambda^2 + q_1\lambda + q_0 = 0 \quad \left(\text{(CE) for } (\star)\right) \text{ (§4.2)} \quad \square$$

<u>Ex. 3.</u> (P. 320) convert $y'' - y' - 2y = \sin x$ to a system.

<u>Sol:</u> $v_1 = y$, $v_2 = y'$ $\Rightarrow$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin x \end{bmatrix}.$$

Wronskian from § 6.1 $= \left| \text{sol1}, \text{sol2}, \dots, \text{soln} \right|$

Eq (✡✡)

$$\begin{vmatrix} v_1^1 & v_1^2 & \cdots & v_1^n \\ v_2^1 & v_2^2 & \cdots & v_2^n \\ \vdots & \vdots & & \vdots \\ v_n^1 & v_n^2 & & v_n^n \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$$

= Wronskian from § 2.5.

That is, § 2.5 and chapter 4 are special cases of chapter 6.